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Balanced C_4-Quatrefoil Designs

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Abstract

In graph theory, the decomposition problems of graphs are very important topics. Various types of decompositions of many graphs can be seen in the literature of gaph theory.
We show that the necessary and sufficient condition for the existence of a balanced C_4-quatrefoil decomposition of the complete multi-graph λK_n is $\lambda(n-1) \equiv 0 \pmod{32}$ and $n \geq 13$.
This decomposition is to be known as a balanced C_4-quatrefoil design.

Key words: Balanced C_4-quatrefoil decomposition, Complete multi-graph, Graph theory

MSC2000 classification: 05B30, 05C70

1. Introduction

Let K_n denote *the complete graph* of n vertices. *The complete multi-graph* λK_n is the complete graph K_n in which every edge is taken λ times. Let C_4 be *the* 4*-cycle* (or *the cycle* on 4 vertices). *The C_4-quatrefoil* is a graph of 4 edge-disjoint C_4's with a common vertex and the common vertex is called *the center of the C_4-quatrefoil.*
When λK_n is decomposed into edge-disjoint sum of C_4-quatrefoils, we say that λK_n *has a C_4-quatrefoil decomposition.* Moreover, when every vertex of λK_n appears in the same number of C_4-quatrefoils, we say that λK_n *has a balanced C_4-quatrefoil decomposition* and this number is called *the replication number.* This balanced C_4-quatrefoil decomposition of λK_n is to be known as *a balanced C_4-quatrefoil design.*

In this paper, it is shown that the necessary and sufficient condition for the existence of such a balanced C_4-quatrefoil decomposition of λK_n is $\lambda(n-1) \equiv 0 \pmod{32}$ and $n \geq 13$.

It is a well-known result that K_n has a C_3 decomposition if and only if $n \equiv 1$ or $3 \pmod 6$. This decomposition is known as *a Steiner triple system.* See Colbourn and Rosa[2] and Wallis[14]. Horák and Rosa[3] proved that K_n has a C_3-bowtie decomposition if and only if $n \equiv 1$ or $9 \pmod{12}$. This decomposition is known as *a C_3-bowtie system.*
For combinatorial designs, see [1,4,5,14]. Another type of foil-decompositions, see [6–13].

2. Balanced C_4-quatrefoil decomposition of λK_n

We use the following notation.

Notation. We consider the vertex set V of λK_n as $V = \{1, 2, ..., n\}$. The vertex operations $i + x$ in the following theorems are taken modulo n with residues $1, 2, ..., n$. We denote a C_4-quatrefoil with 4 C_4's $(1,2,3,4)$, $(1,5,6,7)$, $(1,8,9,10)$, $(1,11,12,13)$ passing through $1-2-3-4-1-5-6-7-$

$1-8-9-10-1-11-12-13-1$ by $\{(1,2,3,4),(1,5,6,7),(1,8,9,10),(1,11,12,13)\}$.

We have the following theorem.

Theorem 1. If λK_n has a balanced C_4-quatrefoil decomposition, then $\lambda(n-1) \equiv 0 \pmod{32}$ and $n \geq 13$.

Proof. Suppose that λK_n has a balanced C_4-quatrefoil decomposition. Since a C_4-trefoil is a subgraph of λK_n, $n \geq 13$. Let b be the number of C_4-quatrefoils and r be the replication number. Then $b = \lambda n(n-1)/32$ and $r = 13\lambda(n-1)/32$. Among r C_4-quatrefoils having a vertex v of λK_n, let r_1 and r_2 be the numbers of C_4-quatrefoils in which v is the center and v is not the center, respectively. Then $r_1 + r_2 = r$. Counting the number of vertices adjacent to v, $8r_1 + 2r_2 = \lambda(n-1)$. From these relations, $r_1 = \lambda(n-1)/32$ and $r_2 = 12\lambda(n-1)/32$. Thus, $\lambda(n-1) \equiv 0 \pmod{32}$.

Note. The condition $\lambda(n-1) \equiv 0 \pmod{32}$ and $n \geq 13$ in Theorem 1 can be classified as follows:
(i) $\lambda \geq 1$, $n \equiv 1 \pmod{32}$, $n \geq 33$
(ii) $\lambda \equiv 0 \pmod{2}$, $n \equiv 1 \pmod{16}$, $n \geq 17$
(iii) $\lambda \equiv 0 \pmod{4}$, $n \equiv 1 \pmod{8}$, $n \geq 17$
(iv) $\lambda \equiv 0 \pmod{8}$, $n \equiv 1 \pmod{4}$, $n \geq 13$
(v) $\lambda \equiv 0 \pmod{16}$, $n \equiv 1 \pmod{2}$, $n \geq 13$
(vi) $\lambda \equiv 0 \pmod{32}$, $n \geq 13$.

We have the following theorem.

Theorem 2. If λK_n has a balanced C_4-quatrefoil decomposition, then $(s\lambda)K_n$ has a balanced C_4-quatrefoil decomposition for every s.

Proof. Obvious. Repeat s times the balanced C_4-quatrefoil decomposition of λK_n.

Definition. *The C_4-t-foil* is a graph of t edge-disjoint C_4's with a common vertex and *the C_4-t-foiloid* is a multi-graph of t C_4's with a common vertex.
For example, $\{(1,2,3,4),(1,5,6,7),(1,8,9,10),(1,11,12,13),(1,14,15,16)\}$ is a C_4-5-foil.
$\{(1,2,3,4),(1,5,6,7),(1,8,9,10),(1,2,3,5),(1,6,8,10)\}$ is a C_4-5-foiloid.
So the appearance number of each vertex in the C_4-t-foil is 1. But the appearance number of each vertex in the C_4-t-foiloid is not always 1.

Notation. The t C_4's in a C_4-t-foil or a C_4-t-foiloid are denoted in order like sequence.
For example, put a C_4-3-foiloid $B = \{(1,2,3,4),\ (1,3,5,6),\ (1,4,6,7)\}$. Then $(1,2,3,4)$ is the first C_4, $(1,3,5,6)$ is the second C_4, and $(1,4,6,7)$ is the third C_4 of B.
Put a C_4-3-foiloid $D = \{(1,3,4,5),\ (1,4,5,6),\ (1,5,6,7)\}$. Then we denote $B \cup B = \{(1,2,3,4),$ $(1,3,5,6),(1,4,6,7),(1,2,3,4),(1,3,5,6),(1,4,6,7)\}$, and $B \cup D = \{(1,2,3,4),(1,3,5,6),(1,4,6,7),$ $(1,3,4,5),(1,4,5,6),(1,5,6,7)\}$, and $D \cup B = \{(1,3,4,5),(1,4,5,6),(1,5,6,7),(1,2,3,4),(1,3,5,6),$ $(1,4,6,7)\}$.

We have the following theorems.

Theorem 3. When $n \equiv 1 \pmod{32}$ and $n \geq 33$, λK_n has a balanced C_4-quatrefoil decomposition for every λ.

Proof. Put $n = 32t + 1$. Construct n C_4-$4t$-foils as follows:

$\{(i,i+1,i+12t+2,i+4t+1),(i,i+2,i+12t+4,i+4t+2),(i,i+3,i+12t+6,i+4t+3),...,(i,i+4t,i+20t,i+8t)\}$ $(i=1,2,...,n)$.
Decompose each C_4-$4t$-foil into t C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of K_n. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.

Example 3.1. Balanced C_4-quatrefoil decomposition of K_{33}.
$\{(i,i+1,i+14,i+5),(i,i+2,i+16,i+6),(i,i+3,i+18,i+7),(i,i+4,i+20,i+8)\}$ $(i=1,2,...,33)$.

Example 3.2. Balanced C_4-quatrefoil decomposition of K_{65}.
$\{(i,i+1,i+26,i+9),(i,i+2,i+28,i+10),(i,i+3,i+30,i+11),(i,i+4,i+32,i+12)\}$
$\{(i,i+5,i+34,i+13),(i,i+6,i+36,i+14),(i,i+7,i+38,i+15),(i,i+8,i+40,i+16)\}$
$(i=1,2,...,65)$.

Example 3.3. Balanced C_4-quatrefoil decomposition of K_{97}.
$\{(i,i+1,i+38,i+13),(i,i+2,i+40,i+14),(i,i+3,i+42,i+15),(i,i+4,i+44,i+16)\}$
$\{(i,i+5,i+46,i+17),(i,i+6,i+48,i+18),(i,i+7,i+50,i+19),(i,i+8,i+52,i+20)\}$
$\{(i,i+9,i+54,i+21),(i,i+10,i+56,i+22),(i,i+11,i+58,i+23),(i,i+12,i+60,i+24)\}$
$(i=1,2,...,97)$.

Theorem 4. When $\lambda\equiv 0 \pmod 2$, $n\equiv 1 \pmod{16}$ and $n\geq 17$, λK_n has a balanced C_4-quatrefoil decomposition.

Proof. We consider 2 cases.
Case 1. $n\equiv 1$ **(mod 32) and** $n\geq 33$. By Theorem 3 and Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2. $n\equiv 17$ **(mod 32) and** $n\geq 17$. Put $n=32t+17$. Construct n C_4-$(8t+4)$-foils as follows:
$\{(i,i+16t+9,i+24t+14,i+24t+13),(i,i+16t+11,i+24t+18,i+24t+15),(i,i+16t+13,i+24t+22,i+24t+17),...,(i,i+20t+9,i+32t+14,i+28t+13)\}\cup\{(i,i+20t+11,i+1,i+28t+15),(i,i+20t+13,i+5,i+28t+17),(i,i+20t+15,i+9,i+28t+19),...,(i,i+24t+11,i+8t+1,i+32t+15)\}$ $\cup$ $\{(i,i+16t+10,i+24t+16,i+2),(i,i+16t+12,i+24t+20,i+4),(i,i+16t+14,i+24t+24,i+6),...,(i,i+20t+10,i+32t+16,i+4t+2)\}\cup\{(i,i+20t+12,i+3,i+4t+4),(i,i+20t+14,i+7,i+4t+6),(i,i+20t+16,i+11,i+4t+8),...,(i,i+24t+12,i+8t+3,i+8t+4)\}$
$(i=1,2,...,n)$.
Decompose each C_4-$(8t+4)$-foil into $(2t+1)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $2K_n$. Applying Theorem 2, λK_n has a balanced C_4-trefoil decomposition.

Example 4.1. Balanced C_4-quatrefoil decomposition of $2K_{17}$.
$\{(i,i+9,i+14,i+13),(i,i+11,i+1,i+15),(i,i+10,i+16,i+2),(i,i+12,i+3,i+4)\}$ $(i=1,2,...,17)$.

Example 4.2. Balanced C_4-quatrefoil decomposition of $2K_{49}$.
$\{(i,i+25,i+38,i+37),(i,i+27,i+42,i+39),(i,i+29,i+46,i+41),(i,i+31,i+1,i+43)\}$
$\{(i,i+33,i+5,i+45),(i,i+35,i+9,i+47),(i,i+26,i+40,i+2),(i,i+28,i+44,i+4)\}$
$\{(i,i+30,i+48,i+6),(i,i+32,i+3,i+8),(i,i+34,i+7,i+10),(i,i+36,i+11,i+12)\}$
$(i=1,2,...,49)$.

Example 4.3. Balanced C_4-quatrefoil decomposition of $2K_{81}$.
$\{(i,i+41,i+62,i+61),(i,i+43,i+66,i+63),(i,i+45,i+70,i+65),(i,i+47,i+74,i+67)\}$
$\{(i,i+49,i+78,i+69),(i,i+51,i+1,i+71),(i,i+53,i+5,i+73),(i,i+55,i+9,i+75)\}$
$\{(i,i+57,i+13,i+77),(i,i+59,i+17,i+79),(i,i+42,i+64,i+2),(i,i+44,i+68,i+4)\}$
$\{(i,i+46,i+72,i+6),(i,i+48,i+76,i+8),(i,i+50,i+80,i+10),(i,i+52,i+3,i+12)\}$

$\{(i,i+54,i+7,i+14),(i,i+56,i+11,i+16),(i,i+58,i+15,i+18),(i,i+60,i+19,i+20)\}$
$(i=1,2,...,81)$.

We use the following permutation for a C_4-t-foil or a C_4-t-foiloid.

Notation. Put $B=\{C_4^{(1)},...,C_4^{(i)},...,C_4^{(j)},...,C_4^{(t)}\}$ be a C_4-t-foil or a C_4-t-foiloid and $\sigma=(i,j)$ be a permutation. Then $\sigma B=\{C_4^{(1)},...,C_4^{(j)},...,C_4^{(i)},...,C_4^{(t)}\}$.

Theorem 5. When $\lambda\equiv 0 \pmod 4$, $n\equiv 1 \pmod 8$ and $n\geq 17$, λK_n has a balanced C_4-quatrefoil decomposition.

Proof. We consider 2 cases.
Case 1. $n\equiv 1$ (mod 16) and $n\geq 17$. By Theorem 4 and Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2. $n\equiv 9$ (mod 16) and $n\geq 25$. Put $n=16t+9$. Construct n C_4-$(8t+4)$-foiloids as follows:
$B_i=\{(i,i+1,i+8t+6,i+8t+5),(i,i+2,i+8t+8,i+8t+6),(i,i+3,i+8t+10,i+8t+7),...,(i,i+4t+2,i+16t+8,i+12t+6)\}\cup\{(i,i+4t+3,i+1,i+12t+7),(i,i+4t+4,i+3,i+12t+8),(i,i+4t+5,i+5,i+12t+9),...,(i,i+8t+4,i+8t+3,i+16t+8)\}$ $(i=1,2,...,n)$.
Put $D_i=\sigma B_i$ and $\sigma=(2,8t+3)$. Decompose each D_i into $(2t+1)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $4K_n$. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.

Example 5.1. Balanced C_4-quatrefoil decomposition of $4K_{25}$.
$\{(i,i+1,i+14,i+13),(i,i+11,i+9,i+23),(i,i+3,i+18,i+15),(i,i+4,i+20,i+16)\}$
$\{(i,i+5,i+22,i+17),(i,i+6,i+24,i+18),(i,i+7,i+1,i+19),(i,i+8,i+3,i+20)\}$
$\{(i,i+9,i+5,i+21),(i,i+10,i+7,i+22),(i,i+2,i+16,i+14),(i,i+12,i+11,i+24)\}$
$(i=1,2,...,25)$.

Example 5.2. Balanced C_4-quatrefoil decomposition of $4K_{41}$.
$\{(i,i+1,i+22,i+21),(i,i+19,i+17,i+39),(i,i+3,i+26,i+23),(i,i+4,i+28,i+24)\}$
$\{(i,i+5,i+30,i+25),(i,i+6,i+32,i+26),(i,i+7,i+34,i+27),(i,i+8,i+36,i+28)\}$
$\{(i,i+9,i+38,i+29),(i,i+10,i+40,i+30),(i,i+11,i+1,i+31),(i,i+12,i+3,i+32)\}$
$\{(i,i+13,i+5,i+33),(i,i+14,i+7,i+34),(i,i+15,i+9,i+35),(i,i+16,i+11,i+36)\}$
$\{(i,i+17,i+13,i+37),(i,i+18,i+15,i+38),(i,i+2,i+24,i+22),(i,i+20,i+19,i+40)\}$
$(i=1,2,...,41)$.

Example 5.3. Balanced C_4-quatrefoil decomposition of $4K_{57}$.
$\{(i,i+1,i+30,i+29),(i,i+27,i+25,i+55),(i,i+3,i+34,i+31),(i,i+4,i+36,i+32)\}$
$\{(i,i+5,i+38,i+33),(i,i+6,i+40,i+34),(i,i+7,i+42,i+35),(i,i+8,i+44,i+36)\}$
$\{(i,i+9,i+46,i+37),(i,i+10,i+48,i+38),(i,i+11,i+50,i+39),(i,i+12,i+52,i+40)\}$
$\{(i,i+13,i+54,i+41),(i,i+14,i+56,i+42),(i,i+15,i+1,i+43),(i,i+16,i+3,i+44)\}$
$\{(i,i+17,i+5,i+45),(i,i+18,i+7,i+46),(i,i+19,i+9,i+47),(i,i+20,i+11,i+48)\}$
$\{(i,i+21,i+13,i+49),(i,i+22,i+15,i+50),(i,i+23,i+17,i+51),(i,i+24,i+19,i+52)\}$
$\{(i,i+25,i+21,i+53),(i,i+26,i+23,i+54),(i,i+2,i+32,i+30),(i,i+28,i+27,i+56)\}$
$(i=1,2,...,57)$.

Theorem 6. When $\lambda\equiv 0 \pmod 8$, $n\equiv 1 \pmod 4$ and $n\geq 13$, λK_n has a balanced C_4-quatrefoil decomposition.

Proof. We consider 3 cases.

Case 1. $n \equiv 1$ **(mod 8) and** $n \geq 17$. By Theorem 5 and Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2.1. $n = 13$. **(Example 6.1. Balanced C_4-quatrefoil decomposition of** $8K_{13}$.)
Construct a balanced C_4-quatrefoil decomposition of $8K_{13}$ as follows:
$\{(i,i+1,i+2,i+4),(i,i+5,i+10,i+7),(i,i+8,i+3,i+6),(i,i+12,i+11,i+9)\}$
$\{(i,i+2,i+4,i+8),(i,i+3,i+6,i+12),(i,i+10,i+7,i+1),(i,i+11,i+9,i+5)\}$
$\{(i,i+4,i+8,i+3),(i,i+6,i+12,i+11),(i,i+7,i+1,i+2),(i,i+9,i+5,i+10)\}$
$(i = 1,2,...,13)$.
Applying Theorem 2, λK_{13} has a balanced C_4-quatrefoil decomposition.
Case 2.2. $n \equiv 5$ **(mod 8) and** $n \geq 21$. Put $n = 8t+5$. Construct n C_4-$(4t+2)$-foiloids as follows:
$B_i = \{(i,i+1,i+4t+4,i+4t+3),(i,i+2,i+4t+6,i+4t+4),(i,i+3,i+4t+8,i+4t+5),...,(i,i+2t+1,i+8t+4,i+6t+3)\} \cup \{(i,i+2t+2,i+1,i+6t+4),(i,i+2t+3,i+3,i+6t+5),(i,i+2t+4,i+5,i+6t+6),...,(i,i+4t+2,i+4t+1,i+8t+4)\}$ $(i = 1,2,...,n)$.
Put $D_i = B_i \cup B_i$. Then each D_i is a C_4-$(8t+4)$-foiloid. Put $E_i = \sigma_1\sigma_2\sigma_3\sigma_4 D_i$, where $\sigma_1 = (2,5)$, $\sigma_2 = (4t, 4t+2)$, $\sigma_3 = (4t+3, 4t+5)$, $\sigma_4 = (8t, 8t+3)$. Decompose each E_i into $(2t+1)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $8K_n$. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.

Example 6.2. Balanced C_4-quatrefoil decomposition of $8K_{21}$.
$\{(i,i+1,i+12,i+11),(i,i+5,i+20,i+15),(i,i+3,i+16,i+13),(i,i+4,i+18,i+14)\}$
$\{(i,i+2,i+14,i+12),(i,i+6,i+1,i+16),(i,i+7,i+3,i+17),(i,i+10,i+9,i+20)\}$
$\{(i,i+9,i+7,i+19),(i,i+8,i+5,i+18),(i,i+3,i+16,i+13),(i,i+2,i+14,i+12)\}$
$\{(i,i+1,i+12,i+11),(i,i+4,i+18,i+14),(i,i+5,i+20,i+15),(i,i+9,i+7,i+19)\}$
$\{(i,i+7,i+3,i+17),(i,i+8,i+5,i+18),(i,i+6,i+1,i+16),(i,i+10,i+9,i+20)\}$
$(i = 1,2,...,21)$.

Example 6.3. Balanced C_4-quatrefoil decomposition of $8K_{29}$.
$\{(i,i+1,i+16,i+15),(i,i+5,i+24,i+19),(i,i+3,i+20,i+17),(i,i+4,i+22,i+18)\}$
$\{(i,i+2,i+18,i+16),(i,i+6,i+26,i+20),(i,i+7,i+28,i+21),(i,i+8,i+1,i+22)\}$
$\{(i,i+9,i+3,i+23),(i,i+10,i+5,i+24),(i,i+11,i+7,i+25),(i,i+14,i+13,i+28)\}$
$\{(i,i+13,i+11,i+27),(i,i+12,i+9,i+26),(i,i+3,i+20,i+17),(i,i+2,i+18,i+16)\}$
$\{(i,i+1,i+16,i+15),(i,i+4,i+22,i+18),(i,i+5,i+24,i+19),(i,i+6,i+26,i+20)\}$
$\{(i,i+7,i+28,i+21),(i,i+8,i+1,i+22),(i,i+9,i+3,i+23),(i,i+13,i+11,i+27)\}$
$\{(i,i+11,i+7,i+25),(i,i+12,i+9,i+26),(i,i+10,i+5,i+24),(i,i+14,i+13,i+28)\}$
$(i = 1,2,...,29)$.

Theorem 7. When $\lambda \equiv 0$ (mod 16), $n \equiv 1$ (mod 2) and $n \geq 13$, λK_n has a balanced C_4-quatrefoil decomposition.

Proof. We consider 7 cases.
Case 1. $n \equiv 1$ **(mod 4) and** $n \geq 13$. By Theorem 6 and Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2.1. $n = 15$. **(Example 7.1. Balanced C_4-quatrefoil decomposition of** $16K_{15}$.)
Construct a balanced C_4-quatrefoil decomposition of $16K_{15}$ as follows:
$\{(i,i+1,i+2,i+9),(i,i+4,i+10,i+7),(i,i+11,i+5,i+8),(i,i+14,i+13,i+6)\}$
$\{(i,i+1,i+14,i+8),(i,i+2,i+9,i+4),(i,i+10,i+7,i+3),(i,i+5,i+11,i+12)\}$
$\{(i,i+3,i+1,i+2),(i,i+5,i+8,i+12),(i,i+13,i+6,i+11),(i,i+9,i+4,i+10)\}$
$\{(i,i+13,i+1,i+14),(i,i+6,i+11,i+5),(i,i+4,i+3,i+10),(i,i+12,i+7,i+2)\}$
$\{(i,i+7,i+3,i+1),(i,i+8,i+12,i+14),(i,i+2,i+9,i+13),(i,i+10,i+6,i+4)\}$
$\{(i,i+9,i+13,i+1),(i,i+3,i+10,i+6),(i,i+14,i+8,i+5),(i,i+11,i+12,i+7)\}$

$\{(i,i+6,i+4,i+3),(i,i+12,i+14,i+13),(i,i+8,i+5,i+11),(i,i+7,i+2,i+9)\}$
$(i=1,2,...,15)$.
Applying Theorem 2, λK_{15} has a balanced C_4-quatrefoil decomposition.
Case 2.2. $n=19$. **(Example 7.2. Balanced C_4-quatrefoil decomposition of** $16K_{19}$.**)**
Construct a balanced C_4-quatrefoil decomposition of $16K_{19}$ as follows:
$\{(i,i+1,i+2,i+4),(i,i+8,i+16,i+13),(i,i+7,i+14,i+9),(i,i+18,i+17,i+15)\}$
$\{(i,i+11,i+3,i+6),(i,i+12,i+5,i+10),(i,i+1,i+2,i+4),(i,i+8,i+16,i+13)\}$
$\{(i,i+7,i+14,i+9),(i,i+18,i+17,i+15),(i,i+11,i+3,i+6),(i,i+12,i+5,i+10)\}$
$\{(i,i+2,i+4,i+8),(i,i+16,i+13,i+7),(i,i+14,i+9,i+18),(i,i+17,i+15,i+11)\}$
$\{(i,i+3,i+6,i+12),(i,i+5,i+10,i+1),(i,i+2,i+4,i+8),(i,i+16,i+13,i+7)\}$
$\{(i,i+14,i+9,i+18),(i,i+17,i+15,i+11),(i,i+3,i+6,i+12),(i,i+5,i+10,i+1)\}$
$\{(i,i+4,i+8,i+16),(i,i+13,i+7,i+14),(i,i+9,i+18,i+17),(i,i+15,i+11,i+3)\}$
$\{(i,i+6,i+12,i+5),(i,i+10,i+1,i+2),(i,i+4,i+8,i+16),(i,i+13,i+7,i+14)\}$
$\{(i,i+9,i+18,i+17),(i,i+15,i+11,i+3),(i,i+6,i+12,i+5),(i,i+10,i+1,i+2)\}$
$(i=1,2,...,19)$.
Applying Theorem 2, λK_{19} has a balanced C_4-quatrefoil decomposition.
Case 2.3. $n=23$. **(Example 7.3. Balanced C_4-quatrefoil decomposition of** $16K_{23}$.**)**
Construct a balanced C_4-quatrefoil decomposition of $16K_{23}$ as follows:
$\{(i,i+1,i+2,i+4),(i,i+8,i+16,i+9),(i,i+18,i+13,i+3),(i,i+7,i+14,i+5)\}$
$\{(i,i+5,i+10,i+20),(i,i+17,i+11,i+22),(i,i+21,i+19,i+15),(i,i+6,i+12,i+1)\}$
$\{(i,i+1,i+2,i+4),(i,i+8,i+16,i+9),(i,i+18,i+13,i+3),(i,i+7,i+14,i+5)\}$
$\{(i,i+5,i+10,i+20),(i,i+17,i+11,i+22),(i,i+21,i+19,i+15),(i,i+6,i+12,i+1)\}$
$\{(i,i+2,i+4,i+8),(i,i+16,i+9,i+18),(i,i+13,i+3,i+6),(i,i+14,i+5,i+10)\}$
$\{(i,i+10,i+20,i+17),(i,i+11,i+22,i+21),(i,i+19,i+15,i+7),(i,i+12,i+1,i+2)\}$
$\{(i,i+2,i+4,i+8),(i,i+16,i+9,i+18),(i,i+13,i+3,i+6),(i,i+14,i+5,i+10)\}$
$\{(i,i+10,i+20,i+17),(i,i+11,i+22,i+21),(i,i+19,i+15,i+7),(i,i+12,i+1,i+2)\}$
$\{(i,i+4,i+8,i+16),(i,i+9,i+18,i+13),(i,i+3,i+6,i+12),(i,i+20,i+17,i+11)\}$
$\{(i,i+22,i+21,i+19),(i,i+15,i+7,i+14),(i,i+4,i+8,i+16),(i,i+9,i+18,i+13)\}$
$\{(i,i+3,i+6,i+12),(i,i+20,i+17,i+11),(i,i+22,i+21,i+19),(i,i+15,i+7,i+14)\}$
$(i=1,2,...,23)$.
Applying Theorem 2, λK_{23} has a balanced C_4-quatrefoil decomposition.
Case 2.4.1. $n\equiv 3$ **(mod** 12) **and** $n\geq 27$. Put $n=12t+3$. Construct n C_4-$2t$-foiloids X_i, n C_4-$(2t+1)$-foils Y_i, n C_4-$2t$-foiloids Z_i as follows:
$X_i=\{(i,i+4,i+7,i+3),(i,i+7,i+13,i+6),(i,i+10,i+19,i+9),...,(i,i+6t-2,i+12t-5,i+6t-3)\}$
$\cup\ \{(i,i+6t,i+12t-1,i+6t-1)\}$
$Y_i=\{(i,i+2,i+3,i+1),(i,i+5,i+9,i+4),(i,i+8,i+15,i+7),...,(i,i+6t-1,i+12t-3,i+6t-2)\}$
$\cup\ \{(i,i+6t+1,i+12t+1,i+6t)\}$
$Z_i=\{(i,i+1,i+6t+2,i+6t+1)\}\ \cup\ \{(i,i+3,i+5,i+2),(i,i+6,i+11,i+5),(i,i+9,i+17,i+8),...,(i,i+6t-3,i+12t-7,i+6t-4)\}$
$(i=1,2,...,n)$.
Put $B_i=X_i\cup X_i\cup X_i\cup X_i\cup Y_i\cup Y_i\cup Y_i\cup Y_i\cup Z_i\cup Z_i\cup Z_i\cup Z_i$. Then each B_i is a C_4-$(24t+4)$-foiloid.
When $n=27$, put $D_i=\sigma_1\sigma_2\sigma_3\sigma_4B_i$, where $\sigma_1=(2,39)$, $\sigma_2=(6,43)$, $\sigma_3=(10,47)$, $\sigma_4=(14,51)$.
When $n=39$, put $D_i=\sigma_1\sigma_2\sigma_3\sigma_4B_i$, where $\sigma_1=(2,55)$, $\sigma_2=(8,61)$, $\sigma_3=(14,67)$, $\sigma_4=(20,73)$.
When $n=12t+3$ and $n\geq 51$, put $D_i=\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8B_i$, where $\sigma_1=(2,16t+7)$, $\sigma_2=(2t+2,18t+7)$, $\sigma_3=(4t+2,20t+7)$, $\sigma_4=(6t+2,22t+7)$, $\sigma_5=(6,16t+11)$, $\sigma_6=(2t+6,18t+11)$, $\sigma_7=(4t+6,20t+11)$, $\sigma_8=(6t+6,22t+11)$.
Decompose each D_i into $(6t+1)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2.4.2. $n\equiv 7$ **(mod** 12) **and** $n\geq 31$. Put $n=12t+7$. Construct n C_4-$(2t+1)$-foiloids X_i, n C_4-$(2t+1)$-foils Y_i, n C_4-$(2t+1)$-foiloids Z_i as follows:

$X_i = \{(i, i+1, i+6t+4, i+6t+3)\} \cup \{(i, i+4, i+7, i+3), (i, i+7, i+13, i+6), (i, i+10, i+19, i+9), ..., (i, i+6t+1, i+12t+1, i+6t)\}$
$Y_i = \{(i, i+2, i+3, i+1), (i, i+5, i+9, i+4), (i, i+8, i+15, i+7), ..., (i, i+6t+2, i+12t+3, i+6t+1)\}$
$Z_i = \{(i, i+3, i+5, i+2), (i, i+6, i+11, i+5), (i, i+9, i+17, i+8), ..., (i, i+6t+3, i+12t+5, i+6t+2)\}$
$(i = 1, 2, ..., n)$.
Put $B_i = X_i \cup X_i \cup X_i \cup X_i \cup Y_i \cup Y_i \cup Y_i \cup Y_i \cup Z_i \cup Z_i \cup Z_i \cup Z_i$. Then each B_i is a C_4-$(24t+12)$-foiloid.
When $n = 31$, put $D_i = \sigma_1\sigma_2\sigma_3\sigma_4 B_i$, where $\sigma_1 = (3, 42)$, $\sigma_2 = (8, 47)$, $\sigma_3 = (13, 52)$, $\sigma_4 = (18, 57)$.
When $n = 12t+7$ and $n \geq 43$, put $D_i = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8 B_i$, where $\sigma_1 = (3, 16t+10)$, $\sigma_2 = (2t+4, 18t+11)$, $\sigma_3 = (4t+5, 20t+12)$, $\sigma_4 = (6t+6, 22t+13)$, $\sigma_5 = (7, 16t+14)$, $\sigma_6 = (2t+8, 18t+15)$, $\sigma_7 = (4t+9, 20t+16)$, $\sigma_8 = (6t+10, 22t+17)$.
Decompose each D_i into $(6t+3)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2.4.3. $n \equiv 11$ **(mod 12) and** $n \geq 35$. Put $n = 12t+11$. Construct n C_4-$(2t+2)$-foiloids X_i, n C_4-$(2t+2)$-foils Y_i, n C_4-$(2t+1)$-foiloids Z_i as follows:
$X_i = \{(i, i+1, i+6t+6, i+6t+5)\} \cup \{(i, i+4, i+7, i+3), (i, i+7, i+13, i+6), (i, i+10, i+19, i+9), ..., (i, i+6t+4, i+12t+7, i+6t+3)\}$
$Y_i = \{(i, i+2, i+3, i+1), (i, i+5, i+9, i+4), (i, i+8, i+15, i+7), ..., (i, i+6t+5, i+12t+9, i+6t+4)\}$
$Z_i = \{(i, i+3, i+5, i+2), (i, i+6, i+11, i+5), (i, i+9, i+17, i+8), ..., (i, i+6t+3, i+12t+5, i+6t+2)\}$
$(i = 1, 2, ..., n)$.
Put $B_i = X_i \cup X_i \cup X_i \cup X_i \cup Y_i \cup Y_i \cup Y_i \cup Y_i \cup Z_i \cup Z_i \cup Z_i \cup Z_i$. Then each B_i is a C_4-$(24t+20)$-foiloid.
When $n = 35$, put $D_i = \sigma_1\sigma_2\sigma_3\sigma_4 B_i$, where $\sigma_1 = (3, 50)$, $\sigma_2 = (9, 55)$, $\sigma_3 = (15, 60)$, $\sigma_4 = (21, 65)$.
When $n = 12t+11$ and $n \geq 47$, put $D_i = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8 B_i$, where $\sigma_1 = (3, 16t+18)$, $\sigma_2 = (2t+5, 18t+19)$, $\sigma_3 = (4t+7, 20t+20)$, $\sigma_4 = (6t+9, 22t+21)$, $\sigma_5 = (7, 16t+22)$, $\sigma_6 = (2t+9, 18t+23)$, $\sigma_7 = (4t+11, 20t+24)$, $\sigma_8 = (6t+13, 22t+25)$.
Decompose each D_i into $(6t+5)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $16K_n$. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.

Theorem 8. When $\lambda \equiv 0 \pmod{32}$ and $n \geq 13$, λK_n has a balanced C_4-quatrefoil decomposition.

Proof. We consider 5 cases.
Case 1. $n \equiv 1$ **(mod 2) and** $n \geq 13$. By Theorem 7 and Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.
Case 2.1. $n = 14$. **(Example 8.1. Balanced** C_4**-quatrefoil decomposition of** $32K_{14}$**.)**
Construct a balanced C_4-quatrefoil decomposition of $32K_{14}$ as follows:
$\{(i, i+1, i+8, i+2), (i, i+4, i+7, i+6), (i, i+5, i+9, i+11), (i, i+10, i+13, i+12)\}$
$\{(i, i+1, i+8, i+2), (i, i+5, i+9, i+11), (i, i+10, i+6, i+4), (i, i+13, i+12, i+3)\}$
$\{(i, i+1, i+8, i+9), (i, i+3, i+5, i+2), (i, i+6, i+4, i+7), (i, i+10, i+13, i+12)\}$
$\{(i, i+1, i+8, i+9), (i, i+3, i+5, i+2), (i, i+7, i+6, i+11), (i, i+12, i+10, i+13)\}$
$\{(i, i+2, i+3, i+12), (i, i+4, i+7, i+6), (i, i+11, i+1, i+8), (i, i+13, i+5, i+9)\}$
$\{(i, i+2, i+3, i+12), (i, i+6, i+4, i+7), (i, i+11, i+1, i+8), (i, i+13, i+5, i+9)\}$
$\{(i, i+2, i+10, i+6), (i, i+9, i+4, i+7), (i, i+11, i+1, i+8), (i, i+12, i+3, i+5)\}$
$\{(i, i+2, i+10, i+6), (i, i+9, i+4, i+7), (i, i+11, i+1, i+8), (i, i+13, i+12, i+3)\}$
$\{(i, i+3, i+12, i+10), (i, i+4, i+7, i+13), (i, i+8, i+2, i+6), (i, i+9, i+11, i+1)\}$
$\{(i, i+3, i+12, i+10), (i, i+6, i+11, i+1), (i, i+7, i+13, i+5), (i, i+8, i+9, i+4)\}$
$\{(i, i+4, i+7, i+13), (i, i+6, i+11, i+1), (i, i+8, i+2, i+10), (i, i+12, i+3, i+5)\}$
$\{(i, i+5, i+2, i+3), (i, i+7, i+6, i+11), (i, i+8, i+9, i+4), (i, i+12, i+10, i+13)\}$
$\{(i, i+5, i+2, i+3), (i, i+7, i+13, i+8), (i, i+9, i+11, i+1), (i, i+10, i+6, i+4)\}$
$(i = 1, 2, ..., 14)$.
Applying Theorem 2, λK_{14} has a balanced C_4-quatrefoil decomposition.
Case 2.2. $n = 16$. **(Example 8.2. Balanced** C_4**-quatrefoil decomposition of** $32K_{16}$**.)**

Construct a balanced C_4-quatrefoil decomposition of $32K_{16}$ as follows:
$\{(i,i+1,i+2,i+9),(i,i+3,i+6,i+11),(i,i+4,i+8,i+12),(i,i+5,i+10,i+13)\}$
$\{(i,i+1,i+2,i+9),(i,i+3,i+6,i+11),(i,i+4,i+8,i+12),(i,i+7,i+14,i+15)\}$
$\{(i,i+1,i+2,i+9),(i,i+3,i+6,i+11),(i,i+5,i+10,i+13),(i,i+7,i+14,i+15)\}$
$\{(i,i+1,i+2,i+9),(i,i+4,i+8,i+12),(i,i+5,i+10,i+13),(i,i+7,i+14,i+15)\}$
$\{(i,i+2,i+4,i+10),(i,i+9,i+1,i+8),(i,i+12,i+7,i+3),(i,i+14,i+11,i+5)\}$
$\{(i,i+2,i+4,i+10),(i,i+9,i+1,i+8),(i,i+12,i+7,i+3),(i,i+15,i+13,i+6)\}$
$\{(i,i+2,i+4,i+10),(i,i+9,i+1,i+8),(i,i+14,i+11,i+5),(i,i+15,i+13,i+6)\}$
$\{(i,i+2,i+4,i+10),(i,i+12,i+7,i+3),(i,i+14,i+11,i+5),(i,i+15,i+13,i+6)\}$
$\{(i,i+3,i+6,i+11),(i,i+4,i+8,i+12),(i,i+5,i+10,i+13),(i,i+7,i+14,i+15)\}$
$\{(i,i+6,i+12,i+14),(i,i+8,i+7,i+15),(i,i+10,i+3,i+1),(i,i+11,i+5,i+2)\}$
$\{(i,i+6,i+12,i+14),(i,i+8,i+7,i+15),(i,i+10,i+3,i+1),(i,i+13,i+9,i+4)\}$
$\{(i,i+6,i+12,i+14),(i,i+8,i+7,i+15),(i,i+11,i+5,i+2),(i,i+13,i+9,i+4)\}$
$\{(i,i+6,i+12,i+14),(i,i+10,i+3,i+1),(i,i+11,i+5,i+2),(i,i+13,i+9,i+4)\}$
$\{(i,i+8,i+7,i+15),(i,i+10,i+3,i+1),(i,i+11,i+5,i+2),(i,i+13,i+9,i+4)\}$
$\{(i,i+9,i+1,i+8),(i,i+12,i+7,i+3),(i,i+14,i+11,i+5),(i,i+15,i+13,i+6)\}$
$(i=1,2,...,16)$.
Applying Theorem 2, λK_{16} has a balanced C_4-quatrefoil decomposition.

Case 2.3. $n=18$. **(Example 8.3. Balanced C_4-quatrefoil decomposition of $32K_{18}$.)**
Construct a balanced C_4-quatrefoil decomposition of $32K_{18}$ as follows:
$\{(i,i+10,i+1,i+9),(i,i+2,i+4,i+11),(i,i+3,i+6,i+12),(i,i+8,i+16,i+17)\}$
$\{(i,i+5,i+10,i+14),(i,i+6,i+12,i+15),(i,i+13,i+7,i+3),(i,i+9,i+8,i+17)\}$
$\{(i,i+4,i+8,i+13),(i,i+7,i+14,i+16),(i,i+11,i+3,i+1),(i,i+12,i+5,i+2)\}$
$\{(i,i+1,i+2,i+10),(i,i+14,i+9,i+4),(i,i+15,i+11,i+5),(i,i+16,i+13,i+6)\}$
$\{(i,i+17,i+15,i+7),(i,i+5,i+10,i+14),(i,i+2,i+4,i+11),(i,i+3,i+6,i+12)\}$
$\{(i,i+4,i+8,i+13),(i,i+1,i+2,i+10),(i,i+6,i+12,i+15),(i,i+7,i+14,i+16)\}$
$\{(i,i+8,i+16,i+17),(i,i+14,i+9,i+4),(i,i+12,i+5,i+2),(i,i+11,i+3,i+1)\}$
$\{(i,i+1,i+2,i+10),(i,i+13,i+7,i+3),(i,i+9,i+8,i+17),(i,i+15,i+11,i+5)\}$
$\{(i,i+16,i+13,i+6),(i,i+17,i+15,i+7),(i,i+5,i+10,i+14),(i,i+2,i+4,i+11)\}$
$\{(i,i+3,i+6,i+12),(i,i+4,i+8,i+13),(i,i+10,i+1,i+9),(i,i+7,i+14,i+16)\}$
$\{(i,i+8,i+16,i+17),(i,i+6,i+12,i+15),(i,i+13,i+7,i+3),(i,i+10,i+1,i+9)\}$
$\{(i,i+11,i+3,i+1),(i,i+12,i+5,i+2),(i,i+17,i+15,i+7),(i,i+14,i+9,i+4)\}$
$\{(i,i+15,i+11,i+5),(i,i+16,i+13,i+6),(i,i+9,i+8,i+17),(i,i+1,i+2,i+10)\}$
$\{(i,i+2,i+4,i+11),(i,i+3,i+6,i+12),(i,i+9,i+8,i+17),(i,i+5,i+10,i+14)\}$
$\{(i,i+6,i+12,i+15),(i,i+7,i+14,i+16),(i,i+11,i+3,i+1),(i,i+4,i+8,i+13)\}$
$\{(i,i+10,i+1,i+9),(i,i+8,i+16,i+17),(i,i+15,i+11,i+5),(i,i+13,i+7,i+3)\}$
$\{(i,i+14,i+9,i+4),(i,i+12,i+5,i+2),(i,i+16,i+13,i+6),(i,i+17,i+15,i+7)\}$
$(i=1,2,...,18)$.
Applying Theorem 2, λK_{18} has a balanced C_4-quatrefoil decomposition.

Case 2.4. $n\equiv 0$ **(mod 2) and** $n\geq 20$. Put $n=2t$. Construct n C_4-$(2t-1)$-foiloids as follows:
$B_i=\{(i,i+1,i+2,i+t+1),(i,i+2,i+4,i+t+2),(i,i+3,i+6,i+t+3),...,(i,i+t-1,i+2t-2,i+2t-1)\}$ $\cup$ $\{(i,i+t,i+t-1,i+2t-1)\}$ $\cup$ $\{(i,i+t+1,i+1,i+t)\}$ $\cup$ $\{(i,i+t+2,i+3,i+1),(i,i+t+3,i+5,i+2),(i,i+t+4,i+7,i+3),...,(i,i+2t-1,i+2t-3,i+t-2)\}$
$(i=1,2,...,n)$.
Put $D_i=B_i\cup B_i\cup B_i\cup B_i$. Then each D_i is a C_4-$(8t-4)$-foiloid.
When $n=20$, put $E_i=\sigma D_i$ and $\sigma=\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}$, where $\sigma_1=(1,5)$, $\sigma_2=(4,76)$, $\sigma_3=(8,9)$, $\sigma_4=(11,13)$, $\sigma_5=(16,19)$, $\sigma_6=(20,21)$, $\sigma_7=(27,30)$, $\sigma_8=(39,41)$, $\sigma_9=(47,50)$, $\sigma_{10}=(53,59)$, $\sigma_{11}=(64,66)$, $\sigma_{12}=(68,69)$.
When $n=22$, first put $E_i'=\sigma D_i$ and $\sigma=\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}$, where $\sigma_1=(1,84)$, $\sigma_2=(2,5)$, $\sigma_3=(8,10)$, $\sigma_4=(11,15)$, $\sigma_5=(16,20)$, $\sigma_6=(23,25)$, $\sigma_7=(31,34)$, $\sigma_8=(51,54)$,

$\sigma_9 = (57, 61)$, $\sigma_{10} = (64, 65)$, $\sigma_{11} = (72, 73)$, $\sigma_{12} = (75, 77)$, $\sigma_{13} = (80, 83)$. Next, put $E_i'' = \sigma_{14}E_i'$ and $\sigma_{14} = (48, 51)$. Last, put $E_i = \sigma_{15}E_i''$ and $\sigma_{15} = (43, 48)$.
When $n = 24$, put $E_i = \sigma D_i$ and $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}$, where $\sigma_1 = (1, 92)$, $\sigma_2 = (2, 7)$, $\sigma_3 = (11, 14)$, $\sigma_4 = (20, 23)$, $\sigma_5 = (24, 25)$, $\sigma_6 = (32, 34)$, $\sigma_7 = (35, 39)$, $\sigma_8 = (47, 49)$, $\sigma_9 = (56, 57)$, $\sigma_{10} = (59, 61)$, $\sigma_{11} = (64, 65)$, $\sigma_{12} = (70, 73)$, $\sigma_{13} = (79, 82)$.
When $n = 26$, put $E_i = \sigma D_i$ and $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}\sigma_{13}$, where $\sigma_1 = (1, 100)$, $\sigma_2 = (2, 5)$, $\sigma_3 = (11, 14)$, $\sigma_4 = (20, 24)$, $\sigma_5 = (27, 29)$, $\sigma_6 = (36, 37)$, $\sigma_7 = (39, 41)$, $\sigma_8 = (51, 53)$, $\sigma_9 = (60, 62)$, $\sigma_{10} = (63, 67)$, $\sigma_{11} = (70, 73)$, $\sigma_{12} = (76, 77)$, $\sigma_{13} = (87, 90)$.
When $n \equiv 4 \pmod 8$ and $n \geq 28$, put $E_i = \sigma D_i$ and $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}$, where $\sigma_1 = (2, 8t-4)$, $\sigma_2 = (t-2, t-1)$, $\sigma_3 = (t+1, t+3)$, $\sigma_4 = (2t-4, 2t-1)$, $\sigma_5 = (2t, 2t+1)$, $\sigma_6 = (3t-3, 3t)$, $\sigma_7 = (4t-1, 4t+1)$, $\sigma_8 = (5t-3, 5t)$, $\sigma_9 = (6t-8, 6t-7)$, $\sigma_{10} = (6t-1, 6t+1)$, $\sigma_{11} = (7t-6, 7t-4)$, $\sigma_{12} = (7t-2, 7t-1)$.
When $n \equiv 6 \pmod 8$ and $n \geq 30$, put $E_i = \sigma D_i$ and $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}$, where $\sigma_1 = (2, 8t-4)$, $\sigma_2 = (t-3, t-1)$, $\sigma_3 = (t, t+4)$, $\sigma_4 = (2t-6, 2t-5)$, $\sigma_5 = (2t+1, 2t+3)$, $\sigma_6 = (3t-2, 3t+1)$, $\sigma_7 = (4t-1, 4t+1)$, $\sigma_8 = (5t-4, 5t-1)$, $\sigma_9 = (6t-6, 6t-3)$, $\sigma_{10} = (6t-2, 6t-1)$, $\sigma_{11} = (7t-5, 7t-4)$, $\sigma_{12} = (7t-2, 7t)$.
When $n \equiv 0 \pmod 8$ and $n \geq 32$, put $E_i = \sigma D_i$ and $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}$, where $\sigma_1 = (2, 8t-4)$, $\sigma_2 = (t-1, t+2)$, $\sigma_3 = (2t-4, 2t-1)$, $\sigma_4 = (2t, 2t+1)$, $\sigma_5 = (3t-4, 3t-2)$, $\sigma_6 = (3t-1, 3t+3)$, $\sigma_7 = (4t-1, 4t+1)$, $\sigma_8 = (5t-4, 5t-3)$, $\sigma_9 = (5t-1, 5t+1)$, $\sigma_{10} = (6t-8, 6t-7)$, $\sigma_{11} = (6t-1, 6t+1)$, $\sigma_{12} = (7t-5, 7t-2)$.
When $n \equiv 2 \pmod 8$ and $n \geq 34$, put $E_i = \sigma D_i$ and $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5\sigma_6\sigma_7\sigma_8\sigma_9\sigma_{10}\sigma_{11}\sigma_{12}$, where $\sigma_1 = (2, 8t-4)$, $\sigma_2 = (t-2, t+1)$, $\sigma_3 = (2t-6, 2t-5)$, $\sigma_4 = (2t+1, 2t+3)$, $\sigma_5 = (3t-3, 3t-2)$, $\sigma_6 = (3t, 3t+2)$, $\sigma_7 = (4t-1, 4t+1)$, $\sigma_8 = (5t-5, 5t-3)$, $\sigma_9 = (5t-2, 5t+2)$, $\sigma_{10} = (6t-6, 6t-3)$, $\sigma_{11} = (6t-2, 6t-1)$, $\sigma_{12} = (7t-4, 7t-1)$.
Decompose each E_i into $(2t-1)$ C_4-quatrefoils. Then they comprise a balanced C_4-quatrefoil decomposition of $32K_n$. Applying Theorem 2, λK_n has a balanced C_4-quatrefoil decomposition.

Therefore, we have the following main theorem and its corollary.

Main Theorem. λK_n has a balanced C_4-quatrefoil decomposition if and only if $\lambda(n-1) \equiv 0 \pmod{32}$ and $n \geq 13$.

Corollary. K_n has a balanced C_4-quatrefoil decomposition if and only if $n \equiv 1 \pmod{32}$, $n \geq 33$.

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