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Endomorphisms of a Module over a Valuation Domain[1]

by

Hiroyuki Ishibashi[2]

Department of Mathematics
Josai University
Sakado, Saitama 350-02, Japan

Unless specified, R is a valuation ring, that is, an integral domain in which either a devides b, or b devides a for any nonzero a, b in R. This shows that R is a local ring with the unique maximal ideal $\mathfrak{m}$ consisting of all nonunits of R. Clearly its unit group is $R^* = R - \mathfrak{m}$.

Let M be a left free module over R of rank n, and $\mathrm{End}_R(M)$ or $\mathrm{End}(M)$ the right R algebra of R-endomorphisms of M. The unit group of $\mathrm{End}_R(M)$ is $\mathrm{Aut}_R(M)$ or simply $\mathrm{Aut}(M)$. We write an endomorphism σ on the right side of a module element $x \in M$.

The special elements in $\mathrm{End}_R(M)$ used here are (a) to (f) following, where $\mathcal{E} = \{e_1, e_2, \cdots, e_n\}$ is a fixed basis for M over R, and $X = \{x_1, x_2, \cdots, x_n\}$ is an arbitrarily chosen basis for M over R.

(a) For $x, y \in M$ and $L \subseteq M$, let $M = Rx \oplus Ry \oplus L$. A transposition $\Delta = \Delta_{x,y,L} \in \mathrm{Aut}(M)$ is defined by

$$x\Delta = y, \quad y\Delta = x \quad \text{and} \quad \Delta = 1 \quad \text{on} \quad L.$$

(b) For $a \in R$, $x, y \in M$ and $U \subseteq M$, let $M = Rx \oplus Ry \oplus U$ and $L = Ry \oplus U$. A transvection $\tau = \tau_{x,ay,U} \in \mathrm{Aut}(M)$ is defined by

$$x\tau = x + ay \quad \text{and} \quad \tau = 1 \quad \text{on} \quad L.$$

(c) For $a \in R$, $x, y \in M$ and $U \subseteq M$, let $M = Rx \oplus Ry \oplus U$.

[1] This is an abstract and the details will be published elsewhere.

[2] e-maile: hishi@math.josai.ac.jp

We define a left transposed transvection or a left skew transvection $\varphi = \varphi_{x,ay,U} \in \mathrm{Aut}(M)$ by

$$\varphi = \Delta_{x,y,U}\tau_{x,ay,U},$$

i.e.,

$$x\varphi = y, \quad y\varphi = x + ay \quad \text{and} \quad \varphi = 1 \quad \text{on} \quad U.$$

Similarly, a right skew transvection $\tau_{x,ay,U}\Delta_{x,y,U}$ is possible to define. However, as we will see, left skew is right skew and right skew is left skew. Therefore, we will often call them just skew transvections.

(d) For any elements $a_1, a_2, \cdots, a_n$ in R and for X a basis for M, we define $\delta = \delta_X(a_1, a_2, \cdots, a_n) \in \mathrm{End}(M)$ by

$$x_i\delta = a_i x_i \quad , \qquad i = 1, 2, \cdots, n.$$

(e) An element $\eta = \eta_X \in \mathrm{Aut}(M)$ is defined by

$$x_1\eta = x_1 \quad \text{and} \quad x_i\eta = x_{i-1} + x_i \quad , \qquad 2 \leq i \leq n.$$

If $n = 1$, i.e., $\mid X \mid = 1$, we define $\eta_X = 1$, i.e., the identity map on M.

(f) For $\pi \in S_n$ we define a permutation automorphism $\pi_X \in \mathrm{End}_R(M)$ by

$$x_i\pi_X = x_{\pi i}$$

The set of such π_X is denoted by S_X. Clearly S_X is a subgroup of $\mathrm{Aut}(M)$ isomorphic to S_n.

If $X = \mathcal{E}$, i.e., the canonical basis, then for $\pi \in S_n$ the permutation automorphism $\pi_{\mathcal{E}}$ is said to be a canonical permutation. For simplicity we may write π and S_n instead of π_X and S_X, respectively.

Moreover, providing these particular elements in $\mathrm{End}_R(M)$, we define the following three subsets of $\mathrm{End}_R(M)$, where $X = \{x_1, x_2, \cdots, x_n\}$ is again an arbitrary chosen basis for M :

The set of transvections relative to X is

$$\tau_{R,X} = \{\tau_{x_i,ax_j,U} \mid a \in R,\ U = \oplus^n_{\substack{i=1 \\ h\neq i,j}} Rx_h,\ 1 \leq i \neq j \leq n\}, \tag{5.1}$$

the set of skew transvections relative to X is

$$\varphi_{R,X} = \{\varphi_{x_i,ax_j,U} \mid a \in R,\ U = \oplus^n_{\substack{i=1 \\ h\neq i,j}} Rx_h,\ 1 \leq i \neq j \leq n\}, \tag{5.2}$$

and for subsets $S_1, S_2, \cdots, S_n$ of R we write

$$\delta_X(S_1, S_2, \cdots, S_n) = \{\delta_X(a_1, a_2, \cdots, a_n) \mid a_i \in S_i\}. \tag{5.3}$$

For any $\sigma \in \mathrm{End}_R M$ we define the fixed submodule M_σ of σ by

$$M_\sigma = \{x \in M \mid x\sigma = x\}.$$

Definition. For $i = 0, 1, \cdots, n$ we define

$$S^{(i)} = \{\sigma \in \mathrm{End}(M) \mid \mathrm{rank}\ M_\sigma = n - i\}.$$

An element σ in $S^{(1)}$ is called a simple element, i.e., σ is simple if and only if σ fixes a hyper plane of M.

By definition, Δ in (a) and τ in (b) are in $S^{(1)}$, and φ in (c) is in $S^{(2)}$. Also δ in (d) is in $S^{(n-i)}$ if exactly i of $\{a_1, a_2, \cdots, a_n\}$ is 1. Further η in (e) belongs to $S^{(n-1)}$.

Main Theorem. Let $0 \neq \sigma \in \mathrm{End}_R M$. Then there exist

(i) $\eta_{Z'}$ with $Z' \subseteq Z$ for some basis Z for M

and

(ii) skew transvections $\psi_1, \psi_2, \cdots, \psi_l$ with $0 \leq l \leq n-1$

such that

$$\psi_l \cdots \psi_2 \psi_1 \eta_{Z'} \sigma = \delta_X(a_1, a_2, \cdots, a_l, a_{l+1}, \cdots, a_n)$$

for some basis X for M with $a_1 \mid a_2 \mid \cdots \mid a_l$ and $a_l \mid a_i$ for $l \leq i \leq n$.

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