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Ground State of the Nelson Model

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1 The Nelson Model

We consider the ground state problem of the Nelson model. The Nelson model is a quantum mechanical model which describes the dynamics of some particle and a scalar Bose field. In this paper, we consider only in the case that the particle number is one.

In this paper, we consider two kind of the Nelson models. The first type Nelson model is the standard Nelson model which appeared in [8]. The second type is the Nelson model in a non-Fock representation which introduced by Arai[1].

The Hilbert space of the Nelson model is defined by

$$\mathcal{F} := L^2(\mathbb{R}^3) \otimes \mathcal{F}_{\mathrm{b}},$$

where

$$\mathcal{F}_{\mathrm{b}} := \bigoplus_{n=0}^{\infty} \left[\bigotimes_{s}^{n} L^2(\mathbb{R}^3) \right]$$

is the Boson Fock space over $L^2(\mathbb{R}^3)$(see [10]). Any state of the Nelson model is described by a non-zero vector in $\mathcal{F}$.

For the Boson mass $m \geq 0$, we define a function $\omega_m(k) := \sqrt{k^2 + m^2}$. The function $\omega_m(k)$ defines a nonnegative self-adjoint operator on $L^2(\mathbb{R}^3)$. The n-Boson Hamiltonian $\omega_m^{[n]}$ is defined by

$$\omega_m^{[n]} := \sum_{j=1}^{n} \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \overset{j\text{-th}}{\omega_m} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1},$$

which is a self-adjoint operator on $\otimes_s^n L^2(\mathbb{R}^3)$. The free Hamiltonian of the Bose field $H_{\mathrm{b}}(m)$ is defined by the direct sum of all n-Boson Hamiltonian:

$$H_{\mathrm{b}}(m) := \bigoplus_{n=0}^{\infty} \omega_m^{[n]},$$

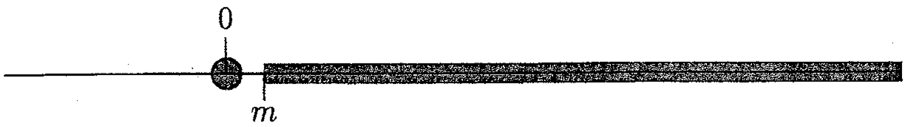

Figure 1: Spectrum of $H_b(m)$

where $\omega_m^{[0]} = 0$. The operator $H_b(m)$ is a nonnegative self-adjoint operator on $\mathcal{F}_b$. The vector $\Omega := (1, 0, 0, \ldots) \in \mathcal{F}_b$ is unique eigenvector of $H_b(m)$ and $\sigma(H_b(m)) = \{0\} \cup [m, \infty)$(Figure 1).

For $f \in L^2(\mathbb{R}^3)$ we define a closed operator $a(f)^*$ on $\mathcal{F}_b$ by

$$D(a(f)^*) := \left\{ \Psi \in \mathcal{F}_b | \sum_{n=1}^{\infty} n \| S_n f \otimes \Psi^{(n-1)} \|^2 < \infty \right\},$$
$$(a(f)^* \Psi)^{(n)} := \sqrt{n} S_n f \otimes \Psi^{(n-1)}, \quad \Psi \in D(a(f)^*),$$

where S_n is the symmetrization operator on $\otimes_s^n L^2(\mathbb{R}^3)$. The operator $a(f)^*$ is called a creation operator. We set $a(f) := (a(f)^*)^*$ the adjoint of $a(f)^*$. The operator $a(f)$ is called an annihilation operator. The operator

$$\Phi_S(f) := \frac{1}{\sqrt{2}} \overline{(a(f) + a(f)^*)}$$

is called a Segal field operator, and is self-adjoint. For $x \in \mathbb{R}^3$ and $\hat{\rho} \in L^2(\mathbb{R}^3) \cap D(|k|^{-1/2})$ we define $v(x) \in L^2(\mathbb{R}^3)$ by

$$v(x)(k) := v(x, k) := \frac{1}{(2\pi)^{3/2}} \frac{\hat{\rho}(k)}{|k|^{1/2}} e^{-ik \cdot x}$$

The Hilbert space $\mathcal{F}$ can be identify with the fibre direct integral of $\mathcal{F}_b$:

$$\mathcal{F} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{F}_b \mathrm{d}x \tag{1}$$

In this identification the operator

$$\phi^{\oplus}(v) := \int_{\mathbb{R}^3}^{\oplus} \Phi_s(v(x)) \mathrm{d}x \tag{2}$$

gives a self-adjoint operator on $\mathcal{F}$. In the context of physics, the function $\hat{\rho}$ is called a ultraviolet cutoff function. The most important example for $\hat{\rho}(k)$ is $\chi_\Lambda(k)$ which is a characteristic function on the ball $\{k \in \mathbb{R}^3 | |k| < \Lambda\}$. The positive constant Λ is called a ultraviolet cutoff.

Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ be a external potential for the particle. In this article, we assume that

Figure 2: Spectrum of H_{p}

[N.1] There exist constants $a < 1$ and $b \in \mathbb{R}$ such that

$$\|V_-^{1/2}\psi\|^2 \le a\|(-\Delta)^{1/2}\psi\|^2 + b\|\psi\|^2, \quad \psi \in C_0^\infty(\mathbb{R}^3),$$

where $V_-(x) := -\min\{0, V(x)\}$ is the negative part of V.

Hypothesis[N.1] defines a semi-bounded symmetric quadratic form

$$\langle\psi, H_{\mathrm{p}}(V)\psi\rangle := \langle\psi, -\Delta\psi\rangle + \|V_+^{1/2}\psi\|^2 - \|V_-^{1/2}\psi\|^2, \quad \psi \in C_0^\infty(\mathbb{R}^3).$$

That quadratic form uniquely defines a semi-bounded self-adjoint operator, and we denote that operator by the same symbol $H_{\mathrm{p}}(V)$. The self-adjoint operator $H_{\mathrm{p}}(V)$ is a Schödinger operator defined as quadratic forms. The typical example of V satisfying [N.1] is

$$V(x) = -C/|x|, \quad V(x) = Cx^2, \quad (C > 0).$$

In the first(Coulomb) case, it is well known that H_{p} has negative eigenvalues $\{e_n\}_{n=0}^\infty$ and $\sigma(H_{\mathrm{p}}) = \{e_n\}_{n=0}^\infty \cup [0, \infty)$ (Figure 2).

We call

$$H_0(m) := H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{b}}(m),$$

the free Hamiltonian. $H_0(m)$ does not include the interaction term between the particle and the Bose field. Therefore we can find spectrum of $H_0(m)$ easily:

$$\begin{aligned}\sigma(H_0(m)) &= \{\lambda + \mu \in \mathbb{R} | \lambda \in \sigma(H_{\mathrm{p}}), \mu \in \sigma(H_{\mathrm{b}}(m))\} \\ \sigma_p(H_0(m)) &= \{\lambda + \mu \in \mathbb{R} | \lambda \in \sigma_p(H_{\mathrm{p}}), \mu \in \sigma_p(H_{\mathrm{b}}(m))\} = \sigma_p(H_{\mathrm{p}}).\end{aligned}$$

In the Coulomb case, we draw $\sigma(H_0(m))$ at Figure 3.

The Hamiltonian of the standard Nelson model H_m^V has the following three parts:

$$H_m^V := H_{\mathrm{p}}(V) \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{b}}(m) + \phi^{\oplus}(v).$$

the second term $H_{\mathrm{b}}(m)$ is the free Boson Hamiltonian with Boson mass m, and the last term $\phi^{\oplus}(v)$ is the interaction Hamiltonian between the particle and the Bose field. For the ultraviolet cutoff function $\hat{\rho}$, we assume

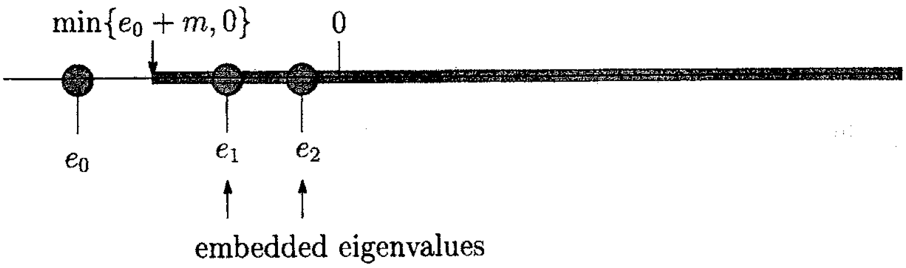

Figure 3: Spectrum of $H_0(m)$

[N.2] $|k|^{-1/2}\hat{\rho} \in D(\omega_m^{-1/2})$.

Note that, if the Boson mass $m > 0$ then ω_m^{-1} is bounded, so Hypothesis[N.2] holds. In the case that $m = 0$ and $\hat{\rho} = \chi_\Lambda$, Hypothesis[N.2] holds.

We define another Hamiltonian

$$\widetilde{H}_m^V := H_{\mathrm{p}}(V) \otimes \mathbb{1} + \mathbb{1} \otimes H_f(m) + \phi^{\oplus}(G) - \mathcal{V}_m(\hat{x}) \otimes \mathbb{1} + \mathcal{W}_m \mathbb{1}, \tag{3}$$

where $G(x,k) := v(x,k) - v(0,k) \in L^2(\mathbb{R}^3) \cap D(|k|^{-1/2})$, $\mathcal{V}_m(\hat{x})$ is the multiplication operator by the function

$$\mathcal{V}_m(x) := \mathrm{Re}\langle \omega_m^{-1/2} v(0), \omega_m^{-1/2} v(x) \rangle,$$

and $\mathcal{W}_m := \|\omega_m^{-1/2} v(0)\|^2$ is a constant (note that, by [N.2], $v(x), v(0) \in D(\omega_m^{-1/2})$).

The Hamiltonian of the quantum system must be self-adjoint. About the Nelson model, we can easily prove the self-adjointness of these Hamiltonian:

Proposition 1.1. *Assume that [N.1] and [N.2]. Then H_m^V and $\widetilde{H}_m^V$ is self-adjoint on $D(H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{b}}(m))$ and bounded below. Moreover H_m^V and $\widetilde{H}_m^V$ is essentially self-adjoint on any core for $H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes H_{\mathrm{b}}(m)$.*

Definition 1.2. *We say that the infrared regular condition holds if and only if*

$$|k|^{-1/2}\hat{\rho} \in D(\omega_m^{-1}),$$

and we say that the infrared singular condition holds if and only if

$$|k|^{-1/2}\hat{\rho} \notin D(\omega_m^{-1}),$$

When the massive case $m > 0$, ω_m^{-1} is bounded, so the infrared regular condition holds. In the case $m = 0$ and $\hat{\rho} = \chi_\Lambda$, it is easy to see that $|k|^{-1/2}|k|^{-1}\chi_\Lambda \notin L^2(\mathbb{R}^3)$, so this case is infrared singular.

The following proposition means that infrared regular condition lead to the equivalence between H_m^V and $\widetilde{H}_m^V$:

Proposition 1.3. *Assume [N.1] and [N.2]. Suppose that the infrared regular condition holds. Then the Hamiltonian H_m^V is unitarily equivalent to $\widetilde{H}_m^V$.*

Proof. By the infrared regular condition, the operator $T_m := \exp[-i\mathbb{1} \otimes \Phi_S(iv(0)/\omega_m)]$ is a unitary operator on $\mathcal{F}$, and we have

$$\begin{aligned} T_m H_{\mathrm{b}}(m) T_m^* &= H_{\mathrm{b}}(m) - \mathbb{1} \otimes \Phi_S(v(0)) + \frac{1}{2}\langle v(0), \omega_m^{-1} v(0)\rangle, \\ T_m \phi^{\oplus}(v) T_m^* &= \phi^{\oplus}(v) + \mathrm{Re}\langle \omega_m^{-1} v(0), v(x)\rangle, \end{aligned}$$

where we use a formula(see [2, Lemma 4-44 and 12-5]). Obviously T_m commutes H_{p}. Therefore $\widetilde{H}_m^V = T_m H_m^V T_m^*$. ∎

For $\Psi \in D(H_{\mathrm{b}}(m))$, we define a Fock space valued function $a(k)\Psi$ by

$$a(k)\Psi = (\Psi^{(1)}(k), \sqrt{2}\Psi^{(2)}(k,\cdot), \dots, \sqrt{n}\Psi^{(n)}(k,\cdot), \dots).$$

Next we define a distribution on the Fock space $\mathcal{F}_{\mathrm{b}}$ by

$$(a(k)^*\Psi)^{(n)} := \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \delta(k - k_j)\Psi^{(n-1)}(k_1, \dots, k_{j-1}, k_{j+1}, \dots, k_n).$$

Here δ is the Dirac delta function. The $a(k)$'s and $a(k)^*$'s satisfy the following 'formal' CCR relations:

$$\begin{aligned} [a(k), a(k')^*] &= \delta(k - k'), \\ [a(k), a(k')] &= [a(k)^*, a(k')^*] = 0. \end{aligned}$$

we define

$$\begin{aligned} b(k) &:= a(k) - \frac{1}{\sqrt{2}}\omega_m(k)^{-1} v(0,k), \\ b(k)^* &:= a(k)^* - \frac{1}{\sqrt{2}}\omega_m(k)^{-1} v(0,k). \end{aligned}$$

The second term of $b(k)$, $b(k)^*$ is constant for each $k \in \mathbb{R}^3$. Hence $b(k)$'s and $b(k)^*$'s satisfy the formal CCR relations:

$$\begin{aligned} [b(k), b(k')^*] &= \delta(k - k'), \\ [b(k), b(k')] &= [b(k)^*, b(k')^*] = 0. \end{aligned}$$

By using $a(k)$, we write the Hamiltonian H_m^V as

$$\begin{aligned} \langle \Psi, H_m^V \Psi\rangle =& \langle \Psi, H_{\mathrm{p}} \otimes \mathbb{1}\Psi\rangle + \int_{\mathbb{R}^3} \omega_m(k) \langle \mathbb{1} \otimes a(k)\Psi, \mathbb{1} \otimes a(k)\Psi\rangle \mathrm{d}k \\ &+ \int_{\mathbb{R}^3} \frac{\hat{\rho}(k)}{|k|^{1/2}} \left[\langle e^{-ik\hat{x}} \otimes a(k)\Psi, \Psi\rangle + \langle \Psi, e^{-ik\hat{x}} \otimes a(k)\Psi\rangle\right] \mathrm{d}k, \end{aligned}$$

and $\widetilde{H}_m^V$ can be written as

$$\langle \Psi, \widetilde{H}_m^V \Psi \rangle = \langle \Psi, H_{\mathrm{p}} \otimes \mathbb{1} \Psi \rangle + \int_{\mathbb{R}^3} \omega_m(k) \langle \mathbb{1} \otimes b(k)\Psi, \mathbb{1} \otimes b(k)\Psi \rangle \mathrm{d}k$$
$$+ \int_{\mathbb{R}^3} \frac{\hat{\rho}(k)}{|k|^{1/2}} \left[\langle e^{-ik\hat{x}} \otimes b(k)\Psi, \Psi \rangle + \langle \Psi, e^{-ik\hat{x}} \otimes b(k)\Psi \rangle \right] \mathrm{d}k,$$

$a(k), a(k)^*$ is called the Fock representation of the CCR. When the infrared regular condition holds, $b(k), b(k)^*$ is unitarily equivalent to the Fock representation of the CCR, but in the infrared singular case, $b(k), b(k)^*$ is not unitarily equivalent to the Fock representation of the CCR(see [2, p.202]). By this reason, when the infrared singular condition holds, we call that the operator $\widetilde{H}_0^V$ is the Nelson Hamiltonian in a non-Fock representation.

2 Existence of Ground State

Let H be self-adjoint and bounded from below. We call the real value $E_0(H) = \inf \sigma(H)$ the ground (state) energy of H. If $E_0(H)$ is an eigenvalue of H, the corresponding eigenvector is called a ground state of H. We set

$$E^V(m) := \inf \sigma(H_m^V), \quad \widetilde{E}^V(m) := \inf \sigma(\widetilde{H}_m^V)$$

the ground state energy of H_m^V and $\widetilde{H}_m^V$ respectively.

Proposition 2.1. *Assume [N.1] and [N.2]. Then*

$$E^V(m) = \widetilde{E}^V(m)$$

for all $m \geq 0$.

Proof. In the case $m > 0$, by Proposition 1.3, H_m^V is unitarily equivalent to $\widetilde{H}_m^V$. Hence $E^V(m) = \widetilde{E}^V(m)$ for $m > 0$. Let $m > m' > 0$, then we have $H^V(m) \geq H^V(m') \geq H^V(0)$. Therefore, by the variational principle $E^V(m) \geq E^V(m') \geq E^V(0)$. Hence $\lim_{m \to +0} E^V(m)$ exists, and $\lim_{m \to +0} E^V(m) \geq E^V(0)$. It is easy to see that $\lim_{m \to +0} H_m^V \Psi = H_0^V \Psi$ for all $\Psi \in D(H_{\mathrm{p}}) \hat{\otimes} D(H_{\mathrm{b}}(0))$, where $\hat{\otimes}$ means algebraic tensor product. Therefore H_m^V converges to H_0^V in the strong resolvent sense. ([9, Theorem VIII.25]). Similarly $\widetilde{H}_m^V$ converges to $\widetilde{H}_0^V$ strong resolvent sense. Hence we have the inverse inequality $\lim_{m \to +0} E^V(m) \leq E^V(0)$ and $\lim_{m \to +0} \widetilde{E}^V(m) \leq \widetilde{E}^V(0)$ ([9, Theorem VIII.23]). ∎

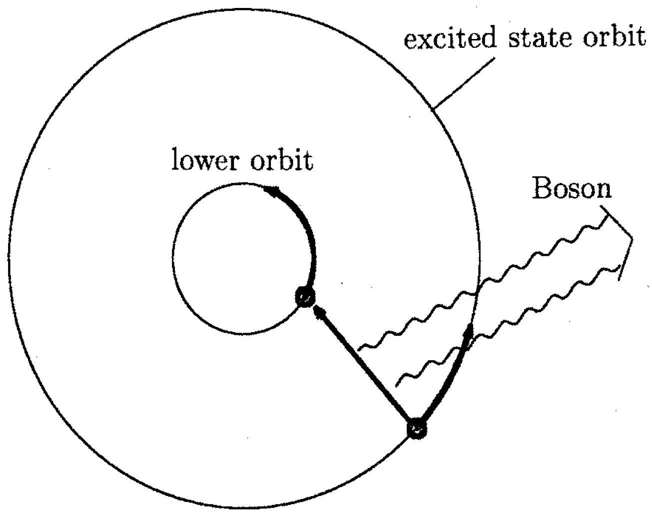

Figure 4: All excited states are unstable

The problem we consider here is

Problem. Do H_m^V and $\widetilde{H}_m^V$ have a ground state?

In the case $m > 0$, the free Hamiltonian $H_0(m)$ has a discrete ground state (see Figure 3). Therefore by the regular perturbation theory, the massive Hamiltonian H_m^V $(m > 0)$ has a ground state for sufficiently small $\|\hat{\rho}\|$. But we show that the massive$(m > 0)$ Hamiltonian H_m^V has a ground state for all coupling constant $\|\hat{\rho}\|$. In the proof about existence of massive ground state, we use a localization estimate technique which is developed by M. Griesemer, E. Lieb, and M. Loss [4]. The condition to have a ground state we give here is essentially same as GLL criterium[4], but our condition contains the case of the oscillator$(V = Cx^{\alpha}, (C, \alpha > 0))$.

In the massless case $m = 0$, all eigenvalue of the free Hamiltonian $H_0(m = 0)$ embedded in the continuum(Figure 3), so we can not apply the regular perturbation theory for any perturbation. Since the embedded eigenvalue may vanish by any small perturbation, the analysis of ground state for H_0^V, $\widetilde{H}_0^V$ is difficult.

When there is no interaction between the Bose field and the particle, all excited states are stable because they are eigenvectors of the free Hamiltonian $H_0(m = 0)$. However, when the particle interact with the Bose field, the excited state particle should emit a Boson and fall to a lower orbit (see Figure 4). Actually, in an atom, any excited state electron emits light spontaneously and falls to a lower orbit. Namely, all excited states should be unstable. Therefore, we expect that H_0^V and $\widetilde{H}_0^V$ have no eigenvalue above the ground state energy.

Incidentally, for a natural potential V, I believe that the Hamiltonian H_0^V and $\widetilde{H}_0^V$ have no singular continuous spectrum. Then all spectrum of the Hamiltonian is absolutely continuous spectrum if the Hamiltonian has no ground state. In this situation, for any state $\Psi \in \mathcal{F}$, the time evolution of Ψ converge weakly to 0 by the Riemann-Lebesgue Theorem, which is a contradiction. Because, the particle must remain near the origin as Figure 4.. Therefore if the particle Hamiltonian H_{p} has ground state and the Nelson model is physically natural, the Hamiltonian of the Nelson model should have a ground state.

In 2001, A. Arai showed that the Nelson Hamiltonian in a non-Fock representation $\widetilde{H}_0^V$ has a ground state, if $V(x) \geq c|x|^\alpha$, $c, \alpha - 2 \geq 0$ and the infrared singular condition holds(see[1]).

However, J. Lörinczi, R. A. Minlos and H. Spohn showed that if $V(x) \geq c|x|^\alpha$, $c, \alpha - 2 \geq 0$ then the standard massless Nelson Hamiltonian H_0^V has no ground state in the infrared singular case([7]).

In Mathematically, the existence of ground state is a phenomenon depending on the representation of the CCR. In the context of this article, in the infrared singular case, the Arai's Nelson Hamiltonian $\widetilde{H}_0^V$ is physically natural more than the standard Nelson Hamiltonian H_0^V.

Let $\theta \in C_0^\infty(\mathbb{R}^3)$, $\widetilde{\theta} \in C^\infty(\mathbb{R}^3)$ be functions which satisfy the following properties (i), (ii):

$$\text{(i)} \quad 0 \leq \theta(x), \widetilde{\theta}(x) \leq 1, \quad \theta(x)^2 + \widetilde{\theta}(x)^2 = 1, \quad x \in \mathbb{R}^3.$$

$$\text{(ii)} \quad \theta(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| \geq 2. \end{cases}$$

For $R > 0$ we define particle cutoff functions θ_R, $\widetilde{\theta}_R$ as follows:

$$\theta_R(x) := \theta(x/R), \quad \widetilde{\theta}_R(x) := \theta(x/R).$$

We abbreviate $\theta_R \otimes \mathbb{1}$, $\widetilde{\theta}_R \otimes \mathbb{1}$ to θ_R, $\widetilde{\theta}_R$. We define a minimal energy in the state where the particle is separated more than R away from the origin:

$$E_\infty(R, m) := \inf_{\substack{\Psi \in Q(H_m^V) \\ \|\widetilde{\theta}_R \Psi\| \neq 0}} \frac{\langle \widetilde{\theta}_R \Psi, H_m^V \widetilde{\theta}_R \Psi \rangle}{\|\widetilde{\theta}_R \Psi\|^2}, \widetilde{E}_\infty(R, m) := \inf_{\substack{\Psi \in Q(\widetilde{H}_m^V) \\ \|\widetilde{\theta}_R \Psi\| \neq 0}} \frac{\langle \widetilde{\theta}_R \Psi, \widetilde{H}_m^V \widetilde{\theta}_R \Psi \rangle}{\|\widetilde{\theta}_R \Psi\|^2},$$

where Q means the form domain. Note that $E_\infty(R, m)$ and $\widetilde{E}_\infty(R, m)$ are monotone increasing in $m \geq 0$. If $m \geq 0$, since $\widetilde{\theta}_R$ commutes with T_m, we have $E_\infty(R,m) = \widetilde{E}_\infty(R,m)$. By the variational principle, we have

$$E_\infty(R,m) \geq E^V(m), \quad \widetilde{E}_\infty(R,m) \geq \widetilde{E}^V(m),$$

for all $m \geq 0$.

Definition 2.2 (binding condition). *We say the inequality*

$$E^V(m) < \limsup_{R\to\infty} E_\infty(R,m)$$

the binding condition.

Now we state that the existence theorem of the massive Nelson model;

Theorem 2.3 (Existence of ground state $(m>0)$). *Let $m>0$. Assume that [N.1] and [N.2] hold. Suppose that the binding condition for $m>0$ holds. Then H_m^V has a ground state.*

Proof. This proof is based on [4]. In this proof, we take the space representation for the Bosons. Namely, we consider

$$\begin{aligned}\hat{H} :=& \mathbb{1} \otimes \Gamma(\mathcal{F}^{-1}) H_m^V \mathbb{1} \otimes \Gamma(\mathcal{F}) \\ =& H_{\mathrm{p}} \otimes \mathbb{1} + \mathbb{1} \otimes d\Gamma_{\mathrm{b}}(\sqrt{-\Delta + m^2}) + \phi^{\oplus}(\mathcal{F}^{-1}v),\end{aligned}$$

where $\mathcal{F}$ is the Fourier transform on the one Boson space, Γ is the second quantization of second type, $d\Gamma_{\mathrm{b}}$ is the second quantization operator (see [2]), and

$$(\mathcal{F}^{-1}v)(x,y) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} \frac{\hat{\rho}(k)}{|k|^{1/2}} e^{-ik(x-y)} dk,$$

where x is the coordinate of the particle, and y is the coordinate of the Boson. For $P>0$ we set $j_1(y) := \theta(y/P)$, $j_2(y) := \widetilde{\theta}(y/R)$ the Boson cutoff functions. We define a new creation and annihilation operators

$$\begin{aligned}c(f) &:= a(j_1 f) \otimes \mathbb{1} + \mathbb{1} \otimes a(j_2 f), \\ c(g)^* &:= a(j_1 g)^* \otimes \mathbb{1} + \mathbb{1} \otimes a(j_2 g)^*, \quad f, g \in L^2(\mathbb{R}^3),\end{aligned}$$

which is a closed operator on $\mathcal{F}_{\mathrm{b}} \otimes \mathcal{F}_{\mathrm{b}}$. Let

$$\begin{aligned}&\mathcal{F}_{\ell} := \overline{\mathcal{F}_{\ell,\mathrm{fin}}} \subset \mathcal{F}_{\mathrm{b}} \otimes \mathcal{F}_{\mathrm{b}}, \\ &\mathcal{F}_{\ell,\mathrm{fin}} := \mathcal{L}\{\Omega \otimes \Omega, c(g_1)^* \cdots c(g_k)^* \Omega \otimes \Omega | g_j \in L^2(\mathbb{R}^3), j = 1, \ldots, k, k \in \mathbb{N}\}.\end{aligned}$$

We define a operator $U_0 : \mathcal{F}_{\mathrm{b}} \to \mathcal{F}_{\mathrm{b}} \otimes \mathcal{F}_{\mathrm{b}}$ by

$$\begin{aligned}D(U_0) &:= \mathcal{F}_{\mathrm{fin}}, \\ U_0 a(g_1)^* \cdots a(g_k)^* \Omega &= c(g_1)^* \cdots c(g_k)^* \Omega \otimes \Omega,\end{aligned}$$

where $\mathcal{F}_{\rm fin}$ means the finite particle subspace (see [2]). It is easy to see that U_0 is isometry, and $U := \overline{U_0}$ is isometric operator from $\mathcal{F}_{\rm b}$ to $\mathcal{F}_{\rm b} \otimes \mathcal{F}_{\rm b}$ with $\mathrm{Ran}(U) = \mathcal{F}_\ell$. Therefore

$$U^*U = \mathbb{1}_{\mathcal{F}_{\rm b}}, \quad UU^* = \text{orthogonal projection on } \mathcal{F}_\ell.$$

We define a dense space by $\mathcal{D} := C_0^\infty(\mathbb{R}^3)\hat{\otimes}\mathcal{F}_{\rm fin}(C_0^\infty(\mathbb{R}^3))$. By the IMS localization formula, we have that

$$\hat{H} = \theta_R \hat{H} \theta_R + \widetilde{\theta}_R \hat{H} \widetilde{\theta}_R - \frac{1}{2}|\nabla\theta_R|^2 - \frac{1}{2}|\nabla\widetilde{\theta}_R|^2, \tag{4}$$

in the sense of quadratic form on $\mathcal{D}$. The key of the proof is the following lemma

Lemma 2.4. *For all $\Psi \in \mathcal{D}$, we have that*

$$\begin{aligned}\langle \Psi, \theta_R \hat{H} \theta_R \Psi\rangle = &\langle \Psi, \theta_R U^* \{\hat{H} \otimes \mathbb{1}_{\mathcal{F}_{\rm b}} + \mathbb{1}_{\mathcal{F}} \otimes H_{\rm b}(m)\} U \phi_R \Psi\rangle \\ &+ o(1)\langle \Psi, (\hat{H} - E^V(m) + 1)\Psi\rangle,\end{aligned}$$

in the sense of quadratic form, where the operator in $\{\}$ is acting on the Hilbert space $\mathcal{F}\otimes\mathcal{F}_{\rm b}$, and $o(1)$ is a constant such that $\lim_{R\to\infty}\lim_{P\to\infty} o(1) = 0$.

We omit the proof of this Lemma(see [4]). By this lemma and (4), we have

$$\begin{aligned}\hat{H} \geq \ &\theta_R U^* \{E^V(m) \otimes \mathbb{1} + \mathbb{1} \otimes m(\mathbb{1} - P_2)\} U \theta_R \\ &+ E_\infty(R, m)\widetilde{\theta}_R^2 + o(1)(\hat{H} - E^V(m) + 1),\end{aligned}$$

in the sense of quadratic form on $\mathcal{D}$, where P_2 is the orthogonal projection on Ω. Therefore

$$\begin{aligned}\hat{H} - E^V(m) \geq &(E_\infty(R, m) - E^V(m))\widetilde{\theta}_R^2 + m\theta_R^2 - m\theta_R U^* I \otimes P_2 U \theta_R \\ &+ o(1)(\hat{H} + \mathbb{1}).\end{aligned}$$

Note that $T := \theta_R U^* \mathbb{1} \otimes P_2 U \theta_R = (\Gamma(j_1)\theta_R)^2$. Since $H_{\rm b}(m)$ is massive, hence T is $(-\Delta \otimes \mathbb{1} + \mathbb{1} \otimes H_{\rm b}(m) + 1)$-form compact(more precisely see [4]). By $\theta_R^2 + \widetilde{\theta}_R^2 = 1$, we have that

$$(E_\infty(R, m) - E^V(m))\widetilde{\theta}_R^2 + m\theta_R^2 \geq \min\{E_\infty(R, m) - E^V(m), m\}.$$

Therefore we get

$$(1 - o(1))\hat{H} - E^V(m) + mT \geq \min\{E_\infty(R, m) - E^V(m), m\} + o(1),$$

in the sense of the quadratic form on $Q(\hat{H})$, where we done the closed extension of the quadratic form. Using the condition [N.1],[N.2], one can show that T is $\hat{H}$ -form compact. So T does not change the essential spectrum of $\hat{H}$, and hence

$$(1-o(1))\Sigma(\hat{H}) - E^V(m) \geq \min E_\infty(R,m) - E^V(m), m + o(1), \qquad (5)$$

where Σ means the bottom of the essential spectrum. Now we take the limit $\lim_{R\to\infty}\lim_{P\to\infty}$, and we get

$$\Sigma(\hat{H}) - E^V(m) > 0.$$

This inequality means that $\hat{H}$ has a discrete ground state. ∎

For existence of the massless ground state, we assume these following conditions:

[N.3] There exists an open set $S \subset \mathbb{R}^3$, such that $\operatorname{supp}\hat{\rho} = \bar{S}$. Moreover, for all $n \in \mathbb{N}$

$$S_n := \{i \in S||k| < n\}$$

has the cone-property (see [6]).

[N.4] There exists a function $\eta \in H^1(\mathbb{R}^3)$, such that $\hat{\rho} = \chi_S\eta$.

[N.5] $\hat{\rho}$ is continuously differentiable in $S\backslash\{0\}$.

[N.6] $|k|^{-3/2}\hat{\rho}, |k|^{-1/2}|\nabla\hat{\rho}| \in L^p(S)$ for all $p \in (1,2)$.

Theorem 2.5 (Existence of ground state $m = 0$). *Let $m = 0$. Assume conditions [N.1]-[N.6]. If the binding condition holds, then $\widetilde{H}_0^V$ has a ground state.*

Remark. If $\lim_{|x|\to\infty} V(x) = \infty$, one can show that $\lim_{R\to\infty} E_\infty(R,m) = \infty$. Therefore the binding condition holds. If $\lim_{|x|\to\infty} V(x) = 0$, $V \in L^2_{\mathrm{loc}}(\mathbb{R}^3)$ and H_{p} has a negative ground state. Then the binding condition holds (see [4, Theorem 3.1]).

Remark. $\hat{\rho} = \chi_\Lambda$ satisfies the conditions [N.1]-[N.6].

Proof. See [12] ∎

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