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Tilings and Fractals from Pisot substitutions

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This is the note for the lecture at RIMS (Kyoto University).

1 Definition of Pisot Unit Subsitutions

$$\begin{aligned} \mathcal{A} &:= \{1,2,\cdots,d\} \quad \text{(alphabet)} \\ \mathcal{A}^* &:= \bigcup_{n=0}^{\infty} \mathcal{A}^n \quad \text{(free monoid, i.e., the set of finite words)} \end{aligned}$$

$(G\{1,\cdots,d\}:$ a free group of rank $d)$

Definition 1.1 $\sigma : \mathcal{A}^* \longrightarrow \mathcal{A}^*$ *is a* substitution *if*

(1)
$$\begin{aligned} \sigma(i) &= W^{(i)} \in \mathcal{A}^*, \quad W^{(i)} \neq \emptyset \\ &= w_1^{(i)} \dots w_k^{(i)} \dots w_{l_i}^{(i)}, \quad w_k^{(i)} \in \mathcal{A} \\ &= P_k^{(i)} w_k^{(i)} S_k^{(i)}; \end{aligned}$$

(2) $\sigma(w_1 \dots w_k) := \sigma(w_1) \cdots \sigma(w_k)$ *for* $w_1 \dots w_k \in \mathcal{A}^*$.

(A substitution σ is invertible *if σ is an automorphism on $G\{1,2,\cdots d\}$.)*

Let L_σ be a matrix of σ, that is,

$$L_\sigma(i,j) := \text{ the number of the letter } i \text{ contained in } \sigma(j).$$

Example 1.1 *(Fibonacci substitution)*

$$\sigma_F : \begin{array}{ccc} 1 & \rightarrow & 12 \\ 1 & \rightarrow & 1 \end{array}, \quad L_{\sigma_F} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example 1.2 *(Rauzy substitution)*

$$\sigma_R : \begin{array}{ccc} 1 & \rightarrow & 12 \\ 1 & \rightarrow & 13 \\ 1 & \rightarrow & 1 \end{array}, \quad L_{\sigma_R} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Assumption For the substitution σ,

(1) L_σ is *primitive*, that is, $\exists N : L_\sigma^N > 0$;

(2) L_σ is *unimodular*, that is, $\det L_\sigma = \pm 1$;

(3) L_σ is *Pisot type*, that is, for eigenvalues $\lambda = \lambda_1, \lambda_2, \cdots, \lambda_d$ of L_σ,

$$\lambda = \lambda_1 > 1 > \mid \lambda_i \mid, \quad i = 2, \cdots, d;$$

(4) the characteristic polynomial $\Phi_\sigma(x)$ of L_σ is *irreducible*;

(5) $\sigma(1) = 1W'$.

We say the substitution σ satisfying Assumption the *Pisot Unit subsitution.*

On Assumption (5),

$$w_\sigma := \lim_{n\to\infty} \sigma^n(1) = s_1 s_2 \cdots s_m \cdots$$

is a fixed point of σ, that is, $\sigma(w_\sigma) \doteq w_\sigma$.

$f : \mathcal{A}^* \to \boldsymbol{Z}^d$ is a homomorphism given by

$$f(i) := \boldsymbol{e}_i \quad \text{and} \quad f(w_1 \cdots w_k) := \sum_{j=1}^{k} f(w_j).$$

Lemma 1.1 *The following relation holds:*

$$\begin{array}{cccccc} & \mathcal{A}^* & \xrightarrow{\sigma} & \mathcal{A}^* & \\ f & \downarrow & & \downarrow & f \\ & \boldsymbol{Z}^d & \xrightarrow{L_\sigma} & \boldsymbol{Z}^{d*} & . \end{array}$$

Let $\boldsymbol{v} > \boldsymbol{0}$ be a positive eigenvector of λ and P be L_σ-invariant contractive plain, that is,

$$\boldsymbol{R}^d := \mathcal{L}(\boldsymbol{v}) \oplus P.$$

The projection π is given by

$$\pi : \boldsymbol{Z}^d \to P \text{ along } \boldsymbol{v}.$$

Lemma 1.2 *The following relation holds:*

$$\begin{array}{cccccc} & \boldsymbol{Z}^d & \xrightarrow{L_\sigma} & \boldsymbol{Z}^d & \\ \pi & \downarrow & & \downarrow & \pi \\ & P & \xrightarrow{L_\sigma} & P & . \end{array}$$

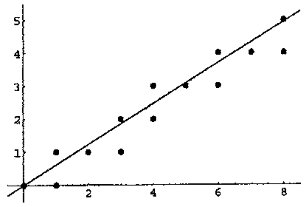


Figure 1: The figure of $\{f(s_0 s_1 \cdots s_k) \mid k = 0, 1, 2, \cdots\}$ on σ_F.

For the fixed point $w_\sigma = s_1 s_2 \cdots s_n \cdots$,

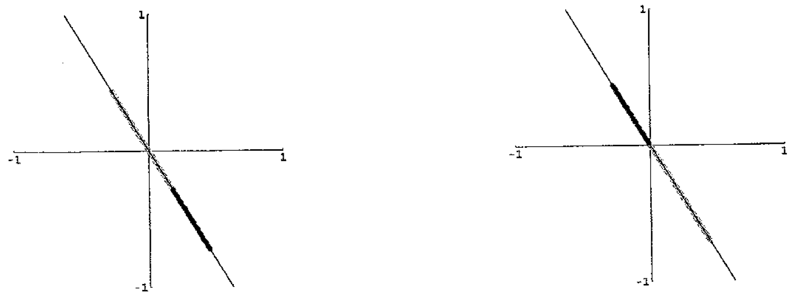
$$\begin{aligned} Y &:= \pi\{f(s_0 s_1 \cdots s_k) \mid k = 0, 1, \cdots\}, \\ Y_i &:= \pi\{f(s_0 s_1 \cdots s_{k-1}) \mid \exists k : s_k = i, \ k = 1, 2, \cdots\}, \\ Y_i' &:= \pi\{f(s_0 s_1 \cdots s_k) \mid \exists k : s_k = i, \ k = 0, 1, \cdots\} \end{aligned}$$

where $s_0 = \varepsilon$ (the empty word).

Definition 1.2 $X_i :=$ *the closure of* πY_i *is called* atomic surfaces ($X_i' :=$ *the closure of* $\pi Y_i'$) *and* $X := \bigcup_{i=1}^d X_i \left(= \bigcup_{i=1}^d X_i'\right)$ *is called* atomic surface *of the substitution* σ.

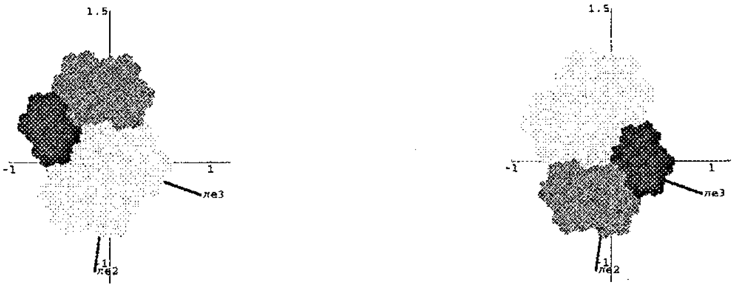
On Example 1.1 and Example 1.2, we have the figures (see Figure 2 and Figure 3 respectively).

Remark There is the theorem. Theorem ([E-I]): On $d = 2$, X_i is interval is the interval iff σ is invertible.



$\bigcup_{i=1,2} X_i$ $\bigcup_{i=1,2} X_i'$

Figure 2: The figure of the atomic surface on $\sigma_F : 1 \mapsto 12, \ 2 \mapsto 1$.



$\bigcup_{i=1,2,3} X_i$ $\bigcup_{i=1,2,3} X_i'$

Figure 3: The figure of the atomic surface on $\sigma_R : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$.

Theorem 1 ([A-I], [I-R]) Atomic surfaces satisfy

(1) $X,\ X_i,\ X_i'$ are compact sets;

(2) $\overline{int.X} = X$;

(3) $L_\sigma^{-1} X_i = \bigcup_{j=1}^{d} \bigcup_{k:W_k^{(j)}=i} \left(X_j + L_\sigma^{-1} \left(f \left(P_k^{(j)} \right) \right) \right)$ (non-overlapping);

(4) ${}^t(\mid X_1 \mid, \ldots, \mid X_d \mid)$ is the eigenvector of L_σ with respect to $\lambda = \lambda_1 > 1$ where $\mid B \mid$ is the volume of the set B.

Sketch of proof. On the notation,

$$\begin{aligned} (\boldsymbol{x},\ i) &:= \{\boldsymbol{x} + \lambda \boldsymbol{e}_i \mid 0 \le \lambda \le 1\}, \\ \sigma(\mathbf{0},\ i) &:= \textstyle\sum_{k=1}^{l_i} \left(f\left(P_k^{(i)}\right),\ W_k^{(i)} \right), \quad 1 \le i \le d, \\ \overline{w_\sigma} &:= \lim_{n\to\infty} \sigma^n(\mathbf{0},\ 1) \quad (\text{ the broken line starting from } \mathbf{0}\ ), \end{aligned}$$

we define

$$\begin{aligned} (Y_i,\ i) &:= \{(\boldsymbol{y},\ i) \mid (\boldsymbol{y},\ i) \in \overline{w}\}, \quad \overline{w} = \bigcup_{i=1}^{d} (Y_i,\ i) \\ &= \left\{ (\boldsymbol{y},\ i) \;\middle|\; (\boldsymbol{y},\ i) \in \sigma\left(\bigcup_{j=1}^{d} (Y_j,\ j) \right) \right\} \\ &= \bigcup_{j=1}^{d} \{(\boldsymbol{y},\ i) \mid (\boldsymbol{y},\ i) \in \sigma(Y_j,\ j)\} \\ &= \bigcup_{j=1}^{d} \bigcup_{k:W_k^{(j)}=i} \left\{ \left(L_\sigma \boldsymbol{y} + f\left(P_k^{(j)}\right),\ i \right) \;\middle|\; \boldsymbol{y} \in Y_j \right\}. \end{aligned}$$

Taking only the starting points of line segments, we get Theorem 1 (3).

Question Are X_i, $i = 1, 2, \ldots, d$ non-overlapping?

Definition 1.3 *Substitution σ satisfies the* coincidence condition *if $\exists n, k$:*

(1) $f\left(P_k^{(n,1)}\right) = f\left(P_k^{(n,2)}\right) = \cdots = f\left(P_k^{(n,d)}\right)$

(2) $w_k^{(n,1)} = w_k^{(n,2)} = \cdots = w_k^{(n,d)}$

where

$$\begin{array}{rcl} \sigma^n(1) & = & w_1^{(n,1)} \cdots w_k^{(n,1)} \cdots w_{l(n,1)}^{(n,1)} \\ \vdots & = & \vdots \\ \sigma^n(d) & = & w_1^{(n,d)} \cdots w_k^{(n,d)} \cdots w_{l(n,d)}^{(n,d)}. \end{array}$$

Proposition 1.1 *If σ satisfies the coincidence condition, then X_i, $i = 1, 2, \ldots, d$ are non-overlapping.*

Conjecture Any Pisot unimodular substitutions satisfy the coincidence condition.

Remark On $d = 2$, the conjecture is proved by Barge and Diamond ([B-D]).

If X_i, $i = 1, 2, \ldots, d$ are non-overlapping, then we have two dynamical systems on X:

(1) $T : X \to X$,

$$Tx = L_\sigma^{-1}x - L_\sigma^{-1}\pi f\left(P_k^{(j)}\right) \quad \text{if } x \in X_i,\ \exists j, k : L_\sigma^{-1}x \in X_j + L_\sigma^{-1}\pi f\left(P_k^{(j)}\right).$$

Therefore, T is Markov endomorphism with the structure matrix ${}^tL_\sigma$.

(2)
$$\begin{array}{rcccl} W : X & \to & X & & \\ \cup & & \cup & & \text{is well-defined} \\ x & \mapsto & x - \pi e_i & \text{if} & x \in X_i \end{array}$$

and W is called the domain exchange transformation (later, we will see W $\simeq$ the rotation on $\mathbf{T}^{d-1}$).

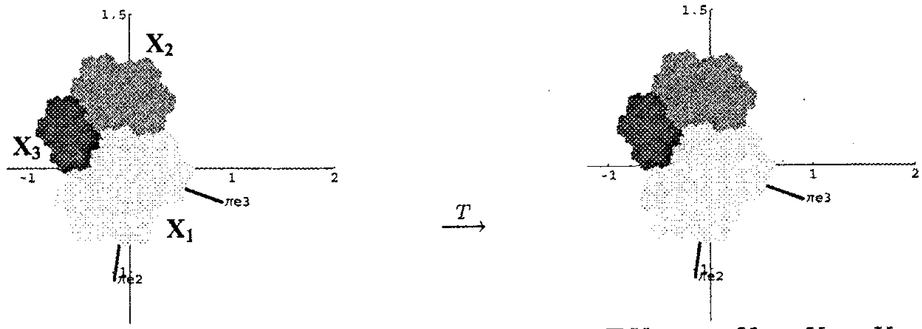
From Markov endomorphism (1), we have the following numerical expression.

Corollary 1.1 *Using Markov endomorphism, X_i is presented by*

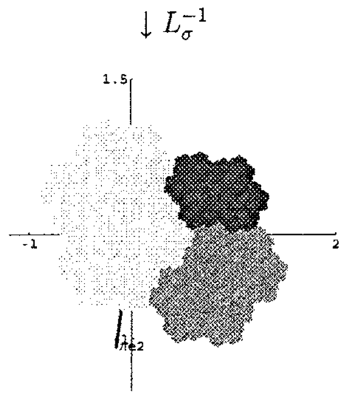
$$X_i = \left\{ \pi_c f\left(P_{k_0}^{(j_0)}\right) + \pi_c L_\sigma f\left(P_{k_1}^{(j_1)}\right) + \pi_c L_\sigma^2 f\left(P_{k_2}^{(j_2)}\right) + \cdots + \pi_c L_\sigma^n f\left(P_{k_n}^{(j_n)}\right) + \cdots \ \middle|\ (*) \right\}$$

where $(*)$ is defined by $\begin{pmatrix} j_0 & j_1 & \cdots & j_n & \cdots \\ k_0 & k_1 & \ldots & k_n & \cdots \end{pmatrix}$ *is given by $w_{k_n}^{(j_n)} = j_{n-1}$ and $w_{k_0}^{(j_0)} = i$.*

(see Figre 4).



$$
\begin{aligned}
TX_1 &= X_1 \cup X_2 \cup X_3 \\
TX_2 &= X_1 \\
TX_3 &= X_2
\end{aligned}
$$



$$
\begin{aligned}
L_\sigma^{-1} X_1 &= X_1 \cup X_2 \cup X_3 \\
L_\sigma^{-1} X_2 &= \pi e_3 + X_1 \\
L_\sigma^{-1} X_3 &= \pi e_3 + X_2
\end{aligned}
$$

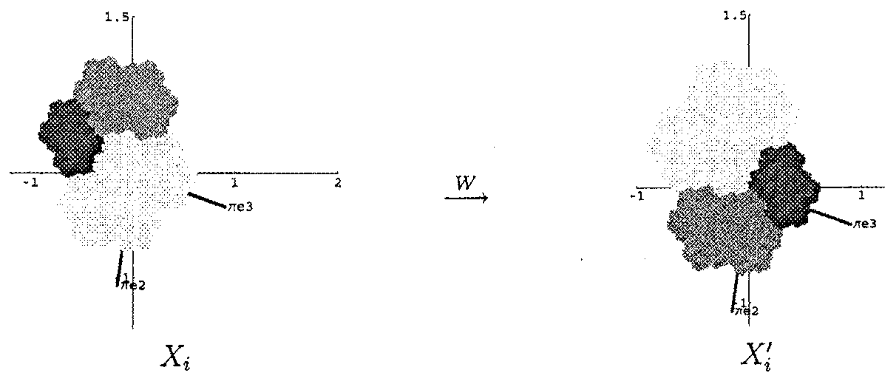


Figure 4: Figures of Markov endomorphism T and the domain exchange transformation W on Example 1.2.

2 Stepped Surfaces and Tiling Subsitutions

For

$$(\boldsymbol{x},\ i^*) \in \boldsymbol{Z}^d \times \{1^*, \ldots, d^*\},$$

we give a geometrical meaning such that

$$(\boldsymbol{x},\ i^*) := \left\{ \sum_{\substack{j\,=\,1,\ldots,d,\\ j\,\neq\, i}} \boldsymbol{x} + \mu_j \boldsymbol{e}_j \;\middle|\; 0 \le \mu_j \le 1 \right\}$$

(see Figure 5).

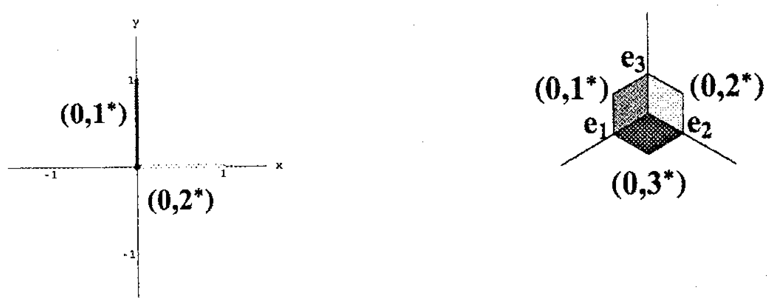


Figure 5: The figures of $(\mathbf{0},\ i^*)$.

Definition 2.1 $S := \{(\boldsymbol{x},\ i^*) \mid (\boldsymbol{x}, \boldsymbol{u}) \ge 0,\ (\boldsymbol{x} - e_i, \boldsymbol{u}) < 0\}$ *is called* the stepped surface *of the contract plain P where the row vector $\boldsymbol{u} >= 0$ is the eigenvector such that $\boldsymbol{u}L_\sigma = \lambda \boldsymbol{u}$ and the contract plain P is given by* $P = \{\boldsymbol{x} \mid (\boldsymbol{x}, \boldsymbol{u}) = 0\}$.

Definition 2.2 $\pi S := \{\pi(\boldsymbol{x},\ i^*) \mid (\boldsymbol{x}, i^*) \in S\}$ *is called a* tiling of P from the stepped surface S *(see Figure 6).*

3 Dual Subsitution σ^* (Tiling Substitution)

$S^* :=$ { the finite sum of elements of S } $\simeq$ { the patches of tiles of the tiling πS }.
Let us define the *dual (tiling) substitution* σ^* by

$$\sigma^* \pi(\boldsymbol{x},\ i^*) = L_\sigma^{-1} \pi \boldsymbol{x} + \sum_{j=1}^{d} \sum_{\left(\begin{smallmatrix} j \\ k \end{smallmatrix}\right):W_k^{(j)}=i} \pi_c\left(L_\sigma^{-1} f\left(P_k^{(j)}\right),\ j^*\right)$$

(see Figure 7 and Figure 8).

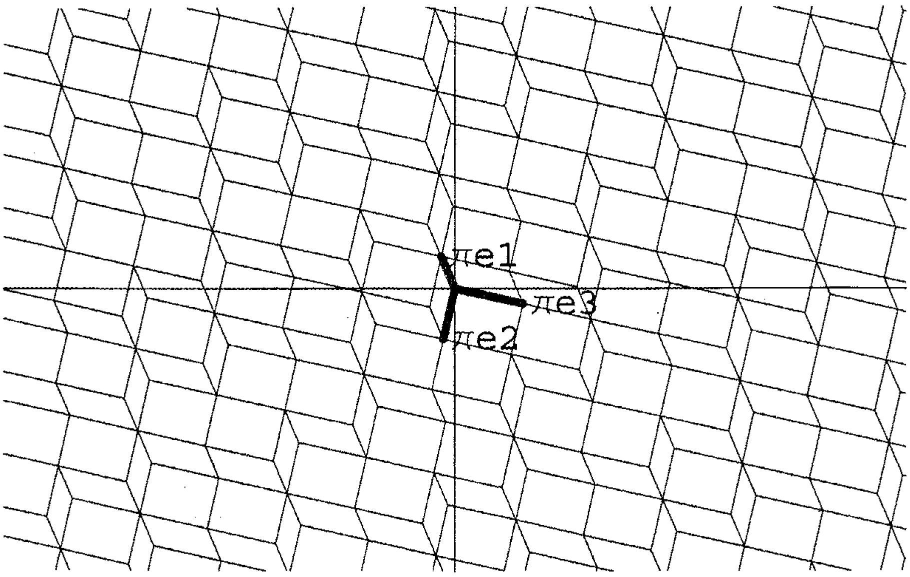


Figure 6: The figure of the tiling πS of P from the stepped surface S on σ_R in Example 1.2.

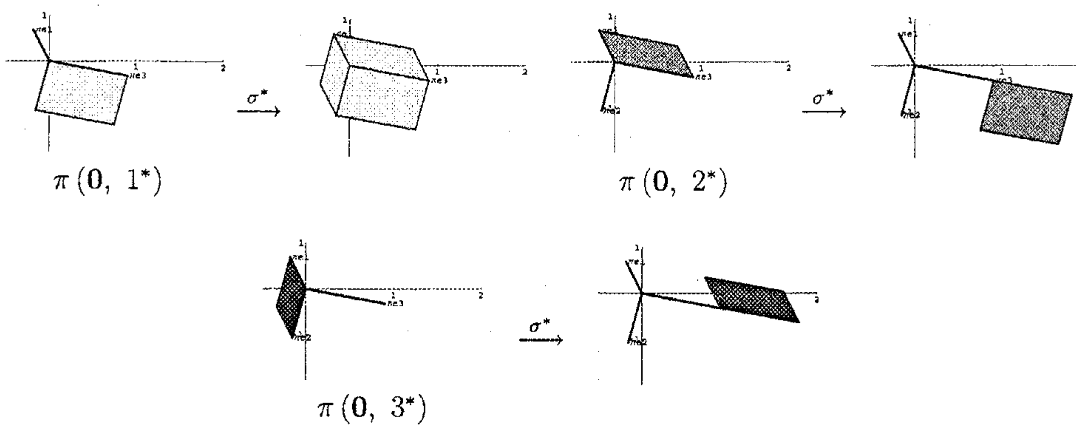


Figure 7: The figure of $\sigma_R^*\pi\,(\mathbf{0},\ i^*)$.

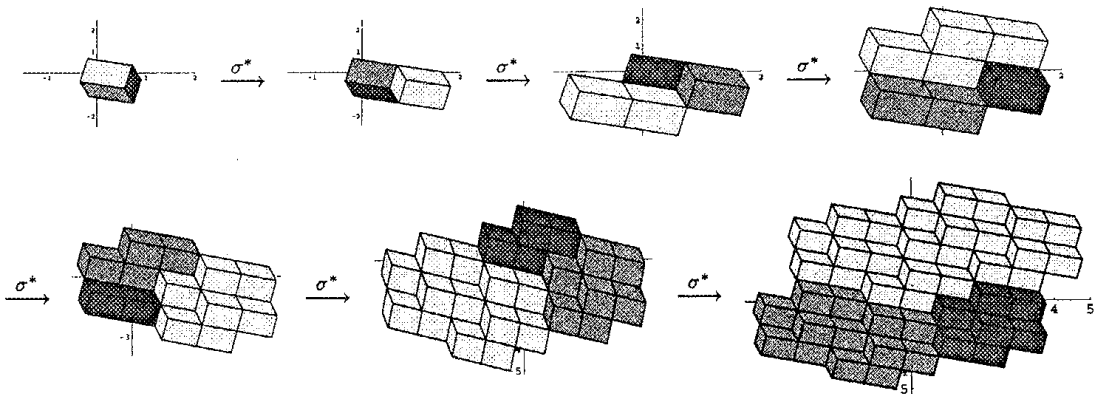


Figure 8: The figure of $\sigma_R^{*n} \bigcup_{i=1,2,3} \pi(\boldsymbol{e}_i,\ i^*)$ on σ_R.

Theorem 2 ([A-I]) σ: An Pisot Unit subsitution, then

(1) $\sigma^* : S^* \to S^*$ is well-defined;

(2) $\forall (\boldsymbol{x},\ i^*) \in S, \exists (\boldsymbol{y},\ j^*) \in S$: $(\boldsymbol{x},\ i^*) \in \sigma^*(\boldsymbol{y},\ j^*)$;

(3) $(\boldsymbol{x},\ i^*) \neq (\boldsymbol{y},\ j^*) \in S \Rightarrow \sigma^*(\boldsymbol{x},\ i^*) \cap \sigma^*(\boldsymbol{y},\ j^*) = \emptyset$.

Theorem 3 ([I-R]) Let $\mathcal{U} = \{\pi(\boldsymbol{e}_i,\ i^*) \mid i = 1, 2, \cdots, d\}$, then,

(1) $\sigma^*\mathcal{U} \succ \mathcal{U}$;

(2) if $d(\partial(\sigma^{*n}\mathcal{U}), \mathbf{0}) \to \infty$ ($n \to \infty$), then

$$\tau' := \{\pi(\boldsymbol{x},\ j^*) \mid \pi(\boldsymbol{x},\ j^*) \in \sigma^{*n}\pi(\boldsymbol{e}_i,\ i^*) \text{ for some } n \text{ and } j^*\}$$

coincides with πS and a quasi-periotdic tiling;

(3) $-X_i = \lim_{n\to\infty} L_\sigma^n \sigma^{*n}\pi(\boldsymbol{e}_i,\ i^*)$;

(4) $\tau := \{\pi\boldsymbol{x} - X_j \mid \pi(\boldsymbol{x},\ j^*) \in \tau'\}$ is also a quasi-periodic tiling of P.

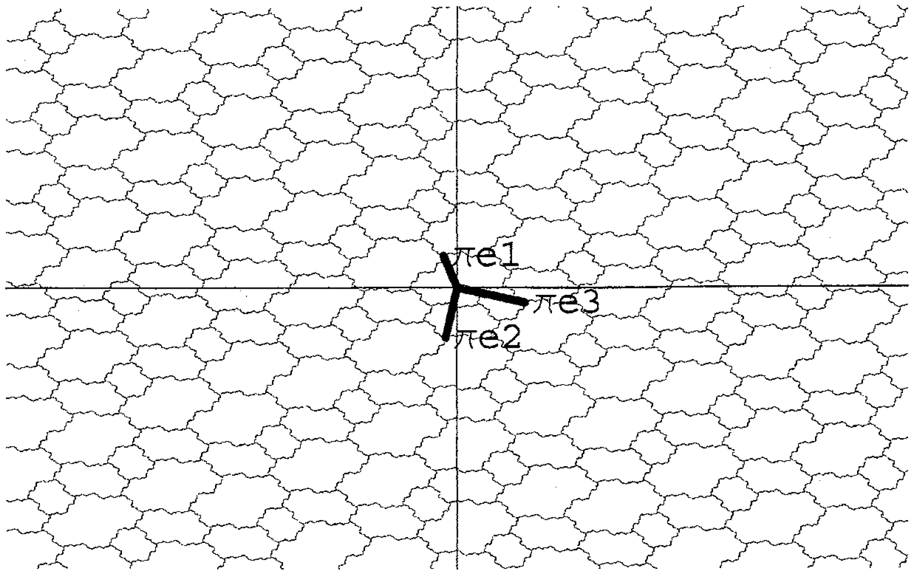


Figure 9: The figure of the quasi-periodic tiling τ.

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