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Spaces of holomorphic maps between complex projective spaces and group actions

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1 Introduction.

The main purpose of this note is to announce the recent results concerning the topology of spaces of maps between complex projective spaces and related group actions on them.

First recall several notations used in this note. Let $j : S^2 = \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^m$ be the inclusion map given by $j([x : y]) = [x : y : 0 : \cdots : 0]$. If $1 \leq m \leq n$ and $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ is a continuous map, the homotopy class of $f \circ j \in \pi_2(\mathbb{C}\mathrm{P}^n) = \mathbb{Z}$ is called *the degree* of f. Let $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ be the space consisting of all continuous maps $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d, and $\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ the subspace of all based continuous maps $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ of degree d. We also denote by $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ (resp. $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \subset \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$) the corresponding subspace consisting of all holomorphic maps (resp. based holomorphic maps). If $m > n$, there is no holomorphic map $\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ except constant maps, and if $f : \mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ is a non-constant holomorphic map, $\deg f = d \geq 1$. So we shall only consider the case $1 \leq m \leq n$ with $d \geq 1$.

The origin of our work derives from the work of G. Segal [17], in which he describes that the Atiyah-Jones type result (cf. [1]) holds for the inclusion map $\mathrm{Hol}(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n)$. For understanding his result intuitively, recall the case $m = n = 1$.

Let $C_d^\infty(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1)$ denote the space of all smooth maps $f : \mathbb{C}\mathrm{P}^1 \to \mathbb{C}\mathrm{P}^1$ of degree d and consider the energy functional $E : C_d^\infty(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1) \to \mathbb{R}$ defined by

$$E(f) = \frac{1}{2}\int_{\mathbb{C}\mathrm{P}^1} \|df\|^2 dvol \quad \text{for } f \in C_d^\infty(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1).$$

The critical point of E is called *a harmonic map* and it is known that the space of all harmonic maps is just $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1)$ in this case. If we believe that Morse theoretical principle would hold for this case, $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1) \subset C_d^\infty \simeq \mathrm{Map}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1)$ is a deformation retract. This is clearly false, but Segal showed that it was true if "$d \to \infty$". More precisely, he proved the following important result:

Theorem 1.1 (G. Segal, [17]). *The inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) = \Omega_d^2\mathbb{C}\mathrm{P}^n \simeq \Omega^2 S^{2n+1} \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences up to dimension $(2n-1)d$. □

Remark. A map $f : X \to Y$ is called *a homotopy equivalence* (resp. *a homology equivalence*) *up to dimension* D if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is bijective when $k < D$ and surjective when $k = D$. Similarly, a map $f : X \to Y$ is called *a homotopy equivalence* (resp. *a homology equivalence*) *through dimension* D if the induced homomorphism $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is bijective whenever $k \leq D$.

In [17], Segal also expected that a similar Atiyah-Jones type result would hold for the inclusion $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ even if $2 \leq m \leq n$, and we would like to investigate this problem. Recently we obtain the following result:

Theorem 1.2 ([12], [22]). *If* $2 \leq m \leq n$, *the inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d; m, n)$, *where* $\lfloor x \rfloor$ *denotes the integer part of a number* x *and the number* $D(d; m, n)$ *is given by* $D(d; m, n) = (2n - 2m + 1)\left(\lfloor \frac{d+1}{2} \rfloor + 1\right) - 1$.

2 The case $d = 1$.

Because the spaces $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ are easily understood when $d = 1$, in this section we review the case $d = 1$.

First, we recall the interesting result due to S. Sasao [16].

Define the map $s'_{m,n} : U_{n+1} \to \mathrm{Map}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ by the usual matrix multiplication

$$s'_{m,n}(A)([x_0 : \cdots : x_m]) = [x_0 : \cdots : x_m : 0 : \cdots : 0] \cdot A$$

for $([x_0 : \cdots : x_m], A) \in \mathbb{C}\mathrm{P}^m \times U_{n+1}$.

If $\Delta_k \subset U_k$ denotes the center of U_k, because the subgroup $\Delta_{m+1} \times U_{n-m} \subset U_{n+1}$ is fixed by the map $s'_{m,n}$, it induces the map

$$s_{m,n} : \mathrm{P}W_{n+1,m+1} \to \mathrm{Map}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n),$$

where $\mathrm{P}W_{n+1,m+1} = U_{n+1}/(\Delta_{m+1} \times U_{n-m})$ denotes *the complex projective Stiefel manifold of orthogonal* $(m+1)$*-frames in* $\mathbb{C}^{n+1}$. Then Sasao showed the following interesting result.

Theorem 2.1 (S. Sasao, [16]). *If* $1 \leq m \leq n$, *the map* $s_{m,n} : \mathrm{P}W_{n+1,m+1} \to \mathrm{Map}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ *is a homotopy equivalence up to dimension* $(4n - 4m + 1$. □

Similarly, define the map $s''_{m,n} : U_n \to \mathrm{Map}^*_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ by the matrix multiplication

$$s''_{m,n}(A)([x_0 : \cdots : x_m]) = [x_0 : \cdots : x_m : 0 : \cdots : 0] \cdot \begin{bmatrix} 1 & \mathbf{0}_n \\ {}^t\mathbf{0}_n & A \end{bmatrix}$$

for $([x_0 : \cdots : x_m], A) \in \mathbb{C}\mathrm{P}^m \times U_n$, where $\mathbf{0}_n = (0, \cdots, 0) \in \mathbb{C}^n$. Since the subgroup $U_{n-m} \subset U_n$ is fixed under this map, it also induces the map

$$\overline{s}_{m,n} : W_{n,m} \to \mathrm{Map}^*_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n),$$

where $W_{n,m} = U_n/U_{n-m}$ denotes *the complex Stiefel manifold of orthogonal* m*-frames in* $\mathbb{C}^n$. Then we also obtain:

Theorem 2.2. *If* $1 \leq m \leq n$, *the map* $\overline{s}_{m,n} : W_{n,m} \to \mathrm{Map}^*_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ *is a homotopy equivalence up to dimension* $4n - 4m + 1$.

Proof. An easy diagram chasing shows that there is a commutative diagram of fibrations

$$\begin{array}{ccccc} W_{n,m} & \longrightarrow & PW_{n+1,m+1} & \longrightarrow & \mathbb{C}\mathrm{P}^n \\ {\scriptstyle \bar{s}_{m,n}}\downarrow & & {\scriptstyle s_{m,n}}\downarrow & & \| \\ \mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) & \xrightarrow{\subset} & \mathrm{Map}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) & \xrightarrow{ev} & \mathbb{C}\mathrm{P}^n \end{array}$$

Then the assertion follows from the Five Lemma and the homotopy exact sequences of fibrations. □

Now we can easily understand the reason why the two spaces $PW_{n+1,m+1}$ and $W_{n,m}$ approximate $\mathrm{Map}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ up to homotopy equivalence as above. This can be explained by the following result.

Theorem 2.3 ([10]). *Let $1 \leq m \leq n$ and $d \geq 1$ be integers.*

(1) *If $d = 1$, there are homotopy equivalences*

$$\begin{cases} \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \simeq \mathrm{W}_{n,m} \\ \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \simeq \mathrm{PV}_{n+1,m+1} \end{cases}$$

(2) *The inclusion maps*

$$\begin{cases} i_1 : \mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_1^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_1 : \mathrm{Hol}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_1(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalence up to dimension $4n - 4m + 1$.

Sketch proof. We shall only give the proof of the existence of the homotopy equivalence $\mathrm{Hol}_1^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \simeq \mathrm{W}_{n,m}$.

We take $\mathbf{e}_j = [1 : 0 : 0 : \cdots : 0] \in \mathbb{C}\mathrm{P}^j$ as the basepoint of $\mathbb{C}\mathrm{P}^j$. An element $f \in \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ is a holomorphic map $\mathbb{C}\mathrm{P}^m \to \mathbb{C}\mathrm{P}^n$ satisfying the condition $f(\mathbf{e}_m) = \mathbf{e}_n$. Such a map can be identified with the $(n+1)$-tuple $f = (f_0, f_1, \cdots, f_n)$ of homogeneous polynomials of the degree d in $\mathbb{C}[z_0, \cdots, z_m]$ such that,

(i) $f_0, f_1, \cdots, f_n$ have no common root except $\mathbf{0}_{m+1} \in \mathbb{C}^{m+1}$,

(ii) the coefficient of $(z_0)^d$ in f_0 is 1, and these in the other polynomials are 0.

If $d = 1$, we can write

$$\begin{aligned} f = (f_0, f_1, \cdots, f_n) &= (z_0 + \sum_{k=1}^{m} b_k z_k, \sum_{k=1}^{m} a_{k,1} z_k, \cdots, \sum_{k=1}^{m} a_{k,n} z_k) \\ &= (z_0, z_1, \cdots, z_m) \cdot \begin{bmatrix} 1 & \mathbf{0}_n \\ {}^t\mathbf{b} & A \end{bmatrix}, \end{aligned}$$

where $\mathbf{b} = (b_1, \cdots, b_m) \in \mathbb{C}^m$, $A = \left(a_{i,j}\right)$: $(m \times n)$-matrix.

First, we note that the polynomials $f_0, f_1, \cdots, f_n$ have no common root except $\mathbf{0}_{m+1}$ if and only if rank $A = m$. Next, we also remark that the space of all $(m \times n)$ matrices of rank m is homeomorphic to the homogenous space $\mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_{n-m}(\mathbb{C})$. Via this identification, we define a map $\gamma : \mathrm{Hol}_1^*(\mathbb{CP}^m, \mathbb{CP}^n) \to \mathbb{C}^m \times (\mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_{n-m}(\mathbb{C}))$ by

$$\gamma(f) = \gamma(f_0 : f_1 : \cdots : f_n) = (\mathbf{b}, A).$$

Next define a map $\beta' : \mathbb{C}^m \times \mathrm{GL}_n(\mathbb{C}) \to \mathrm{Hol}_1^*(\mathbb{CP}^m, \mathbb{CP}^n)$ by matrix multiplication

$$\beta'(\mathbf{b}, B)([x_0 : x_1 : \cdots : x_m]) = [x_0 : x_1 : \cdots : x_m : 0 : \cdots : 0] \cdot \begin{bmatrix} 1 & \mathbf{0}_n \\ {}^t\tilde{\mathbf{b}} & B \end{bmatrix}$$

for $([x_0 : x_1 : \cdots : x_m], \mathbf{b}, B) \in \mathbb{CP}^m \times \mathbb{C}^m \times \mathrm{GL}_n(\mathbb{C})$, where $\tilde{\mathbf{b}} = (\mathbf{b}, \mathbf{0}_{n-m}) \in \mathbb{C}^n$. Since β' maps the subspace $\{\mathbf{0}_m\} \times \mathrm{GL}_{n-m}(\mathbb{C}) \subset \mathbb{C}^m \times \mathrm{GL}_n(\mathbb{C})$ to the basepoint, it induces a map

$$\beta : \mathbb{C}^m \times (\mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_{n-m}(\mathbb{C})) \to \mathrm{Hol}_1^*(\mathbb{CP}^m, \mathbb{CP}^n).$$

Simple computation shows that $\beta \circ \gamma = \mathrm{id}$, $\gamma \circ \beta = \mathrm{id}$, and γ is a homeomorphism. Hence, $\mathrm{Hol}_1^*(\mathbb{CP}^m, \mathbb{CP}^n) \cong \mathbb{C}^m \times (\mathrm{GL}_n(\mathbb{C})/\mathrm{GL}_{n-m}) \simeq \mathrm{W}_{n,m}$. □

Corollary 2.4 ([10]). *There are homotopy equivalences*

$$\begin{cases} \mathrm{Hol}_1^*(\mathbb{CP}^n, \mathbb{CP}^n) \simeq U_n, \\ \mathrm{Hol}_1(\mathbb{CP}^n, \mathbb{CP}^n) \simeq \mathrm{P}(SU_{n+1}) = SU_{n+1}/\Delta_{n+1}. \end{cases}$$

□

3 The space $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$.

Recently, J. Mostovoy [12] obtained the remarkable important result concerning the topology of $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$. In this section we recall his result, and for this purpose, we study the restriction fibration sequence

$$F_d(m,n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \xrightarrow{r} \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^{m-1}, \mathbb{C}\mathrm{P}^n),$$

where the map r is defined by the restriction $r(f) = f|\mathbb{C}\mathrm{P}^{m-1}$, $F_d(m,n)$ denotes the fiber of r defined by

$$F_d(m,n) = r^{-1}(\psi_d^{m-1,n}) = \{f \in \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) : f|\mathbb{C}\mathrm{P}^{m-1} = \psi_d^{m-1,n}\},$$

and $\psi_d^{m,n} \in \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ is the holomorphic map defined by

$$\psi_d^{m,n}([x_0 : \cdots : x_m]) = [(x_0)^d : \cdots : (x_m)^d : 0 : \cdots : 0]$$

for $[x_0 : \cdots : x_m] \in \mathbb{C}\mathrm{P}^m$.

We choose it as the basepoint of $\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$. We remark that there is a homotopy equivalence $F_d(m,n) \simeq \Omega^{2m}\mathbb{C}\mathrm{P}^n$ ([16]).

Let $H_d(m,n) \subset \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ be the subspace defined by $H_d(m,n) = F_d(m,n) \cap \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$. We investigate the homotopy types of the subspaces $H_d(m,n)$, $\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ and $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ with the corresponding inclusion maps

$$\begin{cases} i'_d : H_d(m,n) \to F_d(m,n), & i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n). \end{cases}$$

Theorem 3.1 (J. Mostovoy, [13]). *If $2 \le m \le n$, the inclusion maps*

$$\begin{cases} i'_d : H_d(m,n) \to F_d(m,n), & i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d;m,n)$ when $m < n$, and homology equivalences through dimension $D(d;m,n)$ when $m = n$. □

Since $\lim_{d\to\infty} D(d;m,n) = \infty$, we may regard $H_d(m,n)$ and $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ as finite dimensional homotopy (or homology) models for the infinite dimensional spaces $\Omega^{2m}\mathbb{C}\mathrm{P}^n$ and $\mathrm{Map}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$, respectively. We know that the Atiyah-Jones type Theorem holds for several other cases, and the homotopy stability is usually satisfied for these cases (cf. [2], [3], [5], [6], [9], [19]). So one may expect that the homotopy stability may hold even if $m = n$. We consider the case $m = n$ in the next section.

4 The case $m = n$.

In this section we consider the space $\mathrm{Hol}(\mathbb{C}\mathrm{P}^m, \mathbb{C}\mathrm{P}^n)$ when $m = n$. From now on, we write $F_d(n) = F_d(n, n)$ and $H_d(n) = H_d(n, n)$.

First, recall the following two results.

Theorem 4.1 ([21]). (1) *If $m = n$, there are isomorphisms*

$$\begin{cases} \pi_1(\mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \pi_1(\mathrm{Hol}_d^*) \cong \mathbb{Z} \\ \pi_1(\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)) = \pi_1(\mathrm{Hol}_d) \cong \mathbb{Z}/(n+1)d^n \end{cases}$$

(2) *The inclusion maps induce isomorphisms*

$$\begin{cases} i_{d*} : \pi_1(\mathrm{Hol}_d^*) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d^*) = \pi_1(\mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)), \\ j_{d*} : \pi_1(\mathrm{Hol}_d) \xrightarrow{\cong} \pi_1(\mathrm{Map}_d) = \pi_1(\mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)). \end{cases} \qquad \square$$

Theorem 4.2 ([22]). *If $n \geq 1$ and $d \geq 0$ be integers, the inclusion $i'_d : H_d(n) \to F_d(n) \simeq \Omega^{2n}\mathbb{C}\mathrm{P}^n$ induces an isomorphism*

$$i'_{d*} : \pi_1(H_d(n)) \xrightarrow{\cong} \pi_1(F_d(n)) \cong \pi_1(\Omega^{2n}\mathbb{C}\mathrm{P}^n) \cong \mathbb{Z}.$$

Sketch proof. If we use an easy diagram chasing and Theorem 4.1, we can show that $\pi_1(H_d(n)) \cong \mathbb{Z}$. Then the assertion follows from Theorem 3.1 and Hurewicz Theorem. $\square$

If we use the above two results and some group actions, we obtain the following two results.

Theorem 4.3 ([22]). *If $n \geq 2$, the inclusion $i'_d : H_d(n) \to F_d(n) \simeq \Omega^{2n}\mathbb{C}\mathrm{P}^n$ is a homotopy equivalence through dimension $D(d, n)$, where we take $D(d, n) = D(d; n, n) = \lfloor \frac{d+1}{2} \rfloor$.* $\square$

Theorem 4.4 ([22]). *If $n \geq 2$, the inclusion maps*

$$\begin{cases} i_d : \mathrm{Hol}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d^*(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \\ j_d : \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \to \mathrm{Map}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \end{cases}$$

are homotopy equivalences through dimension $D(d,n) = \lfloor \frac{d+1}{2} \rfloor$. □

Proof of Theorem 1.2. The assertion follows from Theorem 3.1 and Theorem 4.4. □

5 The orbit spaces.

Finally, in this section, we consider the right $\mathrm{PGL}_{n+1}(\mathbb{C})$ action on $\mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)$ given by the matrix multiplication

$$\begin{array}{ccc} \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n) \times \mathrm{PGL}_{n+1}(\mathbb{C}) & \longrightarrow & \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n). \\ ([f_0 : \cdots : f_n], A) & \longrightarrow & [f_0 : \cdots : f_n] \cdot A \end{array}$$

We denote by $X_d(n)$ the orbit space $X_d(n) = \mathrm{Hol}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n)/\mathrm{PGL}_{n+1}(\mathbb{C})$.

Theorem 5.1 (R. Milgram, [11]). *If $n = 1$, there is a homeomorphism $X_d(1) \cong \mathrm{P}(\mathcal{F}_d)$, where $\mathcal{F}_d$ denotes the space consisting of all $(d \times d)$ non-singular Toeplitz matrices.* □

Then our main result of this section is as follows.

Theorem 5.2 ([21]). (1) $\pi_1(X_d(n)) \cong \mathbb{Z}/d^n$.

(2) *There is a fibration sequence (up to homotopy)*

$$(*) \qquad SU_{n+1} \to \widetilde{X_d(n)} \to \widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^n, \mathbb{C}\mathrm{P}^n),$$

where $\tilde{Y}$ denotes the universal covering of a space Y.

(3) *If $n = 1$, the fibration $(*)$ is trivial and there is a homotopy equivalence $\widetilde{X_d(1)} \simeq S^3 \times \widetilde{\mathrm{Hol}}_d(\mathbb{C}\mathrm{P}^1, \mathbb{C}\mathrm{P}^1)$.* □

Corollary 5.3 (J. Havlicek, [8]; [21]). $\pi_1(X_d(1)) \cong \mathbb{Z}/d$. □

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