数理解析研究所講究録 1459 巻 2005 年 1-13

Factors generated by C^*-finitely correlated states

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1 Introduction

The notion of quantum Markov chains was introduced by Accardi in [1]. As special cases, the notion of quantum Markov states was defined by Accardi and Frigerio in [2] and that of C^*-finitely correlated states was discussed by Fannes, Nachtergaele and Werner [5]. Further discussions on quantum Markov states are found in [3], [8] and [10] for example.

In [7], Fidaleo and Mukhamedov showed that the von Neumann algebras generated by faithful translation-invariant quantum Markov states are factors of type II_1 or type III_λ with $\lambda \in (0,1]$. In the present paper we discuss the von Neumann algebras generated by C^*-finitely correlated states. In the case where the states are Markov states, it is known ([8], [10] for example) that the states are unique KMS states, and the exact form of local density matrices is also known. Hence, we can see that the von Neumann algebras are factors, and the types of factors can be determined in terms of the local density matrices. But, in the case where the states are C^*-finitely correlated states, we have to find a different method.

A C^*-finitely correlated state is a state on the UHF algebra $\bigotimes_{\mathbb{Z}} M_d$ defined by a triplet $(\mathfrak{C}, E, \rho)$, where $\mathfrak{C}$ is a finite dimensional C^*-algebra, E is a completely positive map from $M_d \otimes \mathfrak{C}$ to $\mathfrak{C}$ and ρ is a state on $\mathfrak{C}$.

In section 2, we show that a C^*-finitely correlated state is a factor state if and only if it satisfies the strong mixing property. To see this, we look at the eigenvectors of $E(I \otimes \cdot)$ with eigenvalues of modulus 1. In section 3, we show that the factors generated by C^*-finitely correlated states are of type I_∞ or type II_1 or type II_∞ or type III_λ for some $\lambda \in (0,1]$.

2 Equivalent condition for factor

Let $\mathfrak{B}_i = M_d = M_d(\mathbb{C})$, the $d \times d$ complex matrix algebra, for $i \in \mathbb{Z}$ and $\mathfrak{B}$ be the infinite C^*-tensor product $\bigotimes_{i\in\mathbb{Z}} \mathfrak{B}_i$. We denote $\mathfrak{B}_\Lambda = \bigotimes_{n\in\Lambda} \mathfrak{B}_n$ for arbitrary subset $\Lambda \subset \mathbb{Z}$. The translation γ is the right shift on $\mathfrak{B}$. We write $\phi_{[1,n]}$ for the localization $\phi|\mathfrak{B}_{[1,n]}$. The following definition is from [5].

Definition 2.1 A state ϕ on $\mathfrak{B}$ is called a *C^*-finitely correlated state* if there exist a finite dimensional C^*-algebra $\mathfrak{C}$, a completely positive map $E : M_d \otimes \mathfrak{C} \to \mathfrak{C}$ and a state ρ on $\mathfrak{C}$ such that

$$\rho(E(I_d \otimes C)) = \rho(C)$$

for all $C \in \mathfrak{C}$ and

$$\phi(A_1 \otimes \cdots \otimes A_n) = \rho(E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_n \otimes I_{\mathfrak{C}}) \cdots)))$$

for all $A_1, \ldots, A_n \in M_d$.

Let ϕ be a C^*-finitely correlated state generated by the triplet $(\mathfrak{C}, E, \rho)$. For any $n \in \mathbb{N}$, we define the completely positive map $E^{(n)}$ from $\mathfrak{B}_{[1,n]} \otimes \mathfrak{C}$ to $\mathfrak{C}$ by

$$E^{(n)}(A_1 \otimes \cdots \otimes A_n \otimes C) = E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_n \otimes C) \cdots))$$

for all $A_1, \ldots, A_n \in M_d$ and $C \in \mathfrak{C}$. We will also need the linear space $\mathfrak{C}_0 \subset \mathfrak{C}$ which is the smallest subspace of $\mathfrak{C}$ containing I and invariant under $E(A \otimes \cdot)$ for all $A \in M_d$. Since $\mathfrak{C}$ is finite dimensional, there exists an integer N such that

$$\mathfrak{C}_0 = \{E^{(N)}(A_{[1,N]} \otimes I) \mid A \in \mathfrak{B}_{[1,N]}\}.$$

Moreover, we assume that the triplet $(\mathfrak{C}, E, \rho)$ is minimal, that is, $\mathfrak{C}_0$ generates $\mathfrak{C}$ in the sense of algebra.

Let $(\mathcal{H}, \pi, \xi)$ be the GNS representation of ϕ. Then, we can extend ϕ to $\pi(\mathfrak{B})''$. In the following, we omit π, if there is no confusion. We want to show the condition that $\pi(\mathfrak{B})''$ is a factor. To this end, we introduce two subspaces of $\mathfrak{C}_0$. We define the subspaces $L(E)$ and $L_1(E)$ by

$$L(E) = \{C \in \mathfrak{C} \mid E_I(C) = \lambda C, \ \lambda \in \mathbb{T}\}$$

and

$$L_1(E) = \{C \in \mathfrak{C} \mid E_I(C) = C\},$$

where $E_I = E(I \otimes \cdot)$. $L_1(E)$ is the eignespace of E_1 with eigenvalue 1 and $L(E)$ is the space generated by eigenspaces with eigenvalues of modulus 1. From [6], $L(E)$ and $L_1(E)$ are algebras containd in the center of $\mathfrak{C}$. Moreover, there exists an integer M such that $\lambda^M = 1$ for any eigenvalue λ of E_I with modulus 1.

The following argument is in [6]. For any minimal projection P of $L_1(E)$, we consider the algebra $\mathfrak{C}_P = P\mathfrak{C}P$. Obviously, $\mathfrak{C} = \bigoplus \mathfrak{C}_P$, where the sum is taken over all minimal projections in $L_1(E)$. Since E is a completely positive map, we have $E(M_d \otimes \mathfrak{C}_P) \subset \mathfrak{C}_P$. Therefore, we can define the restriction $E_P : M_d \otimes \mathfrak{C}_P \to \mathfrak{C}_P$. We can assume $\rho(P) \neq 0$. Then, with $\rho_P = \rho(P)^{-1}\rho|\mathfrak{C}_P$, we have a triplet $(\mathfrak{C}_P, E_P, \rho_P)$ generating a C^*-finitely correlated state ϕ_P. A direct expression of ϕ_P is

$$\phi_P(A_1 \otimes \cdots \otimes A_n) = \rho(P)^{-1}\rho(E(A_1 \otimes \cdots \otimes E(A_n \otimes P)\cdots)) \qquad (1)$$

for all $A_1, \ldots, A_n \in M_d$. Then, we have the decomposition

$$\phi = \sum \rho(P)\phi_P,$$

where the sum is taken over all minimal projections in $L_1(E)$.

Let Π denote the set of minimal projections in $L(E)$. Then, $E_I|\Pi$ defines a bijective map from Π to Π. For any projection Q in Π, we have $E_I^M(Q) = Q$. Hence, Q is in $L_1(E_I^{(M)})$, where $E_I^{(M)} = E^{(M)}(I^{\otimes M} \otimes \cdot)$, and we have a C^*-finitely correlated state ϕ_Q on a regrouped chain generated by the triplet $(\mathfrak{C}_Q, E_Q^{(M)}, \rho_Q)$, where $\mathfrak{C}_Q$ and ρ_Q are defined as above and $E_Q^{(M)}$ is the completely positive map from $\bigotimes^M M_d \otimes \mathfrak{C}_Q$ to $\mathfrak{C}_Q$ defined by

$$E_Q^{(M)}(A_1 \otimes A_2 \otimes \cdots \otimes A_M \otimes C_Q) = E(A_1 \otimes E(A_2 \otimes \cdots \otimes E(A_M \otimes C_Q)\cdots))$$

for any $A_1, \ldots, A_M \in M_d$ and $C_Q \in \mathfrak{C}_Q$. A direct expression of ϕ_Q is

$$\phi_Q(A_1 \otimes \cdots \otimes A_n) = \rho(Q)^{-1}\rho(E^{(M)}(A_1 \otimes \cdots \otimes E^{(M)}(A_n \otimes Q)\cdots)) \qquad (2)$$

for all $A_1, \dots, A_n \in \bigotimes_{i=1}^M M_d$. Then, we have the decomposition

$$\phi = \sum_{Q \in \Pi} \rho(Q) \phi_Q.$$

Moreover, ϕ_Q is strongly clustering for γ^M, that is,

$$\lim_{n \to \infty} \phi_Q(A \gamma^{nM}(B)) = \phi(A)\phi(B)$$

for all $A, B \in \mathfrak{B}$. Indeed, we consider the Jordan decomposition of $(E_Q^{(M)})_I = E_Q^{(M)}(I^{\otimes M} \otimes \cdot)$, i.e.,

$$(E_Q^{(M)})_I = \sum_\lambda (\lambda P_\lambda + R_\lambda),$$

where the sum is taken over all eigenvalues, $P_\lambda P_{\lambda'} = \delta_{\lambda\lambda'} P_\lambda$ and R_λ is nilpotent with $P_\lambda R_{\lambda'} = R_{\lambda'} P_\lambda = \delta_{\lambda\lambda'} R_\lambda$. Since $\|(E_Q^{(M)})_I\| \leq 1$ and $(E_Q^{(M)})_I$ has trivial peripheral spectrum ([6]), i.e., the only eigenvector of $(E_Q^{(M)})_I$ with eigenvalue of modulus 1 is Q, $R_1 = 0$ and $P_\lambda = R_\lambda = 0$ for λ with $|\lambda| \geq 1$ and $\lambda \neq 1$. Hence, for any $\varepsilon > 0$, there exists a number $m \in \mathbb{N}$ such that $\|P_1 - (E_Q^{(M)})_I^m\| < \varepsilon$. Furthermore, for any $A \in \mathfrak{B}_{[1,nM]}$, we obtain

$$\phi_Q(A) = \rho_Q(E^{(nM)}(A \otimes Q)) = \lim_{l \to \infty} \rho_Q((E_Q^{(M)})_I^l (E^{(nM)}(A \otimes Q))).$$

Therefore, we have

$$\lim_{l \to \infty} (E_Q^{(M)})_I^l (E^{(nM)}(A \otimes Q)) = \phi_Q(A) Q.$$

This implies that ϕ_Q is strongly clustering for γ^M. In particular, if $\Pi = \{I\}$, we obtain

$$\lim_{l \to \infty} (E_I^l (E^{(n)}(A \otimes I)) = \phi(A) I \tag{3}$$

for all $A \in \mathfrak{B}_{[1,n]}$.

For each $Q \in \Pi$, we set the projection $\bar{Q} \in L(E)$ by

$$\bar{Q} = \sum \{R \in \Pi \mid \phi_Q = \phi_R\}$$

and the set $\bar{\Pi}$ by

$$\bar{\Pi} = \{\bar{Q} \mid Q \in \Pi\}.$$

Lemma 2.2 *With the above notation, we have*

$$L(E) \cap \mathfrak{C}_0 = \mathrm{span}\bar{\Pi}.$$

Proof. For any $T \in L(E) \cap \mathfrak{C}_0$, there exists an element $B \in \mathfrak{B}_{[1,nM]}$ such that $E^{(nM)}(B \otimes I) = T$. From the above argument, we have

$$\begin{aligned} T &= E^{(nM)}(B \otimes I) = \lim_{l\to\infty} E_I^{lM}(E^{(nM)}(B \otimes I)) \\ &= \lim_{l\to\infty} \sum_{Q\in\Pi} (E_Q^{(M)})_I^l (E_Q^{(nM)}(B \otimes Q)) \\ &= \sum_{Q\in\Pi} \phi_Q(B) Q = \sum_{\bar{Q}\in\bar{\Pi}} \phi_Q(B)\bar{Q}. \end{aligned} \tag{4}$$

This implies $L(E) \cap \mathfrak{C}_0 \subset \mathrm{span}\bar{\Pi}$.

To prove the converse, we show that $\bar{Q} \in \mathfrak{C}_0$ for any $Q \in \Pi$. For each $P, Q \in \Pi$, $\bar{P} \neq \bar{Q}$ implies $\phi_P \neq \phi_Q$. Since ϕ_P and ϕ_Q are γ^M-ergodic, $\phi_P \neq \phi_Q$ implies that ϕ_P and ϕ_Q are mutually disjoint ([4, 4.3.19]). Hence, for any $\varepsilon > 0$, there exists an element $A \in \mathfrak{B}_{[-nM+1,nM]}$ such that $|\phi_P(A) - 1| < \varepsilon$ and $|\phi_Q(A)| < \varepsilon$ for any $Q \in \Pi$ with $\bar{P} \neq \bar{Q}$. Since ϕ_Q is γ^M-invariant, we can assume that $A \in \mathfrak{B}_{[1,nM]}$ for some $n \in \mathbb{N}$. Moreover, from (4), there exists a number $l \in \mathbb{N}$ such that

$$\|E_I^{lM}(E^{(nM)}(A \otimes I)) - \sum_{\bar{Q}\in\bar{\Pi}} \phi_Q(A)\bar{Q}\| < \varepsilon.$$

Therefore we have

$$\begin{aligned} & \|\bar{P} - E_I^{lM}(E^{(nM)}(A \otimes I))\| \\ \leq\ & \|\bar{P} - \sum_{\bar{Q}\in\bar{\Pi}} \phi_Q(A)\bar{Q}\| + \|\sum_{\bar{Q}\in\bar{\Pi}} \phi_Q(A)\bar{Q} - E_I^{lM}(E^{(nM)}(A \otimes I))\| \\ <\ & 2\varepsilon \end{aligned}$$

Since $\mathfrak{C}_0$ is closed and $E_I^{lM}(E^{(nM)}(A \otimes I))$ is in $\mathfrak{C}_0$, we have $\bar{P} \in \mathfrak{C}_0$. □

Now, we have the next theorem.

Theorem 2.3 *For any C^*-finitely correlated state ϕ generated by the triplet $(\mathfrak{C}, E, \rho)$, the following conditions are equivalent.*
(i) *$\pi(\mathfrak{B})''$ is a factor.*
(ii) *ϕ is strongly clustering for γ.*
(iii) *$L(E) \cap \mathfrak{C}_0 = \mathbb{C}I$.*
(iv) *$\bar{\Pi} = \{I\}$, that is, ϕ_Q's in (2) with projections $Q \in \Pi$ are same.*

Proof. (iii) $\Leftrightarrow$ (iv) follows from Lemma 2.2.

(iii) $\Rightarrow$ (ii). Since $L(E) \cap \mathfrak{C}_0 = \mathbb{C}I$ implies $\phi = \phi_Q$ for any $Q \in \Pi$, ϕ is strongly clustering for γ^M. Moreover, ϕ is γ-invariant. Therefore, we have

$$\lim_{n\to\infty} \phi(A\gamma^{nM+l}(B)) = \lim_{n\to\infty} \phi(A\gamma^{nM}(\gamma^l(B))) = \phi(A)\phi(\gamma^l(B)) = \phi(A)\phi(B)$$

for any $A, B \in \mathfrak{B}$ and $0 \le l \le k-1$. Hence, ϕ is strongly clustering for γ.

(i) $\Rightarrow$ (iv). For any $P, Q \in \Pi$, $\bar{P} \neq \bar{Q}$ implies ϕ_P and ϕ_Q are disjoint. This contradicts $Z(\pi(\mathfrak{B})'') = \mathbb{C}I$. Hence, we obtain $\bar{\Pi} = \{I\}$.

(ii) $\Rightarrow$ (iii). We assume that ϕ is strongly clustering for γ. Then, ϕ is strongly clustering for γ^M and hence γ^M-ergodic. Since ϕ_Q is γ^M-ergodic for any $Q \in \Pi$, we have $\bar{\Pi} = \{I\}$.

(ii) $\Rightarrow$ (i). Since $Z(\pi(\mathfrak{B})'') = \bigcap_{n\in\mathbb{N}} \pi(\mathfrak{B}_{(-\infty,-n]\cup[n,\infty)})''$ (see e.g. [4, 2.6.10]), for any $X \in Z(\pi(\mathfrak{B})'')$ with $\|X\| = 1$, there exists a sequence $\{X_n\}$ with $X_n \in \mathfrak{B}_{[-l(n),n]\cup[n,l(n)]}$, $\|X_n\| \le 1$ and $\lim_{n\to\infty} X_n = X$ in the weak operator topology. We can write

$$X_n = \sum Y_i^{(n)} \gamma^{n-1}(Z_i^{(n)})$$

for some $Y_i^{(n)} \in \mathfrak{B}_{[-l(n),-n]}$ and $Z_i^{(n)} \in \mathfrak{B}_{[1,l(n)-n+1]}$. For any element $A \in \mathfrak{B}_{[1,p]}$, $p \in \mathbb{N}$, there exists an element $A' \in \mathfrak{B}_{[1,N]}$ such that

$$E^{(p)}(A \otimes I) = E^{(N)}(A' \otimes I).$$

We write $A' = \theta(A)$. For any element $B_m, B'_m \in \mathfrak{B}_{[1,m]}$ with $m < n$, we have

$$\begin{aligned} & \langle B_m\xi, (I^{\otimes n} \otimes A)B'_m\xi\rangle = \phi(B_m^*(I^{\otimes n} \otimes A)B'_m) \\ = \; & \rho(E^{(n)}(B_m^* B'_m \otimes I^{\otimes n-m} \otimes E^{(p)}(A \otimes I))) \\ = \; & \rho(E^{(n)}(B_m^* B'_m \otimes I^{\otimes n-m} \otimes E^{(N)}(\theta(A) \otimes I))) \\ = \; & \langle B_m\xi, (I^{\otimes n} \otimes \theta(A))B'_m\xi\rangle. \end{aligned}$$

Therefore, $X'_n = \sum Y_i^{(n)} \gamma^{n-1}(\theta(Z_i^{(n)}))$ converges to X in the weak operator topology. Moreover, since $\theta(Z_i^{(n)}) \in \mathfrak{B}_{[1,N]}$, we can write

$$X'_n = \sum_{i=1}^{d^{2N}} S_i^{(n)} \gamma^n(T_i)$$

for some $S_i^{(n)} \in \mathfrak{B}_{[-l(n),-n]}$ and a system of matrix units $\{T_i\}$ of $\mathfrak{B}_{[1,N]}$. Since X'_n converges to X in the weak operator topology, there exists some constant $C > 0$ such that $\|X'_n\| \le C$ for any $n \in \mathbb{N}$. Then, we have $\|S_i^{(n)}\| \le C$.

From the proof of (3), for $\varepsilon > 0$ there exists $L \in \mathbb{N}$ such that

$$\|E_I^L(E^{(p)}(A \otimes I)) - \phi(A)I\| < \varepsilon\|A\|$$

for any $A \in \mathfrak{B}_{[1,p]}$ and $p \in \mathbb{N}$. Using this uniform convergence, for any $B_m, B'_m \in \mathfrak{B}_{[1,m]}$ we have

$$\begin{aligned}
&\langle B_m\xi, XB'_m\xi\rangle = \lim_{n\to\infty}\langle B_m\xi, X'_nB'_m\xi\rangle \\
=\ & \lim_{n\to\infty}\sum_{i=1}^{d^{2N}}\langle B_m\xi, S_i^{(n)}\gamma^n(T_i)B'_m\xi\rangle = \lim_{n\to\infty}\sum_{i=1}^{d^{2N}}\phi(B_m^*S_i^{(n)}\gamma^n(T_i)B'_m) \\
=\ & \lim_{n\to\infty}\sum_{i=1}^{d^{2N}}\rho(E^{l(n)-n+1}(S_i^{(n)} \otimes E_I^n(E^{(m)}(B_m^*B'_m \otimes E_I^{n-m}(E^{(N)}(T_i \otimes I)))))) \\
=\ & \lim_{n\to\infty}\sum_{i=1}^{d^{2N}}\phi(S_i^{(n)})\phi(B_m^*B'_m)\phi(T_i) = \phi(B_m^*B'_m)\lim_{n\to\infty}\sum_{i=1}^{d^{2N}}\phi(S_i^{(n)}\gamma^n(T_i)) \\
=\ & \phi(B_m^*B'_m)\lim_{n\to\infty}\phi(X'_n) = \langle B_m\xi, \phi(X)B'_m\xi\rangle.
\end{aligned}$$

Therefore, we obtain $X = \phi(X)I$. □

By the theorem, for any $P, Q \in \Pi$ such that $\phi_P \neq \phi_Q$, ϕ_P and ϕ_Q are disjoint and factor states. Therefore, for any $P \in \Pi$, there exists a minimal projection T in $Z(\pi(\mathfrak{B})'')$, such that

$$\phi_P(B) = \langle\xi, T\xi\rangle^{-1}\langle\xi, BT\xi\rangle$$

for any $B \in \pi(\mathfrak{B})''$. In fact, T is the support projection of ϕ_P. We define a bijective map η from $\bar{\Pi}$ to a set of minimal projections in $Z(\pi(\mathfrak{B})'')$ by

$$\eta(\bar{P}) = T.$$

Now, we have the next corollary.

Corollary 2.4 *We obtain*

$$Z(\pi(\mathfrak{B})'') = \mathrm{span}\{\eta(\bar{P}) \mid \bar{P} \in \bar{\Pi}\}.$$

In particular, the dimension of the center $Z(\pi(\mathfrak{B})'')$ is finite and not greater than the dimension of the center of $\mathfrak{C}$.

3 Types of factors generated by C^*-finitely correlated states

In this section, we examine the types of factors generated by strongly clustering C^*-finitely correlated states. In the following, we assume that ϕ is a C^*-finitely correlated state generated by a triplet $(\mathfrak{C}, E, \rho)$ and it is strongly clustering.

Since ϕ is γ-invariant, we can extend γ to $\pi(\mathfrak{B})''$. Let P be the support projection of ϕ. Then, $\gamma(P) = P$. Indeed, $\phi(\gamma(P)) = \phi(P)$ implies $\gamma(P) \geq P$. Similary, we have $\gamma^{-1}(P) \geq P$. This means $\gamma(P) = P$. Therefore, we can define the automorphism $\gamma|P\mathfrak{B}P$. Here, the normal extension of ϕ to $\pi(\mathfrak{B})''$ is denoted by the same ϕ and $\pi(\mathfrak{B})$ is identified with $\mathfrak{B}$.

Let $S(\pi(\mathfrak{B})'')$ be the Connes invariant. The next proposition is in [7]. The proof is given for convenience.

Proposition 3.1 *Let $\phi^P = \phi|P\mathfrak{B}P$. Then, we have*

$$S(\pi(\mathfrak{B})'')\backslash\{0\} = \mathrm{Sp}(\Delta_{\phi^P})\backslash\{0\},$$

where Δ_{ϕ^P} is a modular operator of ϕ^P.

Proof. Since $\pi(\mathfrak{B})''$ is a factor, we know that $S(\pi(\mathfrak{B})'') = S(P\pi(\mathfrak{B})''P)$. $P\mathfrak{B}P$ is asymptotically abelian with respect to γ and ϕ^P is strongly clustering for γ. Therefore, if a state ω on $P\mathfrak{B}P$ is quasi-containd in ϕ^P, then we have $\mathrm{Sp}(\Delta_{\phi^P}) \subset \mathrm{Sp}(\Delta_\omega)$ ([13]). In particular, for a projection $Q \in \pi(\mathfrak{B})''$ with $0 \neq Q \leq P$, we have $\mathrm{Sp}(\Delta_{\phi^P}) \subset \mathrm{Sp}(\Delta_{\phi^Q})$, where $\phi^Q = \phi^P(Q)^{-1}\phi^P(Q\,\cdot\,)$. Moreover, ϕ^P is faithful on $P\pi(\mathfrak{B})''P$ and $P\mathfrak{B}P$ is weakly dense in $P\pi(\mathfrak{B})''P$. Hence, we have

$$S(P\pi(\mathfrak{B})''P)\backslash\{0\} = \mathrm{Sp}(\Delta_{\phi^P})\backslash\{0\}.$$

□

In the following, we examine the type of $\pi(\mathfrak{B})''$. In the case where $\mathrm{Sp}(\Delta_{\phi^P}) \neq \{1\}$, since ϕ^P is faithful, $\mathrm{Sp}(\Delta_{\phi^P})$ contains a number which is neither 0 nor 1. Therefore, $S(\pi(\mathfrak{B})'') \neq \{0,1\}$. Hence, $\pi(\mathfrak{B})''$ is a III_λ factor for some $\lambda \in (0,1]$.

If $\mathrm{Sp}(\Delta_{\phi^P}) = \{1\}$, then ϕ^P is a tracial state on $P\pi(\mathfrak{B})''P$. Hence, P is a finite projection. Therefore, $\pi(\mathfrak{B})''$ is not a III factor. If ϕ is faithful, then $\pi(\mathfrak{B})''$ is a II_1 factor. If ϕ is pure, then $\pi(\mathfrak{B})''$ is a I_∞ factor. From [6], ϕ is pure if and only if ϕ is strongly clustering and the mean entropy of ϕ is zero.

Proposition 3.2 *If* $\mathrm{Sp}(\Delta_{\phi^P}) = \{1\}$ *and* ϕ *is neither faithful nor pure, then* $\pi(\mathfrak{B})''$ *is a* II_∞ *factor.*

Proof. From the assumption, ϕ is not pure. Hence, $\pi(\mathfrak{B})''$ is a II_1 factor or a II_∞ factor. Now, we assume that $\pi(\mathfrak{B})''$ is a II_1 factor. Then, there is a faithful tracial state τ on $\pi(\mathfrak{B})''$. Since ϕ is not faithful, there exist a support projection P of ϕ with $0 < \tau(P) < 1$. Then, we can get the decomposition

$$\tau = \tau(P)\tau(P\,\cdot\,) + \tau(I-P)\tau((I-P)\,\cdot\,).$$

But, since P is invariant under γ, this contradicts to the ergodicity of τ. Therefore, $\pi(\mathfrak{B})''$ is a II_∞ factor. □

In the rest of this section, we present examples of III_λ factors for $\lambda \in (0,1]$ which are generated by translation-invariant quantum Markov states.

Definition 3.3 [2] A state ϕ on $\mathfrak{B}$ is said to be a *quantum Markov state*, if there exists a conditional expectation E_n from $\mathfrak{B}_{[1,n+1]}$ to $\mathfrak{B}_{[1,n]}$ such that $\mathfrak{B}_{[1,n-1]} \subset \mathrm{ran}(E_n)$ and

$$\phi \circ E_n = \phi_{[1,n+1]}$$

for each $n \in \mathbb{N}$.

Although the above definition is a bit different from the original one of Accardi and Frigerio in [2], it is known that both definitions are equivalent ([8]).

In the case where the quantum Markov state ϕ is translation-invariant, we can assume that $E_n = \mathrm{id}_{\mathfrak{B}_{[1,n-1]}} \otimes E$ for some conditional expectation E from $M_d \otimes M_d$ into M_d ([10]). Therefore, translation-invariant quantum Markov states are C^*-finitely correlated states.

In the following, we assume that ϕ is a locally faithful translation-invariant quantum Markov state generated by (E, ρ) with $\rho = \phi|\mathfrak{B}_1$ and that ϕ is not a tracial state. Let $\mathfrak{D} = \mathrm{ran}(E)$. Since $\mathfrak{D}$ is a finite dimensional C^*-algebra, we can write

$$\mathfrak{D} = \bigoplus_{i=1}^{p} M_{d_i}.$$

Let m_i be the multiplicity of M_{d_i} as a C^*-subalgebra of M_d, and we define

$$\bar{\mathfrak{D}} = \bigoplus_{i=1}^{p} M_{m_i},$$

$\mathfrak{C}_n = \bar{\mathfrak{D}} \otimes \mathfrak{B}_{[1,n-1]} \otimes \mathfrak{D}$ and $\mathfrak{C}_n^{xy} = M_{m_x} \otimes \mathfrak{B}_{[1,n-1]} \otimes M_{d_y}$ for $1 \le x, y \le p$. From [3], there exist positive operators $T_{ij} \in M_{m_i} \otimes M_{d_j}$ for any $1 \le i, j \le p$ such that the density matrix of $\phi|\mathfrak{C}_n$ is written by

$$D_n = \bigoplus_{i_1,\dots,i_n} \rho(I_{m_{i_1}}) T_{i_1 i_2} \otimes T_{i_2 i_3} \otimes \cdots \otimes T_{i_{n-1} i_n}. \tag{5}$$

Since T_{ij} is positive, we can choose a system of matrix units $\{e_{kl}^{(ij)}\}$ for $M_{m_i} \otimes M_{d_j}$ and write

$$T_{ij} = \mathrm{diag}(e^{t_1^{(ij)}}, e^{t_2^{(ij)}}, \dots, e^{t_{m_i d_j}^{(ij)}}).$$

To calculate $S(\pi(\mathfrak{B})'')$, we consider $\mathrm{sp}(\Delta_\phi)$. Since ϕ is faithful, we obtain

$$\mathrm{sp}(\Delta_\phi) \backslash \{0\} = \exp(\mathrm{sp}(\sigma^\phi)),$$

where σ^ϕ is the modular automorphism group of ϕ and $\mathrm{sp}(\sigma^\phi)$ is the Arveson spectrum of σ^ϕ. Since $\mathfrak{B}$ is weakly dense in $\pi(\mathfrak{B})''$, we have

$$\begin{aligned} \mathrm{sp}(\sigma^\phi) &= \overline{\bigcup_{B \in \mathfrak{B}} \mathrm{sp}_{\sigma^\phi}(B)} = \overline{\bigcup_{n=1}^{\infty} \bigcup_{B \in \mathfrak{C}_n} \mathrm{sp}_{\sigma^\phi}(B)} \\ &= \overline{\bigcup_{n=1}^{\infty} \bigcup_{x,y=1}^{p} \bigcup_{B \in \mathfrak{C}_n^{xy}} \mathrm{sp}_{\sigma^\phi}(B)}. \end{aligned}$$

From [2], we know that

$$\sigma_t^\phi | \mathfrak{C}_n = \mathrm{Ad} D_n^{it}.$$

Therefore, $\mathfrak{C}_n^{xy}$ is invariant under σ^ϕ and we have

$$\bigcup_{B \in \mathfrak{C}_n^{xy}} \mathrm{sp}_{\sigma^\phi}(B) = \mathrm{sp}(\sigma^\phi | \mathfrak{C}_n^{xy}).$$

Lemma 3.4 *Let ψ be a state on M_k with the density matrix $D = \mathrm{diag}(e^{t_1}, \dots, e^{t_k})$. Then the Arveson spectrum of σ^ψ is written as*

$$\mathrm{sp}(\sigma^\psi) = \{t_i - t_j \mid 1 \le i, j \le k\}.$$

Proof. This is obvious from the fact that

$$\sigma_t^{\psi} = \mathrm{Ad}(D^{it}).$$

□

Since the density matrix of $\phi|\mathfrak{E}_n$ is written as in (5), the density matrix of $\phi|\mathfrak{E}_n^{xy}$ is written as

$$\bigoplus_{i_2,\cdots,i_{n-1}} \rho(I_{m_x}) T_{xi_2} \otimes T_{i_2 i_3} \otimes \cdots \otimes T_{i_{n-2} i_{n-1}} \otimes T_{i_{n-1} y}.$$

Therefore, we have

$$\begin{aligned} & \mathrm{sp}(\sigma^{\phi}|\mathfrak{E}_n^{xy}) \\ = \; & \{ t_{q_1}^{(xi_2)} + \sum_{k=2}^{n-2} t_{q_k}^{(i_k i_{k+1})} + t_{q_{n-1}}^{(i_{n-1}y)} - t_{r_1}^{(xj_2)} - \sum_{l=2}^{n-2} t_{r_l}^{(j_l j_{l+1})} - t_{r_{n-1}}^{(j_{n-1}y)} \\ & \mid \text{all possible } i_k, j_l, q_k, r_l \}. \end{aligned} \tag{6}$$

Since $\exp(\mathrm{sp}(\sigma^{\phi})) = S(\pi(\mathfrak{B})'')\backslash\{0\}$, $\mathrm{sp}(\sigma^{\phi})$ is a group. Hence, we obtain $\mathrm{sp}(\sigma^{\phi}) = \mathbb{R}$ or else there exists a number $\lambda \in (0,1)$ such that

$$\mathrm{sp}(\sigma^{\phi}) = (\log \lambda)\mathbb{Z}.$$

Let G be a closed subgroup of $\mathbb{R}$ generated by

$$\{ t_{j_1}^{(i_1 i_2)} + t_{j_2}^{(i_2 i_4)} - t_{j_3}^{(i_1 i_3)} - t_{j_4}^{(i_3 i_4)} \mid \text{all possible } i_k, j_l \}.$$

Proposition 3.5 *We obtain*

$$G = \mathrm{sp}(\sigma^{\phi})$$

Proof. By (6), for any i_k, j_l, we obtain

$$t_{j_1}^{(i_1 i_2)} + t_{j_2}^{(i_2 i_4)} - t_{j_3}^{(i_1 i_3)} - t_{j_4}^{(i_3 i_4)} \in \mathrm{sp}(\sigma^{\phi}|\mathfrak{E}_2^{i_1 i_4}).$$

Therefore, $G \subset \mathrm{sp}(\sigma^{\phi})$.

We show the converse. From definition, we obtain $t_{j_1}^{(i_1 i_1)} - t_{j_4}^{(i_4 i_4)} \in G$. Then, for any

$$t_{j_1}^{(xi_1)} + t_{j_2}^{(i_1 i_2)} + t_{j_3}^{(i_2 y)} - t_{j_4}^{(xi_3)} - t_{j_5}^{(i_3 i_4)} - t_{j_6}^{(i_4 y)} \in \mathrm{sp}(\sigma^{\phi}|\mathfrak{E}_3^{xy}),$$

by adding

$$\begin{aligned} & t_{k_1}^{(i_1i_1)} + t_{j_5}^{(i_3i_4)} - t_{k_2}^{(i_3i_1)} - t_{k_3}^{(i_1i_4)} \\ = & (t_{k_1}^{(i_1i_1)} - t_{k_4}^{(i_4i_4)}) + (t_{j_5}^{(i_3i_4)} + t_{k_4}^{(i_4i_4)} - t_{k_2}^{(i_3i_1)} - t_{k_3}^{(i_1i_4)}) \in G, \end{aligned}$$

we have

$$\begin{aligned} & (t_{j_1}^{(xi_1)} + t_{j_2}^{(i_1i_2)} + t_{j_3}^{(i_2y)} - t_{j_4}^{(xi_3)} - t_{j_5}^{(i_3i_4)} - t_{j_6}^{(i_4y)}) \\ + & (t_{k_1}^{(i_1i_1)} + t_{j_5}^{(i_3i_4)} - t_{k_2}^{(i_3i_1)} - t_{k_3}^{(i_1i_4)}) \\ = & (t_{j_1}^{(xi_1)} + t_{k_1}^{(i_1i_1)} - t_{j_4}^{(xi_3)} - t_{k_2}^{(i_3i_1)}) + (t_{j_2}^{(i_1i_2)} + t_{j_3}^{(i_2y)} - t_{k_3}^{(i_1i_4)} - t_{j_6}^{(i_4y)}) \in G. \end{aligned}$$

Hence, we get $\mathrm{sp}(\sigma^\phi|\mathfrak{C}_3^{xy}) \subset G$. The idea of the above calculation is to split (xi_1i_2y, xi_3i_4y) to (xi_1i_1, xi_3i_1) and (i_1i_2y, i_1i_4y). The same can be applied to longer words. For example, split $(xi_1i_2i_3y, xi_4i_5i_6y)$ to (xi_1i_1, xi_4i_1), $(i_1i_2i_1, i_1i_5i_1)$ and (i_1i_3y, i_1i_6y). In this way, we obtain $\mathrm{sp}(\sigma^\phi|\mathfrak{C}_n^{xy})$ for all $1 \leq x, y \leq p$ and $n \in \mathbb{N}$, so that $\mathrm{sp}(\sigma^\phi) \subset G$. □

Now, we define a number $\lambda \in \mathbb{R}$ to be 1 if $G = \mathbb{R}$ or to be t if $G = (\log t)\mathbb{Z}$. Then, we have the next proposition.

Proposition 3.6 *With the above definition, if ϕ is not a tracial state, $\pi(\mathfrak{B})''$ is a type III_λ factor.*

It was shown in [7] that $\pi(\mathfrak{B})''$ is a type III_λ factor for some $\lambda \in (0, 1]$ as far as ϕ is not tracial. But, the above proposition enables us to determine the λ from the density matrices T_{ij}'s.

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