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Some recent results on inverse problems arising from phase-field models

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Abstract

In this paper we collect some results recently obtained by the authors, in various settings for inverse problems arising from a phase-field model of parabolic and parabolic-hyperbolic type with memory kernels. In each case we present results of existence, together with results of uniqueness (more common in literature of inverse problems) of solutions.

1 Introduction

This paper contains some of the recent results about identification problems for a phase-field model recently obtained by the authors. We do not intend to give the detailed proofs of our main results (for this we refer the reader to [16] and [17]), but we want to give an overview of them.

Without claiming at completeness, we mention the following papers [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12] for phase field problems, and, for phase-field models with memory [13], [14], [18], [19] and the literature there in. In [15] there is a unified approach for a class of nonlinear parabolic inverse problems.

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If we consider phase-field models with memory, the thermal memory is taken into account modifying the Fourier's heat conduction law by the introduction of a convolution kernel k, that takes memory effects into account. Unfortunately, k cannot be measured directly, but it can be recovered only with additional measurements of the temperature. Consequently, we have to face the problem to determine the temperature θ, the phase-field χ (which takes approximately value $+1$ in the solid and -1 in the liquid) and k simultaneously, i.e., we have to solve an identification problem.
The model we have introduced and studied in [17] and [16] is the same and leads to a system which is of parabolic type (see (3.57) and (4.7)): it has two unknown convolution kernels, but the inverse problems we have solved are different and are set (concerning the space variables), in continuous functions and L^p respectively. In both cases the temperature, the phase-field and the two convolution memory kernels are assumed to be unknown, but, the problems differ as follows: in [17] the heat source is considered (partially) unknown, while in [16] the fifth unknown is the instantaneous conductivity, while the heat source is given.

More interestingly, in order to determine all the unknowns, the continuous space setting, used in [17], has the advantage to allow to make additional measurements of the temperature which can, in principle, be carried out on the boundary, while in the L^p setting, for technical reasons, one is compelled to look for further measurements of the temperature inside the body, which are clearly more difficult to carry out. This can be seen comparing the additional conditions (3.58) (continuous functions spaces) and (4.1)(L^p spaces): in the first case μ_j $(1 \leq j \leq 3)$ is a Borel function in $\overline{\Omega}$, for example a surface integral in $\partial\Omega$, the second case, ϕ_j $(1 \leq j \leq 2)$ is an element of $L^{p'}(\Omega)$.

On the other hand, the L^p setting has the good property of requiring less strong compatibility conditions and regularity on the data.

Finally, we shall consider a (formally) slight variation of the two previous systems. The main difference lies in the fact that we assume that the instantaneous conductivity vanishes. This has the effect (under suitable assumptions) to transform the first equation of the system into a hyperbolic equation, so that we have a hyperbolic equation combined with a parabolic one. So the analysis is quite different and more difficult and, for the sake of simplicity, we have considered only the determination of a single convolution kernel. In this note we report the main results in [21].

In each case, our aim is to find conditions such that the inverse problems we define are well-posed problem in the sense of Hadamard, in the sense that, for each choice of (appropriate) data we get a unique solution in a suitable class, depending continuously on them. We point out that it is not common, for inverse problem, to study existence together with uniqueness. In general many results we can find in the literature deal only with this second aspect.

To conclude, the plan of the paper is the following.

- In Section 2 we introduce the model.
- In Section 3 we consider the parabolic case in spaces of continuous functions.
- In Section 4 we consider the parabolic case in L^p spaces.
- In Section 5 we consider the hyperbolic-parabolic case.

2 The model

In this section we suppose that Ω is an open bounded set in $\mathbb{R}^3$ with sufficiently regular boundary $\partial\Omega$ occupied by an isotropic, rigid and homogeneous heat conductor. We consider only small variations of the absolute temperature and its gradient. A material, which exhibits phase transitions due to the temperature variations, is described by two state variables: the absolute temperature Θ and the phase-field χ at each point $x \in \Omega$ and $t \in [0,T]$ for $T > 0$, where χ takes approximately value -1 in the liquid and $+1$ in the solid.

We now recall the models introduced in [19]. We work with the *temperature variational field* θ defined by:

$$\theta = \frac{\Theta - \Theta_c}{\Theta_c}, \tag{2.1}$$

where Θ_c is the reference temperature at which the transition occurs.

Following Novick-Cohen [24], we assume that the phase-field is ruled by a Cahn–Hilliard type equation of the form (see [3], [7], [23])

$$\partial_t \chi(t,x) = \Delta[-\Delta\chi(t,x) + \chi(t,x)^3 + \gamma'(\chi(t,x)) - \lambda'(\chi)\theta(t,x)], \quad t \in [0,T], \; x \in \Omega, \tag{2.2}$$

where γ and λ are smooth given functions. Let us come to the evolution equation for the temperature. The energy balance equation is

$$\partial_t \mathcal{E}(t,x) + \nabla \cdot \mathbf{J}(t,x) = \mathcal{F}(t,x), \tag{2.3}$$

where $\mathcal{E}$ is the internal energy, $\mathbf{J}$ is the heat flux and $\mathcal{F}$ is the external heat supply. Taking into account a linearized version of the Coleman–Gurtin theory, we assume, as in [10] and [18], the constitutive equations:

$$\mathcal{E}(t,x) = e_c + c_v\Theta_c\theta(t,x) + \int_0^t h(s)\theta(t-s,x)\,ds + \Theta_c\lambda(\chi(t,x)), \tag{2.4}$$

$$\mathbf{J} = -\mathcal{K}\nabla\theta(t,x) - \int_0^t k(s)\nabla\theta(t-s,x)\,ds, \tag{2.5}$$

where h and k account the memory effects, e_c, c_v and $\mathcal{K}$ are the internal energy at equilibrium, the specific heat and the instantaneous conductivity, respectively. Moreover, λ is the already mentioned regular given function. By (2.3), (2.4) and (2.5) we get

$$\begin{aligned}&\partial_t\Big(e_c + c_v\Theta_c\theta(t,x) + \textstyle\int_0^t h(s)\theta(t-s,x)\,ds + \Theta_c\lambda(\chi(t,x))\Big)\\ &-\nabla\cdot\Big(\mathcal{K}\nabla\theta(t,x) + \textstyle\int_0^t k(s)\nabla\theta(t-s,x)\,ds\Big) = \mathcal{F}(t,x).\end{aligned} \tag{2.6}$$

In several papers it is often assumed that λ and γ' are linear functions of their arguments and we set the physical constants equal to one for the sake of simplicity. We get the following system with the initial–boundary conditions:

$$
\begin{cases}
\partial_t\Big(\theta(t,x)+\chi(t,x)+(h*\theta)(t,x)\Big) \\
=\mathcal{K}\Delta\theta(t,x)+k*\Delta\theta(t,x)+\mathcal{F}(t,x), \quad t\in[0,T],\ x\in\Omega, \\
\partial_t\chi(t,x)=-\Delta^2\chi(t,x)+\Delta[\chi^3(t,x)-\chi(t,x)-\theta(t,x)], \quad t\in[0,T],\ x\in\Omega, \\
\theta(0,x)=\theta_0(x),\ x\in\Omega, \\
\chi(0,x)=\chi_0(x),\ x\in\Omega, \\
D_\nu\theta(t,x)=0, \quad t\in[0,T],\ x\in\partial\Omega, \\
D_\nu\chi(t,x)=D_\nu\Delta\chi(t,x)=0, \quad t\in[0,T],\ x\in\partial\Omega,
\end{cases}
\tag{2.7}
$$

where D_ν denotes the outward unit normal vector to $\partial\Omega$ and the symbol $*$ denotes the convolution with respect to the time. We suppose that $\theta_0,\ \chi_0 : \Omega \to \mathbb{R}$ are known functions.

3 The parabolic case in spaces of continuous functions

In this section we shall study an abstract version of problem (2.7). We shall assume that

$$\mathcal{F}(t,x)=f(t)g(x), \tag{3.1}$$

with g known and f unknown. We shall try to determine θ, χ, h, k, f, while the other terms will be considered as known. We start by introducing some notations. Let X be a Banach space with the norm $\|\cdot\|$ and let $T>0$. We denote by $C([0,T];X)$ the space of continuous functions with values in X equipped with the norm:

$$\|u\|_{0,T,X}=\sup_{t\in[0,T]}\|u(t)\|. \tag{3.2}$$

For $\beta\in(0,1)$ we define

$$
\begin{aligned}
C^\beta([0,T];X)=\{u\in C([0,T];X):|u|_{\beta,T,X} \\
=\sup\nolimits_{0\le s<t\le T}(t-s)^{-\beta}\|u(t)-u(s)\|<\infty\},
\end{aligned}
\tag{3.3}
$$

and we endow it with the norm

$$\|u\|_{\beta,T,X}=\|u(0)\|+|u|_{\beta,T,X}. \tag{3.4}$$

If $k\in\mathbb{N}$, we set

$$C^{k+\beta}([0,T];X):=\{u\in C^k([0,T];X):u^{(k)}\in C^\beta([0,T];X)\}. \tag{3.5}$$

If $f\in C^{k+\beta}([0,T];X)$, we define

$$\|u\|_{k+\beta,T,X}:=\sum_{j=0}^{k}\|u^{(j)}(0)\|+|u^{(k)}|_{\beta,T,X}. \tag{3.6}$$

For any pair of Banach spaces X and X_1, $\mathcal{L}(X;X_1)$ denotes the space of all bounded linear operators from X to X_1 equipped with the sup–norm. When $X = X_1$, we set $\mathcal{L}(X) = \mathcal{L}(X;X)$ and we denote by X' the space of all bounded and linear functionals on X. We now give the definition of sectorial operator in order to introduce analytic semigroups.

Definition 3.1. *Let $A : \mathcal{D}(A) \subset X \to X$, be a linear operator, possibly with $\overline{\mathcal{D}(A)} \neq X$. Operator A is said to be sectorial if it satisfies the following assumptions:*

1. *there exist two constants $R > 0$ and $\phi \in (\pi/2, \pi)$ such that any $\lambda \in \mathbb{C}$ with $|\lambda| \geq R$ and $|\arg\lambda| \leq \phi$ belongs to the resolvent set of A;*
2. *there exists $M > 0$ such that $\|\lambda(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq M$ for any $\lambda \in \mathbb{C}$ with $|\lambda| \geq R$, $|\arg\lambda| \leq \phi$.*

The fact that the resolvent set of A is not void implies that A is closed, so that $\mathcal{D}(A)$ endowed with the graph norm

$$\|x\|_{\mathcal{D}(A)} := \|x\| + \|Ax\| \tag{3.7}$$

becomes a Banach space.
If A satisfies Definition 3.1, it is possible to define the semigroup $\{e^{tA}\}_{t\geq 0}$ of bounded linear operators in $\mathcal{L}(X)$ so that $t \to e^{tA}$ is an analytic function from $(0,\infty)$ to $\mathcal{L}(X)$ (for more details see [22], [25], [26], [27], [28]). Let us define the family of interpolation spaces $\mathcal{D}_A(\beta,\infty)$, $\beta \in (0,1)$, between $\mathcal{D}(A)$ and X by

$$\mathcal{D}_A(\beta,\infty) = \Big\{x \in X : \ |x|_{\mathcal{D}_A(\beta,\infty)} := \sup_{0<t<1} t^{1-\beta}\|Ae^{tA}x\| < \infty\Big\}, \tag{3.8}$$

equipped with the norms

$$\|x\|_{\mathcal{D}_A(\beta,\infty)} = \|x\| + |x|_{\mathcal{D}_A(\beta,\infty)} \tag{3.9}$$

that make $\mathcal{D}_A(\beta,\infty)$ Banach spaces. The continuous inclusions

$$\mathcal{D}(A) \subseteq \mathcal{D}_A(\beta,\infty) \subseteq X \tag{3.10}$$

hold for every $\beta \in (0,1)$. We remark that $\mathcal{D}_A(\beta,\infty)$ coincides with the real interpolation space $(X, D(A))_{\beta,\infty}$, with equivalent norms.
We are now in the position to state the inverse problem in the desired abstract setting:

Problem 3.1. (Inverse Abstract Problem (IAP)). *Let X be a Banach space. Determine five functions $\theta : [0,T] \to X$, $\chi : [0,T] \to X$, $f : [0,T] \to \mathbb{R}$, $h : [0,T] \to \mathbb{R}$ and $k : [0,T] \to \mathbb{R}$ satisfying the system*

$$\begin{cases} \theta'(t) + \chi'(t) + (h * \theta)'(t) = A\theta(t) + k * A\theta(t) + f(t)g, \\ \chi'(t) = B\chi(t) + F(\chi(t)) - A\theta(t), \\ \theta(0) = \theta_0, \\ \chi(0) = \chi_0, \end{cases} \tag{3.11}$$

with the additional conditions

$$\Phi_j[\theta(t)] = b_j(t), \quad \forall t \in [0,T], \quad j = 1,2,3, \tag{3.12}$$

where $A : \mathcal{D}(A) \to X$ *and* $B : \mathcal{D}(B) \to X$ *are linear, closed, not necessarily commuting operators, and* θ_0, χ_0, g, F, Φ_j, b_j *(j=1,2,3) are given data whose properties are listed in the sequel.*

(H1) X, $\mathcal{D}(A)$ and $\mathcal{D}(B)$ are Banach spaces with the following continuous embedding $\mathcal{D}(B) \hookrightarrow \mathcal{D}_B(\gamma,\infty) \hookrightarrow \mathcal{D}(A) \hookrightarrow X$, for some $\gamma \in (0,1)$.

(H2) $A : \mathcal{D}(A) \to X$ and $B : \mathcal{D}(B) \to X$ are linear, sectorial operators in X.

(H3) $F \in C^2(\mathcal{D}_B(\gamma,\infty), X)$ and F'' is uniformly Lipschitz continuous from $\mathcal{D}_B(\gamma,\infty)$ to $\mathcal{L}(\mathcal{D}_B(\gamma,\infty), \mathcal{L}(\mathcal{D}_B(\gamma,\infty), X))$ in every bounded subset of $\mathcal{D}_B(\gamma,\infty)$.

(H4)

$$\begin{array}{l} \theta_0 \in \mathcal{D}(A), \quad \chi_0 \in \mathcal{D}(B), \\ v_0 := B\chi_0 + F(\chi_0) - A\theta_0 \in \mathcal{D}(B). \end{array} \tag{3.13}$$

(H5) $g,\ A\theta_0 \in \overline{\mathcal{D}(A)}$ (closure in X).

(H6) $\Phi_j \in X'$, $j = 1,2,3$.

(H7) $b_j \in C^{2+\beta}([0,T])$, $\beta \in (0,1)$, for $j = 1,2,3$.

(H8) $rank(N) = rank(\widetilde{N}) = 2$, where

$$N := \begin{bmatrix} \Phi_1(\theta_0) & -\Phi_1(g) \\ \Phi_2(\theta_0) & -\Phi_2(g) \\ \Phi_3(\theta_0) & -\Phi_3(g) \end{bmatrix}, \tag{3.14}$$

$$\widetilde{N} := \begin{bmatrix} \Phi_1(\theta_0) & -\Phi_1(g) & \Phi_1[A\theta_0] - \Phi_1[v_0] - b_1'(0) \\ \Phi_2(\theta_0) & -\Phi_2(g) & \Phi_2[A\theta_0] - \Phi_2[v_0] - b_2'(0) \\ \Phi_3(\theta_0) & -\Phi_3(g) & \Phi_3[A\theta_0] - \Phi_3[v_0] - b_3'(0) \end{bmatrix}. \tag{3.15}$$

As a consequence of H5 and H8, the system

$$\Phi_j(\theta_0)h_0 - \Phi_j[g]f_0 = \Phi_j[A\theta_0] - \Phi_j[v_0] - b_j'(0), \quad j = 1,2,3.$$

has a unique solution (h_0, f_0).

(H9) $u_0 := A\theta_0 + f_0 g - v_0 - h_0\theta_0 \in \mathcal{D}(A)$.

(H10) $DetM \neq 0$ where

$$M := \begin{bmatrix} \Phi_1(\theta_0) & -\Phi_1(A\theta_0) & -\Phi_1(g) \\ \Phi_2(\theta_0) & -\Phi_2(A\theta_0) & -\Phi_2(g) \\ \Phi_3(\theta_0) & -\Phi_3(A\theta_0) & -\Phi_3(g) \end{bmatrix}. \tag{3.16}$$

Now we set

$$v_1 := [B + F'(\chi_0)]v_0 - Au_0. \tag{3.17}$$

As a consequence of H10, the linear system

$$\begin{aligned} b_j''(0) + \Phi_j[v_1] &= \Phi_j[Au_0 + k_0 A\theta_0] \\ &+\Phi_j[z_0 g - h_0 u_0 - w_0\theta_0], \quad j = 1,2,3 \end{aligned} \tag{3.18}$$

has a unique solution (w_0, k_0, z_0).

(H11) $[2A - h_0]u_0 - [B + F'(\chi_0)]v_0 - w_0\theta_0 + k_0 A\theta_0 + z_0 g \in \mathcal{D}_A(\beta, \infty)$.

(H12) $v_1 \in \mathcal{D}_B(\beta, \infty)$.

(H13) $\Phi_j[\theta_0] = b_j(0)$, $\Phi_j[u_0] = b_j'(0)$, $j = 1, 2, 3$.

We state a *local in time* existence result (giving a sketch of the proof) for the Inverse Abstract Problem and a *global in time* uniqueness result for the same problem. Then we give an application of the above mentioned theorems to the concrete case.

Theorem 3.1. (Existence local in time). *Let the assumptions H1-H13 hold for $\beta \in (0,1)$. Then there exists $\tau \in (0,T]$ such that problem (3.11)-(3.12) has a solution (θ, χ, h, k, f), with*

$$\theta \in C^{2+\beta}([0,\tau]; X) \cap C^{1+\beta}([0,\tau]; \mathcal{D}(A)), \tag{3.19}$$

$$\chi \in C^{2+\beta}([0,\tau]; X) \cap C^{1+\beta}([0,\tau]; \mathcal{D}(B)), \tag{3.20}$$

$$h \in C^{1+\beta}([0,\tau]), \tag{3.21}$$

$$k \in C^{\beta}([0,\tau]), \tag{3.22}$$

$$f \in C^{1+\beta}([0,\tau]). \tag{3.23}$$

Sketch of the proof. For a complete proof, see Section 6 in [17]. Here we give only a sketch. Assume that the conditions $H1 - H13$ are satisfied and a solution (θ, χ, h, k, f), satisfying the regularity assumptions $(3.19) - (3.23)$ in some interval $[0, \tau]$, exists.

We observe that, from the second equation in (3.11), we get

$$\chi'(0) = v_0. \tag{3.24}$$

Applying Φ_j $(1 \le j \le 3)$ to the first equation in (3.11) and using the last for $t = 0$, we get

$$b_j'(0) + \Phi_j(v_0) + h(0)\Phi_j(\theta_0) = \Phi_j(A\theta_0) + f(0)\Phi_j(g), \quad j = 1, 2, 3.$$

From H8 it follows

$$h(0) = h_0, \qquad f(0) = f_0. \tag{3.25}$$

Using again the first equation in (3.11) for $t = 0$, we get

$$\theta'(0) = u_0. \tag{3.26}$$

Now we set

$$\begin{aligned} &u(t) := \theta'(t) \in C^{1+\beta}([0,\tau], X) \cap C^{\beta}([0,\tau], \mathcal{D}(A)), \\ &v(t) := \chi'(t) \in C^{1+\beta}([0,\tau], X) \cap C^{\beta}([0,\tau], \mathcal{D}(B)), \\ &w(t) := h'(t) \in C^{\beta}([0,\tau]), \\ &z(t) := f'(t) \in C^{\beta}([0,\tau]), \end{aligned} \tag{3.27}$$

so that, differentiating system (3.11), we obtain:

$$\begin{cases} u'(t) + v'(t) = (A - h_0)u(t) + k(t)A\theta_0 + k * Au(t) \\ +z(t)g - w(t)\theta_0 - w * u(t), \\ v'(t) = Bv(t) + F'(\chi_0 + 1 * v(t))v(t) - Au(t), \\ u(0) = u_0, \\ v(0) = v_0. \end{cases} \tag{3.28}$$

From (3.28), we have that

$$v'(0) = v_1. \tag{3.29}$$

Applying Φ_j $(1 \le j \le 3)$ to the first equation in (3.28) and using $\Phi_j[u'] = b_j''$, we obtain, for $t = 0$,

$$\begin{aligned} &b_j''(0) + \Phi_j[v_1] = \Phi_j[(A - h_0)u_0] + k(0)\Phi_j[A\theta_0] \\ &+z(0)\Phi_j[g] - w(0)\Phi_j[\theta_0]. \end{aligned} \tag{3.30}$$

Because of condition H10, we get $w(0) = w_0$, $z(0) = z_0$ and $k(0) = k_0$. Using now the second equation in (3.28), we get

$$\begin{cases} u'(t) = (2A - h_0)u(t) - [B + F'(\chi_0)]v(t) - [F'(\chi_0 + 1 * v(t)) \\ -F'(\chi_0)]v(t) - w(t)\theta_0 + k(t)A\theta_0 + k * Au(t) + z(t)g - w * u(t), \\ v'(t) = [B + F'(\chi_0)]v(t) - Au(t) + [F'(\chi_0 + 1 * v(t)) - F'(\chi_0)]v(t), \\ u(0) = u_0, \\ v(0) = v_0. \end{cases} \tag{3.31}$$

Now we consider the system

$$\begin{cases} \mathcal{U}_0'(t) = (2A - h_0)\mathcal{U}_0(t) - [B + F'(\chi_0)]\mathcal{V}_0(t) - w_0\theta_0 + k_0A\theta_0 + z_0g, \quad t \in [0, T], \\ \mathcal{V}_0'(t) = [B + AF'(\chi_0)]\mathcal{V}_0(t) - A\mathcal{U}_0(t), \\ \mathcal{U}(0) = u_0, \\ \mathcal{V}(0) = v_0. \end{cases} \tag{3.32}$$

We introduce the following linear operator $\mathcal{A}$ in the space $X \times X$:

$$\mathcal{D}(\mathcal{A}) := \mathcal{D}(A) \times \mathcal{D}(B), \tag{3.33}$$

$$\mathcal{A}(u,v) := ([2A - h_0]u - [B + F'(\chi_0)]v, \ -Au + [B + F'(\chi_0)]v). \tag{3.34}$$

One can show that the operator $\mathcal{A}$ is sectorial in $X \times X$. Using the assumptions H4, H9, H11, H12, one can show also that (3.32) has a unique solution $(\mathcal{U}_0, \mathcal{V}_0)$ belonging to $(C^{1+\beta}([0,T];X) \cap C^{\beta}([0,T];\mathcal{D}(A))) \times (C^{1+\beta}([0,T];X) \cap C^{\beta}([0,T];\mathcal{D}(B)))$. We introduce the semigroup

$$e^{t\mathcal{A}} := \begin{bmatrix} S_{11}(t) & S_{12}(t) \\ S_{21}(t) & S_{22}(t) \end{bmatrix}, \tag{3.35}$$

and define the operators

$$\begin{aligned} \mathcal{N}_1(u,v,w,k,z)(t) \ :=& \ \int_0^t S_{11}(t-s)\{(w_0 - w(s))\theta_0 + (k(s) - k_0)A\theta_0 + (z(s) - z_0)g \\ & -[F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s) + k * Au(s) - w * u(s)\}\, ds \\ & + \int_0^t S_{12}(t-s)[F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s)\, ds, \end{aligned} \tag{3.36}$$

$$\begin{aligned} \mathcal{N}_2(u,v,w,k,z)(t) \ :=& \ \int_0^t S_{21}(t-s)\{(w_0 - w(s))\theta_0 + (k(s) - k_0)A\theta_0 + (z(s) - z_0)g \\ & -[F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s) + k * Au(s) - w * u(s)\}\, ds \\ & + \int_0^t S_{22}(t-s)[F'(\chi_0 + 1 * v(s)) - F'(\chi_0)]v(s)\, ds. \end{aligned} \tag{3.37}$$

Then, from (3.31), we get (for $t \in [0,\tau]$):

$$\begin{cases} u(t) = \mathcal{U}_0(t) + \mathcal{N}_1(u,v,w,k,z)(t), \\ v(t) = \mathcal{V}_0(t) + \mathcal{N}_2(u,v,w,k,z)(t). \end{cases} \tag{3.38}$$

We now set, for sake of simplicity

$$\widetilde{B} := B + F'(\chi_0). \tag{3.39}$$

Applying Φ_j $(1 \le j \le 3)$ to the first equation in (3.31), we have also

$$\begin{aligned} b_j''(t) =& \ \Phi_j[(2A - h_0)u(t)] - \Phi_j[\widetilde{B}v(t)] - \Phi_j[[F'(\chi_0 + 1 * v(t)) - F'(\chi_0)]v(t)] \\ & -w(t)\Phi_j[\theta_0] + k(t)\Phi_j[A\theta_0] + \Phi_j[k * Au](t) + z(t)\Phi_j[g] - \Phi_j[w * u](t), \quad j = 1,2,3 \end{aligned} \tag{3.40}$$

which implies

$$w(t)\Phi_j[\theta_0] - k(t)\Phi_j[A\theta_0] - z(t)\Phi_j[g] = \Gamma_{0j}(t) + \Gamma_j(u,v,w,k,z)(t), \tag{3.41}$$

where we have set:

$$\Gamma_{0j}(t) := -b_j''(t) + \Phi_j[[2A - h_0]\mathcal{U}_0(t)] - \Phi_j[\widetilde{B}(\mathcal{V}_0)(t)], \tag{3.42}$$

$$\begin{aligned}\Gamma_j(u,v,w,k,z)(t) \quad := \quad & \Phi_j[[2A-h_0]\mathcal{N}_1(u,v,w,k,z)(t)] - \Phi_j[\widetilde{B}\mathcal{N}_2(u,v,w,k,z)(t)] \\ & -\Phi_j[[F'(\chi_0+1*v(t)) - F'(\chi_0)]v(t)] + \Phi_j[k*Au](t) \\ & +\Phi_j[w*u](t), \quad j=1,2,3.\end{aligned} \tag{3.43}$$

From assumption H10, we obtain

$$\begin{bmatrix} w(t) \\ k(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} \widetilde{\Gamma}_{01}(t) \\ \widetilde{\Gamma}_{02}(t) \\ \widetilde{\Gamma}_{03}(t) \end{bmatrix} + \begin{bmatrix} \widetilde{\Gamma}_1(u,v,w,k,z)(t) \\ \widetilde{\Gamma}_2(u,v,w,k,z)(t) \\ \widetilde{\Gamma}_3(u,v,w,k,z)(t) \end{bmatrix}, \tag{3.44}$$

with

$$\begin{bmatrix} \widetilde{\Gamma}_1(w,k,z,u,v) \\ \widetilde{\Gamma}_2(w,k,z,u,v) \\ \widetilde{\Gamma}_3(w,k,z,u,v) \end{bmatrix} := M^{-1} \begin{bmatrix} \Gamma_1(w,k,z,u,v) \\ \Gamma_2(w,k,z,u,v) \\ \Gamma_3(w,k,z,u,v) \end{bmatrix}, \tag{3.45}$$

$$\begin{bmatrix} \widetilde{\Gamma}_{01}(t) \\ \widetilde{\Gamma}_{02}(t) \\ \widetilde{\Gamma}_{03}(t) \end{bmatrix} := M^{-1} \begin{bmatrix} \Gamma_{01}(t) \\ \Gamma_{02}(t) \\ \Gamma_{03}(t) \end{bmatrix}. \tag{3.46}$$

Then we can consider the system (3.38)-(3.44), which can be (locally) solved through a fixed point argument. □

Theorem 3.2. (Uniqueness global in time). *Let the assumptions H1-H13 hold for $\beta \in (0,1)$. If $(\theta_1,\chi_1,h_1,k_1,f_1)$ and $(\theta_2,\chi_2,h_2,k_2,f_2)$ are solutions of (3.11)-(3.12) both satisfying the regularity conditions*

$$\theta_j \in C^{2+\beta}([0,\tau];X) \cap C^{1+\beta}([0,\tau];\mathcal{D}(A)), \tag{3.47}$$

$$\chi_j \in C^{2+\beta}([0,\tau];X) \cap C^{1+\beta}([0,\tau];\mathcal{D}(B)), \tag{3.48}$$

$$h_j \in C^{1+\beta}([0,\tau]), \tag{3.49}$$

$$k_j \in C^{\beta}([0,\tau]), \tag{3.50}$$

$$f_j \in C^{1+\beta}([0,\tau]), \tag{3.51}$$

for $j=1,2$ and for some $\tau \in (0,T]$, then they coincide.

Proof. See Section 6 in [17]. □

3.1 An application of the abstract results

We wish to apply Theorems 3.1 and 3.2 in the case X is the continuous functions space, so we set:

$$X = C(\overline{\Omega}). \tag{3.52}$$

We define

$$\begin{cases} \mathcal{D}(A) = \{\theta \in \bigcap_{1\le p<+\infty} W^{2,p}(\Omega) \ : \ \Delta\theta \in C(\overline{\Omega}), \quad D_\nu\theta|_{\partial\Omega} = 0\}, \\ A\theta := \Delta\theta, \quad \forall\theta \in \mathcal{D}(A), \end{cases} \tag{3.53}$$

and

$$\begin{cases} \mathcal{D}(B) = \{\chi \in \bigcap_{1\le p<+\infty} W^{4,p}(\Omega) : \ \Delta^2\chi \in C(\overline{\Omega}), \quad D_\nu\chi|_{\partial\Omega} = D_\nu\Delta\chi|_{\partial\Omega} = 0\}, \\ B\chi := -\Delta^2\chi, \quad \forall\chi \in \mathcal{D}(B). \end{cases} \tag{3.54}$$

We recall the following characterizations concerning the interpolation spaces related to A and B (see [22]):

$$\mathcal{D}_A(\beta,\infty) = \begin{cases} C^{2\beta}(\overline{\Omega}), \ \text{if} \ 0<\beta<1/2, \\ \{u \in C^{2\beta}(\overline{\Omega}) : \ D_\nu u|_{\partial\Omega} = 0\}, \ \text{if} \ 1/2<\beta<1, \end{cases} \tag{3.55}$$

$$\mathcal{D}_B(\gamma,\infty) = \begin{cases} C^{4\gamma}(\overline{\Omega}), \ \text{if} \ 0<\gamma<1/4, \\ \{u \in C^{4\gamma}(\overline{\Omega}) : \ D_\nu u|_{\partial\Omega} = 0\}, \ \text{if} \ 1/4<\gamma<3/4, \ \gamma \ne 1/2, \\ \{u \in C^{4\gamma}(\overline{\Omega}) : \ D_\nu u|_{\partial\Omega} = D_\nu\Delta u|_{\partial\Omega} = 0\}, \ \text{if} \ 3/4<\gamma<1. \end{cases} \tag{3.56}$$

We consider the following system:

$$\begin{cases} \partial_t\Big(\theta + \chi + (h*\theta)\Big)(t,x) \\ = \Delta\theta(t,x) + (k*\Delta\theta)(t,x) + f(t)g(x), \ \ t\in[0,T], \ x\in\Omega, \\ \partial_t\chi(t,x) = -\Delta^2\chi(t,x) + \Delta[\phi\circ\chi - \theta](t,x), \ \ t\in[0,T], \ x\in\Omega, \\ \theta(0,x) = \theta_0(x), \ x\in\Omega, \\ \chi(0,x) = \chi_0(x), \ x\in\Omega, \\ D_\nu\theta(t,x) = 0, \ \ t\in[0,T], \ x\in\partial\Omega, \\ D_\nu\chi(t,x) = D_\nu\Delta\chi(t,x) = 0, \ \ t\in[0,T], \ x\in\partial\Omega, \end{cases} \tag{3.57}$$

which, for $\phi(\chi) = \chi^3 - \chi$, is exactly system (2.7) with $\mathcal{K} = 1$. The functions: $\phi : \mathbb{R} \to \mathbb{R}$, and θ_0, $\chi_0 : \Omega \to \mathbb{R}$ are known and their properties will be specified in the sequel. Now we prescribe the further measurements of θ already mentioned in the introduction:

$$\Phi_j[\theta(t,.)] := \int_{\overline{\Omega}} \theta(t,x)\mu_j(dx) = g_j(t), t\in[0,\tau], \tag{3.58}$$

for $j = 1,2,3$, where μ_j is a Borel measure in $\overline{\Omega}$ and g_j is a known function of domain $[0,T]$.

To apply the abstract theorems we have to verify that conditions $H1-H13$ are satisfied in this case. We make the following assumptions, which imply H1-H13, as it is proved in [17]:

(K1) Ω is an open bounded set in $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a sub-manifold of $\mathbb{R}^n$ of class C^4.

(K2) We set

$$F(\chi) := \Delta(\phi \circ \chi), \tag{3.59}$$

and we assume that $\phi \in C^4(\mathbb{R})$ with $\phi^{(4)}$ Lipschitz continuous on bounded subsets of $\mathbb{R}$.

(K3) $\theta_0 \in \mathcal{D}(A)$ (defined in (3.53)), $\quad \chi_0 \in \mathcal{D}(B)$ (defined in (3.54)).

(K4) $\quad v_0 := -\Delta^2\chi_0 + \Delta(\phi \circ \chi_0) - \Delta\theta_0 \in \mathcal{D}(B)$.

(K5) $g \in C(\overline{\Omega})$.

(K6) μ_j is a Borel measure in $\overline{\Omega}$, for $j = 1, 2, 3$.

(K7) $b_j \in C^{2+\beta}([0, T])$, for $j = 1, 2, 3$.

(K8) Set, for $\phi \in C(\overline{\Omega})$,

$$\Phi_j[\phi] := \int_{\overline{\Omega}} \phi(x)\mu_j(dx), \text{ for } j = 1, 2, 3. \tag{3.60}$$

Now, taking into account (3.60), $K3$, $K5$, $K7$, we can consider the matrices N and $\widetilde{N}$ defined in (3.14) and (3.15), respectively. We require that $H8$ holds. This allows to introduce h_0 and f_0 (as in $H8$).

(K9) $u_0 \in \mathcal{D}(A)$, with u_0 defined as in $H9$ and $\mathcal{D}(A)$ defined as in (3.53)).

Next, we can consider the matrix M as in (3.16), and we require that:

(K10) condition H10 holds.

Now we can introduce w_0, k_0, z_0 as in H10.

(K11) Suppose that H11, H12 and H13 hold.

(K12) For $j = 1, 2, 3$,

$$\int_{\overline{\Omega}} \theta_0(x)\mu_j(dx) = b_j(0), \quad j = 1, 2, 3, \tag{3.61}$$

$$\int_{\overline{\Omega}} u_0(x)\mu_j(dx) = b_j'(0), \quad j = 1, 2, 3. \tag{3.62}$$

So, applying Theorems 3.1 and 3.2, we can conclude that:

Theorem 3.3. (Existence local in time). *Suppose that $K1-K12$ hold. Then there exists τ, such that the inverse problem (3.57) and (4.1), has a solution (θ,χ,h,k,f), with*

$$\theta \in C^{2+\beta}([0,\tau];C(\overline{\Omega})) \cap C^{1+\beta}([0,\tau];\mathcal{D}(A)), \tag{3.63}$$

$$\chi \in C^{2+\beta}([0,\tau];C(\overline{\Omega})) \cap C^{1+\beta}([0,\tau];\mathcal{D}(B)), \tag{3.64}$$

$$h \in C^{1+\beta}([0,\tau]), \tag{3.65}$$

$$k \in C^{\beta}([0,\tau]), \tag{3.66}$$

$$f \in C^{1+\beta}([0,\tau]). \tag{3.67}$$

Theorem 3.4. (Uniqueness global in time). *Suppose that $K1-K12$ hold.*
If $(\theta_1,\chi_1,h_1,k_1,f_1)$ and $(\theta_2,\chi_2,h_2,k_2,f_2)$ are solutions of the inverse problem (3.57) and (3.58) both satisfying the regularity conditions

$$\theta_j \in C^{2+\beta}([0,\tau];C(\overline{\Omega})) \cap C^{1+\beta}([0,\tau];\mathcal{D}(A)), \tag{3.68}$$

$$\chi_j \in C^{2+\beta}([0,\tau];C(\overline{\Omega})) \cap C^{1+\beta}([0,\tau];\mathcal{D}(B)), \tag{3.69}$$

$$h_j \in C^{1+\beta}([0,\tau]), \tag{3.70}$$

$$k_j \in C^{\beta}([0,\tau]), \tag{3.71}$$

$$f_j \in C^{1+\beta}([0,\tau]) \tag{3.72}$$

$j=1,2$, *for some $\tau \in (0,T]$, then they coincide.*

4 The case of L^p Spaces

In this section we consider another inverse problem related to (2.7) in the framework of L^p spaces. We start by introducing some notations.

- If $s \in \mathbb{Z}$, $s \geq 2$, $W^{s,p}_B(\Omega) := \{f \in W^{s,p}(\Omega) : D_\nu f \equiv 0\}$.
- If $s \in \mathbb{Z}$, $s \geq 4$, $W^{s,p}_{BB}(\Omega) := \{f \in W^{s,p}(\Omega) : D_\nu f \equiv D_\nu \Delta f \equiv 0\}$.
- $B^s_{p,q}(\Omega)$ $(s>0,\ 1 \leq p,q \leq +\infty)$ are the Besov spaces.
- $(.,.)_{\theta,p}$ is the real interpolation functor $(0<\theta<1,\ 1\leq p \leq +\infty)$.

- If $p \in [1, +\infty)$, $T \in \mathbb{R}^+$, $m \in \mathbb{N}_0$, X Banach space, $f \in W^{m,p}(0,T;X)$, we set

$$\|f\|_{W^{m,p}(0,T;X)} := \sum_{j=0}^{m-1} \|f^{(j)}(0)\|_X + \|f^{(m)}\|_{L^p(0,T;X)},$$

where $W^{m,p}(0,T;X)$ is the vector valued Sobolev space.

We consider additional measurements on the temperature which can be represented as

$$\Phi_j(\theta)(t) := \int_\Omega \phi_j(x)\theta(t,x)\,dx, \qquad \forall t \in [0,T], \quad j = 1,2. \tag{4.1}$$

With the above notations we state our inverse problem.

Definition 4.1. *Let $p \in (1,+\infty)$. Determine θ, χ, h, k, and $\mathcal{K}$ with*

$$\theta \in W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{2,p}(\Omega)) \tag{4.2}$$

$$\chi \in W^{2,p}(0,\tau;L^p(\Omega)) \cap W^{1,p}(0,\tau;W^{4,p}(\Omega)), \tag{4.3}$$

$$h \in W^{1,p}(0,\tau), \tag{4.4}$$

$$k \in L^p(0,\tau), \tag{4.5}$$

$$\mathcal{K} \in \mathbb{R}^+, \tag{4.6}$$

satisfying the system

$$\begin{cases} D_t(\theta + \chi + h * \theta)(t,x) \\ -\Delta[\mathcal{K}\theta(t,x) + k * \theta](t,x) = \mathcal{F}(t,x), \quad (t,x) \in [0,\tau] \times \Omega, \\ D_t\chi(t,x) = \Delta[-\Delta\chi + \gamma'(\chi) - \theta](t,x), \quad (t,x) \in [0,\tau] \times \Omega, \\ D_\nu\theta(t,x') = D_\nu\chi(t,x') = D_\nu\Delta\chi(t,x') = 0, \quad (t,x') \in [0,\tau] \times \partial\Omega, \\ \theta(0,x) = \theta_0(x), \quad x \in \Omega, \\ \chi(0,x) = \chi_0(x), \quad x \in \Omega, \\ \Phi_j[\theta(t)] = g_j(t), \quad j \in \{1,2\}, \quad t \in [0,\tau], \end{cases} \tag{4.7}$$

under suitable regularity and compatibility conditions on the data, where we have set $\lambda'(\chi) = 1$ in (2.2).

Let us introduce the set of conditions that allow us to make this reformulation:

(C1) Ω is an open bounded subset of $\mathbb{R}^n$, lying on one side of its boundary $\partial\Omega$, which is a submanifold of $\mathbb{R}^n$ of class C^4.

(C2) $p \in (1,+\infty)$, $n \in \mathbf{N}$, $n < 4p$.

(C3) $\gamma \in C^\infty(\mathbb{R})$.

(C4) $\chi_0 \in W^{4,p}_{BB}(\Omega)$.

(C5) $\theta_0 \in W^{2,p}_{B}(\Omega)$.

(C6) for some $T \in \mathbb{R}^+$, $\mathcal{F} \in W^{1,p}(0,T; L^p(\Omega))$.

(C7) for $j \in \{1,2\}$, $u \in L^p(\Omega)$, $\Phi_j[u] = \int_\Omega \phi_j(x)u(x)dx$, with $\phi_j \in L^{p'}(\Omega)$.

(C8) for $j \in \{1,2\}$, $g_j \in W^{2,p}(0,T)$.

(C9) $v_0 := \Delta[-\Delta\chi_0 + \gamma'(\chi_0) - \theta_0] \in (L^p(\Omega), W^{4,p}_{BB}(\Omega))_{1-1/p,p}$.

(C10) $\chi^{-1} := \Phi_2[\theta_0]\Phi_1[\Delta\theta_0] - \Phi_1[\theta_0]\Phi_2[\Delta\theta_0] \neq 0$.

(C11) $\mathcal{K}_0 := \chi[\Phi_1[\theta_0]\{\Phi_2[\mathcal{F}(0,.) - v_0] - g_2'(0)\} - \Phi_2[\theta_0]\{\Phi_1[\mathcal{F}(0,.) - v_0] - g_1'(0)\}] \in \mathbb{R}^+$.

(C12) $\Phi_j(\theta_0) = g_j(0)$, $j \in \{1,2\}$.

(C13) $u_0 := \mathcal{F}(0,.) - v_0 - a_0\theta_0 + \mathcal{K}_0\Delta\theta_0 \in (L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p}$, with

$$a_0 := \chi[\{\Phi_2[\mathcal{F}(0,.) - v_0] - g_2'(0)\}\Phi_1[\theta_0] - \{\Phi_1[\mathcal{F}(0,.) - v_0] - g_1'(0)\}\Phi_2[\theta_0]].$$

Remark 4.1. From Theorem 3.5 in [20], we have that, for $p \in (1, +\infty)$,

$$(L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p} = \begin{cases} B^{2(1-1/p)}_{p,p}(\Omega) & \text{if} \quad 1 < p < 3, \\ \{f \in B^{2(1-1/p)}_{p,p}(\Omega) : D_\nu f \equiv 0\} & \text{if} \quad 3 < p < +\infty. \end{cases} \tag{4.8}$$

Moreover,

$$(L^p(\Omega), W^{4,p}_{BB}(\Omega))_{1-1/p,p} = \begin{cases} B^{4(1-1/p)}_{p,p}(\Omega) & \text{if} \quad 1 < p < 5/3, \\ \{f \in B^{4(1-1/p)}_{p,p}(\Omega) : D_\nu f \equiv 0\} & \text{if} \quad 5/3 < p < 5, \\ \{f \in B^{4(1-1/p)}_{p,p}(\Omega) : D_\nu f \equiv D_\nu \Delta f \equiv 0\} & \text{if} \quad 5 < p < +\infty. \end{cases} \tag{4.9}$$

Now we can state the main result of this section.

Theorem 4.1. *Assume that conditions $C1 - C13$ are satisfied. Then there exists $\tau \in (0,T]$ such that the problem (4.7) has a unique solution $(\theta, \chi, h, k, \mathcal{K})$ satisfying the conditions (4.2)-(4.6).*

The idea of the proof of Theorem 4.1 is not so different from the idea of the proof of Theorem 3.1. There are, of course, some technical differences. Here we limit ourselves to state the main result concerning the linerized parabolic system which gives the fundamental estimate to apply fixed point argument to the non linear problem.

Theorem 4.2. *Let $p \in (1,+\infty)$, $\mathcal{K}_0 \in \mathbb{R}^+$. Then the following the problem*

$$\begin{cases} D_t u(t,x) + D_t v(t,x) = \mathcal{K}_0 \Delta u(t,x) - a_0 u(t,x) + f(t,x), & (t,x) \in [0,T] \times \Omega, \\ D_t v(t,x) = -\Delta^2 v(t,x) - \Delta u(t,x) + g(t,x), & (t,x) \in [0,T] \times \Omega, \\ D_\nu u(t,x') = D_\nu v(t,x') = D_\nu \Delta\chi(t,x') = 0, & (t,x') \in [0,T] \times \partial\Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \\ v(0,x) = v_0(x), & x \in \Omega. \end{cases} \tag{4.10}$$

has a unique solution $(u,v) \in (W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{2,p}(\Omega))) \times (W^{1,p}(0,T;L^p(\Omega)) \cap L^p(0,T;W^{4,p}(\Omega)))$ if and only if the following conditions holds:

(I) $f, g \in L^p(0,T;L^p(\Omega))$;

(II) $u_0 \in (L^p(\Omega), W^{2,p}_B(\Omega))_{1-1/p,p}$;

(III) $v_0 \in (L^p(\Omega), W^{4,p}_{BB}(\Omega))_{1-1/p,p}$.

Moreover, $\forall T_0 \in \mathbb{R}^+$ there exists $C(T_0) \in \mathbb{R}^+$, such that, if $0 < T \leq T_0$ we have the estimate:

$$\begin{aligned} \|u\|_{X(T,p)} + \|v\|_{Y(T,p)} &\leq C(T_0)[\|f\|_{L^p(0,T;L^p(\Omega))} + \|g\|_{L^p(0,T;L^p(\Omega))} \\ &+ \|u_0\|_{(L^p(\Omega),W^{2,p}_B(\Omega))_{1-1/p,p}} + \|v_0\|_{(L^p(\Omega),W^{4,p}_{BB}(\Omega))_{1-1/p,p}}]. \end{aligned} \tag{4.11}$$

5 A parabolic-hyperbolic system

Now we consider the case when the instantaneous conductivity vanishes. We consider the system

$$\begin{cases} \partial_t\Big(\theta(t,x) + \chi(t,x)\Big) \\ = k * \Delta\theta(t,x) + \mathcal{F}(t,x), \quad t \in [0,T], \ x \in \Omega, \\ \partial_t\chi(t,x) = -\Delta^2\chi(t,x) + \Delta[\sigma'(\chi(t,x)) - \theta(t,x)], \quad t \in [0,T], \ x \in \Omega, \\ \theta(0,x) = \theta_0(x), \ x \in \Omega, \\ \chi(0,x) = \chi_0(x), \ x \in \Omega, \\ D_\nu\theta(t,x) = 0, \quad t \in [0,T], \ x \in \partial\Omega, \\ D_\nu\chi(t,x) = D_\nu\Delta\chi(t,x) = 0, \qquad t \in [0,T], \ x \in \partial\Omega, \end{cases} \tag{5.1}$$

with the supplementary condition

$$\Phi[\theta(t,.)] := \int_\Omega \phi(x)\theta(t,x)dx = g(t), \qquad t \in [0,\tau], \tag{5.2}$$

where we have set $\lambda'(\chi) = 1$ and we have defined $\chi^3 - \gamma'(\chi) := \sigma(\chi)$ in (2.2) . The inverse problem is: determine θ, χ and k satisfying system (5.1)-(5.2) under suitable conditions on the data.

To state our result we introduce the following notations:

- we shall indicate with V' the dual space of $H^1(\Omega)$;

- we introduce the following operator $\mathcal{B}$:

$$\begin{cases} \mathcal{B} : H^3_B(\Omega) \to V', \\ (\mathcal{B}u, v) = \int_\Omega \nabla(\Delta u)(x) \cdot \overline{\nabla v(x)} dx, \quad \forall v \in H^1(\Omega). \end{cases}$$

Then the following theorem holds (see [21]):

Theorem 5.1. *Assume that the following conditions hold:*
(D1) Ω *open, bounded in* $\mathbb{R}^+$, $\partial\Omega$ *of class* C^4.
(D2) $n \leq 7$.
(D3) $\sigma \in C^\infty(\mathbb{R})$.
(D4) $p \in [2, +\infty[$.
(D5) $T \in \mathbb{R}^+$, $\mathcal{F} \in W^{3,p}(0, T; H^1(\Omega))$.
(D6) $\phi \in H^1(\Omega)$.
(D7) $\theta_0 \in H^3_B(\Omega)$.
(D8) $\chi_0 \in H^4_{BB}(\Omega)$.
(D9) $v_0 := -\Delta^2_x \chi_0 + \Delta_x[\sigma'(\chi_0) - \theta_0] \in H^4_{BB}(\Omega)$.
(D10) $u_0 := \mathcal{F}(0, .) - v_0 \in H^3_B(\Omega)$.
(D11) $v_1 := -\Delta^2_x v_0 + \Delta_x[\sigma''(\chi_0) v_0 - u_0] \in (H^3_B(\Omega), H^4_{BB}(\Omega))_{1-\frac{1}{p},p}$.
(D12) $g \in W^{4,p}(0, T)$.
(D13) $\Phi[\Delta_x \theta_0] \neq 0$.
(D14) $b_0 := [g''(0) + \Phi[v_1] - \Phi[D_t \mathcal{F}(0, .)]][\Phi[\Delta_x \theta_0]^{-1} \in \mathbb{R}^+$.
(D15) $u_1 := b_0 \Delta_x \theta_0 + D_t \mathcal{F}(0, .) - v_1 \in H^2_B(\Omega)$.
(D16) $\int_\Omega \phi(x) \theta_0(x) dx = g(0)$, $\int_\Omega \phi(x) u_0(x) dx = g'(0)$, $\int_\Omega \phi(x) u_1(x) dx = g''(0)$.
(D17) $v_2 := \mathcal{B} v_1 + \Delta_x[\sigma^{(3)}(\chi_0) v_0^2 + \sigma''(\chi_0) v_1] - \Delta_x u_1 \in (V', H^3_B(\Omega))_{1-\frac{1}{p},p}$.
Then, there exists $\tau \in (0, T]$, *such that a unique solution* (θ, χ, b) *of problem (5.1)-(5.2) of domain* $[0, \tau]$ *exists, with*

$$\theta \in W^{4,p}(0, \tau; V') \cap C^3([0, \tau]; H^1(\Omega)) \cap C^2([0, \tau]; H^2(\Omega)) \cap C^1([0, \tau]; H^3(\Omega)),$$

$$\chi \in W^{4,p}(0, \tau; V') \cap W^{3,p}(0, \tau; H^3(\Omega)) \cap W^{2,p}(0, \tau; H^4(\Omega)),$$

$$k \in W^{2,p}(0, \tau).$$

Remark 5.1. We explain why we think of the first equation in (5.1) as a hyperbolic equation: if we differentiate it twice with respect to t and set $u := D_t\theta$, $v := D_t\chi$, $h := D_t k$, we get

$$D^2_t u + D^2_t v = k(0)\Delta u + h\Delta\theta_0 + h * \Delta u + D^2_t \mathcal{F}. \tag{5.3}$$

It turns out that $k(0) = b_0$ (see $(D14)$). So, owing to $(D14)$, (5.3) is hyperbolic in the unknown u.

Remark 5.2. We characterize the interpolation spaces appearing in $(D11)$ and $(D17)$. To this aim, we introduce the operator S in $L^2(\Omega)$ defined as follows:

$$\begin{cases} D(S) = \{v \in H^2(\Omega) : D_\nu v \equiv 0\}, \\ Sv = (1 - \Delta)v, \quad v \in D(S). \end{cases} \tag{5.4}$$

It turn sout that S is a positive self-adjoint operator and that $S^{1/2}$, with domain $H^1(\Omega)$, can be extended to a linear bounded operator from $L^2(\Omega)$ to V', which we shall continue to indicate with $S^{1/2}$.

The following identities hold:

$$(H^3_B(\Omega), H^4_{BB}(\Omega))_{1-\frac{1}{p},p} = \begin{cases} B^{4-\frac{1}{p}}_{2,p,B}(\Omega) & \text{if} \quad 1 < p < 2, \\ B^{4-\frac{1}{p}}_{2,p,BB}(\Omega) & \text{if} \quad 2 < p < +\infty; \end{cases} \tag{5.5}$$

$$(V', H^3_B(\Omega))_{1-\frac{1}{p},p} = \begin{cases} S^{\frac{1}{2}}(B^{4(1-\frac{1}{p})}_{2,p}(\Omega)) & \text{if} \quad 1 < p < \frac{4}{3}, \\ B^{3-\frac{4}{p}}_{2,p}(\Omega) & \text{if} \quad \frac{4}{3} < p < \frac{8}{3}, \\ B^{3-\frac{4}{p}}_{2,p,B}(\Omega) & \text{if} \quad \frac{8}{3} < p < +\infty. \end{cases} \tag{5.6}$$

Sketch of the proof of Theorem 5.1. Assume that a solution (θ, χ, k) with the required properties exists. Then, differentiating with respect to t the first equation in (5.1), we obtain, for $(t,x) \in [0,\tau] \times \Omega$,

$$D^2_t\theta(t,x) + D^2_t\chi(t,x) = k(0)\Delta_x\theta(t,x) + (k' * \Delta_x\theta)(t,x) + D_t\mathcal{F}(t,x). \tag{5.7}$$

Differentiating again (5.7) and the second equation in (5.1) and setting

$$u := D_t\theta, \qquad v := D_t\chi, \qquad h := D_tk, \tag{5.8}$$

we get

$$\begin{aligned} &D^2_tu(t,x) + D^2_tv(t,x) = k(0)\Delta_xu(t,x) + h(t)\Delta_x\theta_0(x) \quad (t,x) \in [0,\tau] \times \Omega, \\ &+(h * \Delta_xu)(t,x) + D^2_t\mathcal{F}(t,x), \end{aligned} \tag{5.9}$$

and

$$D_tv(t,x) - \Delta_x[-\Delta_xv + \sigma''(\chi)v - u](t,x) = 0. \quad (t,x) \in [0,\tau] \times \Omega, \tag{5.10}$$

Moreover,

$$v(0,.) = D_t\chi(0,.) = v_0, \tag{5.11}$$

$$u(0,.) = \mathcal{F}(0,.) - D_t\chi(0,.) = u_0, \tag{5.12}$$

$$D^2_t\chi(0,.) = D_tv(0,.) = v_1. \tag{5.13}$$

If $w \in L^2(\Omega)$, we set

$$\Phi[w] := \int_\Omega w(x)\phi(x)dx \tag{5.14}$$

so that (5.2) can be written in the form

$$\Phi[\theta(t,.)] = g(t), \quad t \in [0,\tau].$$

Applying Φ to (5.7), we obtain

$$g''(t) + \Phi[D_t^2\chi(t,)] = k(0)\Phi[\Delta_x\theta(t,.)] + \Phi[(h * \Delta_x\theta)(t,.)] + \Phi[D_t\mathcal{F}(t,.)] \quad t \in [0,\tau],$$

and, for $t = 0$,

$$g''(0) + \Phi[v_1] = k(0)\Phi[\Delta_x\theta_0] + \Phi[D_t\mathcal{F}(0,.)].$$

It follows from $(D13)$ that

$$k(0) = \frac{g''(0) + \Phi[v_1] - \Phi[D_t\mathcal{F}(0,.)]}{\Phi[\Delta_x\theta_0]} = b_0. \tag{5.15}$$

Next,

$$\begin{aligned} D_tu(0,.) &= D_t^2\theta(0,.) \\ &= k(0)\Delta_x\theta_0 + D_t\mathcal{F}(0,.) - D_t^2\chi(0,.) \\ &= b_0\Delta_x\theta_0 + D_t\mathcal{F}(0,.) - v_1 \\ &= u_1. \end{aligned} \tag{5.16}$$

Finally,

$$D_\nu u(t,x) \equiv D_\nu v(t,x) \equiv D_\nu\Delta_x v(t,x) \equiv 0, \quad (t,x) \in [0,\tau] \times \partial\Omega. \tag{5.17}$$

From these consideration, if (θ,χ,k) solves (5.1)-(5.2) and satisfies the required conditions of regularity, the triple (u,v,h) is such that

$(\alpha) u \in W^{3,p}(0,\tau;V') \cap C^2([0,\tau];H^1(\Omega)) \cap C^1([0,\tau];H^2_B(\Omega)) \cap C([0,\tau];H^3_B(\Omega))$;

(β) $v \in W^{3,p}(0,\tau;V') \cap W^{2,p}([0,\tau];H^3_B(\Omega)) \cap W^{1,p}([0,\tau];H^4_{BB}(\Omega))$;

(σ) $h \in W^{1,p}(0,\tau)$

and solves the system

$$\begin{cases} D_t^2u(t,.) + D_t^2v(t,.) = -b_0Au(t,) - h(t)A\theta_0, & t \in [0,\tau], \\ -(h * Au)(t,.) + D_t^2\mathcal{F}(t,.), & \\ D_tv(t,.) - Bv(t,.) + A[\sigma''((\chi_0 + 1 * v)(t,.))v(t,.)] - Au(t,.) = 0, & t \in [0,\tau], \\ u(0,.) = u_0, & \\ D_tu(0,.) = u_1, & \\ v(0,.) = v_0, & \\ g^{(3)}(t) + \Phi[D_t^2v(t,.)] = -b_0\Phi[Au(t,.)] - h(t)\Phi[A\theta_0], & t \in [0,\tau]. \\ -\Phi[(h * Au)(t,.)] + \Phi[D_t^2\mathcal{F}(t,.)] & \end{cases} \tag{5.18}$$

if we set

$$\begin{cases} D(A) = \{v \in H^2(\Omega) : D_\nu v \equiv 0\}, \\ Av = -\Delta v, \quad v \in D(A), \end{cases} \tag{5.19}$$

$$\begin{cases} D(B) = \{v \in H^4(\Omega) : D_\nu v \equiv D_\nu(\Delta v) \equiv 0\}, \\ Bv = -\Delta^2 v, \quad v \in D(B). \end{cases} \tag{5.20}$$

Given $\tau \in (0, T]$ and (U, V, H), such that
(α') $U \in W^{3,p}(0,\tau;V') \cap C^2([0,\tau];H^1(\Omega)) \cap C^1([0,\tau];H^2_B(\Omega)) \cap C([0,\tau];H^3_B(\Omega))$,
$U(0) = u_0$, $D_tU(0) = u_1$,
(β') $V \in W^{3,p}(0,\tau;V') \cap W^{2,p}([0,\tau];H^3_B(\Omega)) \cap W^{1,p}([0,\tau];H^4_{BB}(\Omega))$,
$V(0) = v_0$, $D_tV(0) = v_1$, $D_t^2V(0) = v_2$,
(σ') $h \in W^{1,p}(0,\tau)$, $h(0) = h_0$,
consider the problem

$$\begin{cases} D_t^2u(t,.) + b_0Au(t,) = -D_t^2V(t,.) - H(t)A\theta_0 & t \in [0,\tau], \\ -(H * AU)(t,.) + D_t^2\mathcal{F}(t,.), & \\ D_tv(t,.) - Bv(t,.) = -A[\sigma''((\chi_0 + 1 * V)(t,.))V(t,.)] + AU(t,.), & t \in [0,\tau], \\ u(0,.) = u_0, & \\ D_tu(0,.) = u_1, & \\ v(0,.) = v_0, & \\ h(t)\Phi[A\theta_0] = -g^{(3)}(t) - \Phi[D_t^2V(t,.)] - b_0\Phi[AU(t,.)], & t \in [0,\tau]. \\ -\Phi[(H * AU)(t,.)] + \Phi[D_t^2\mathcal{F}(t,.)]. & \end{cases} \tag{5.21}$$

Then one can show that (5.21) has a unique solution (u, v, h) such that (α), (β), (σ) hold. We set $\mathcal{S}(U, V, H) := (u, v, h)$. Then one can show that, if τ is sufficiently small, $\mathcal{S}$ has a unique fixed point. □

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