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AUTOMORPHIC GREEN FUNCTIONS FOR SYMMETRIC SPACES

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1. CLASSICLA CASE

Let $\mathfrak{H} = \{\tau \in \mathbb{C} | \operatorname{Im}(\tau) > 0\} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$ be the Poincaré upper half-plane, and Γ a Fuchsian group of the first kind which acts on $\mathfrak{H}$ by the usual Möbius transformation. Since the volume form $\frac{dx\wedge dy}{y^2}$ and the Laplacian $-y^2(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2})$ associated with the Poincare metric $y^{-2}(dx^2+dy^2)$ is $\mathrm{SL}_2(\mathbb{R})$-invariant, they yield the volume from ω_X and the Laplacian $\triangle_X$ of the Riemanian surface $X = \Gamma\backslash\mathfrak{H}$. The resolvent operator $R_s = (\triangle_X + s(s-1))^{-1}$ of the shifted Laplacian $\triangle_X + s(s-1)$ is an integral operator, whose kernel function $G_s(z,w) : X\times X - \Delta X \to \mathbb{C}$ is constructed as

$$G_s(z,w) = \sum_{\gamma\in\Gamma} \phi_s^{\mathfrak{H}}(\gamma z, w), \quad (\operatorname{Re}(s) > 1,\ z \not\equiv w \pmod{\Gamma}), \tag{1.1}$$

$$\phi_s^{\mathfrak{H}}(z,w) = \frac{-1}{4\pi}\frac{\Gamma(s)^2}{\Gamma(2s)}\left(1-\left|\frac{z-w}{\bar z - w}\right|^2\right)^s {}_2F_1\left(s,s;2s;1-\left|\frac{z-w}{\bar z - w}\right|^2\right).$$

The series (1.1) is absolutely convergent if $\operatorname{Re}(s) > 1$, and the convergence is locally uniform for s and $(z,w) \in X\times X - \Delta X$. The function $\phi_s^{\mathfrak{H}}$ is called *the free space Green function of* $\mathfrak{H}$ and $G_s(z,w)$ *the automorphic Green function of* X, which has been an important object of reserch in the analytic theory of automorphic functions ([2],[3], [4]). Among many properties of $G_s(z,w)$, we focus on the following two.

(a) (Poisson equation) For each $w \in X$,

$$(\triangle_{X,z} + s(s-1))G_s(z,w) = \delta_w(z).$$

(b) (square-integrability) $G_s(z,w) \in L^2(X\times X)$.

These two properties are important because they enable us to have the spectral expansion of $G_s(z,w)$ in the space $L^2(X)$ in terms of basic wave functions, i.e., Maass wave functions and Eisenstein series.

The aim of this article is first to provide a proper definition of automorphic Green function for a pair of a higher dimensional locally symmetric space and its modular subvariety generalising the classical construction, and then to state the basic properties of Green funciton generalizing (a) and (b) above.

2. GREEN FUNCTIONS

2.1. **Notations and assumptions.** Let G be a reductive Lie group with compact center. Let θ and σ be involutions of G such that

(1) θ and σ are commutative, i.e, $\theta\sigma = \sigma\theta$.
(2) θ is a Cartan involution of G.

Then $K = G^\theta$ is a maximal compact subgroup of G and $H = G^\sigma$ is a reductive closed subgroup of G such that $H \cap K$ is maximally compact in H. We further make two assumptions. The first is that

(3) the symmetric pair (G, H) has $\mathbb{R}$-rank one,

which means there exists a vector $Y_0 \in \mathfrak{g}$ such that $\mathbb{R}Y_0$ is a maximal abelian subspace of $\mathfrak{g}^{-\sigma} \cap \mathfrak{g}^{-\theta}$; the vector Y_0 is supposed to be taken so that the eigenvalues of $\mathrm{ad}(Y_0)$ belong to $\{0, \pm 1, \pm 2\}$. For $j \in \{0, \pm 1, \pm 2\}$, let $\mathfrak{g}_j$ be the corresponding eigenspace of $\mathrm{ad}(Y_0)$ and set $m_j = \dim_{\mathbb{R}}(\mathfrak{g}_j)$ and $m_j^\pm = \dim_{\mathbb{R}}(\mathfrak{g}_j \cap \mathfrak{g}^{\pm\sigma\theta})$. The second assumption is

(4) $m := 2^{-1}(m_1^+ + m_2^+ + 1) \in \mathbb{Z}$.

Note m is the half of the $\mathbb{R}$-codimension of $H/H \cap K$ in G/K.

2.2. **Free space Green function.** Set

$$\tilde{Y}_0 = \begin{cases} Y_0 & (m_2^- = 0), \\ 2^{-1}Y_0 & (m_2^- > 0) \end{cases}$$

By a general theory, the set $\{a_t = \exp(t\tilde{Y}_0) \,|\, t \geqslant 0\}$ comprises a complete set of representatives for the double coset space $H\backslash G/K$ and the natural smooth map $H \times \{a_t | t > 0\} \times K \to G - HK$ is a submersion. Let us define a function $\phi_s(g) \in C^\infty(H\backslash(G-HK)/K)$ depending on a complex one parameter s by

$$\phi_s(a_t) = C_m \frac{\Gamma(\frac{s+\rho_0}{2})\Gamma(\frac{s-\rho_0}{2}+m)}{\Gamma(s+1)} (\cosh t)^{-(s+\rho_0)} {}_2F_1\left(\tfrac{s+\rho_0}{2}, \tfrac{s-\rho_0}{2}+m; s+1; \tfrac{1}{\cosh^2 t}\right), \quad (t \neq 0)$$

with $\rho_0 = 2^{-1}\mathrm{tr}(\mathrm{ad}(\tilde{Y}_0)|\mathfrak{g}_1 + \mathfrak{g}_2)$,

$$C_m = \begin{cases} -2^{-1} & (m = 1), \\ \Gamma(m-1)^{-1} & (m > 1). \end{cases}$$

Proposition 1. *Let* $\mathrm{Re}(s) > 0$. *The function* ϕ_s *has the following three properties, which characterize* ϕ_s.

(1) *Let* $B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ *be a* G*-invariant symmetric* $\mathbb{R}$*-bilinear form on* $\mathfrak{g}$ *such that* $-B(X, \theta Y)$ *is* θ*-invariant and positive definite and such that* $B(\tilde{Y}_0, \tilde{Y}_0) = 1$. *Let* $C_{\mathfrak{g}}$ *be the Casimir element corresponding to* B. *Then*

$$C_{\mathfrak{g}}\phi_s(g) = (s^2 - \rho_0^2)\,\phi_s(g), \quad g \in G - HK.$$

(2) *If* $m = 1$, *then* $\phi_s(a_t) - \log t = O(1)$, $(t \in (0, \epsilon))$ *for a small interval* $(0, \epsilon)$. *If* $m > 1$, *then* $\lim_{t \to +0} t^{m-1}\phi_s(a_t) = 1$.

(3) $\phi_s(a_t) = O(e^{-(\mathrm{Re}(s)+\rho_0)t})$ $(t > R)$ *for a large poisitive* R.

The behavior of the function $\phi_s(a_t)$ *for small* t *is described more precisely than (2): There exists polynomial functions* $\{A_n(s)\}_{n \geqslant 0}$ *and* $\{C_j(s)\}_{j=0}^{m-2}$ *such that* $\deg C_j(s) = j$ $(0 \leqslant j \leqslant m-2)$, $\deg A_n(s) = n$ $(n \geqslant 0)$, $A_0(s) = C_0(s) = 1$ *and* $\phi_s(a_t) - F(\tanh^2 t)$ *has a continuous extension to a small neighborhood of* $t = 0$, *where*

$$F(z) = \sum_{j=0}^{m-2} \frac{C_j(s^2)}{z^{m-1-j}} + \log z \left(\sum_{n=0}^{\infty} A_n(s^2) z^n\right).$$

For integer $r \geqslant 0$, we set

$$\phi_s^{[r]}(g) = \frac{1}{r!}\left(-\frac{1}{2s}\frac{d}{ds}\right)^r \phi_s(g), \quad g \in G - HK, \, \mathrm{Re}(s) > \rho_0.$$

If $r > 0$, then, by the last half of the proposition above, $g \mapsto \phi_s^{[r]}(g)$ has a *continuous* extension to the whole G.

2.3. **Automorphic Green function.** From now on we assume that our G and the involution σ are both defined over $\mathbb{Q}$. Given an arithmetic subgroup Γ of G (allowed not to be cocompact), we form the Poincaré series

$$\mathcal{G}_s^{[r]}(g) = \sum_{\gamma \in \Gamma \cap H \backslash \Gamma} \phi_s^{[r]}(\gamma g), \qquad \mathrm{Re}(s) > \rho_0, \, r \geqslant 0,$$

which converges in the following sense.

Proposition 2. *The series $\mathcal{G}_s^{[r]}(g)$ converges absolutely and locally uniformly in $(s, g) \in \{\mathrm{Re}(s) > \rho_0\} \times (G - \Gamma HK)$ to yield a right K-invariant integrable function on $\Gamma\backslash G$. If $r \geqslant m$, then $\mathcal{G}_s^{[r]}(g)$ converges absolutely and locally uniformly in $\{\mathrm{Re}(s) > \rho_0\} \times G$ to yields a right K-invariant continuous function on $\Gamma\backslash G$.*

Let $G = \mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ with the maximal compact subgroup $K = \mathrm{SO}(2) \times \mathrm{SO}(2)$ and take the involution $\sigma : (g_1, g_2) \mapsto (g_2, g_1)$ of G. Then the function $\mathcal{G}_s^{[0]}(g)$ regarded as a two variable funciton on $\Gamma\backslash\mathfrak{H} \times \Gamma\backslash\mathfrak{H} \cong (\Gamma \times \Gamma)\backslash G/K$ is essentially the original automorphic Green function of $X = \Gamma\backslash\mathfrak{H}$ recalled in the first section. Thus our series $\mathcal{G}_s^{[r]}(g)$ yields a generalization of the classical construction. The next two propositions say $\mathcal{G}_s^{[r]}(g)$ has properties analogous to those (a) and (b) in the first section.

Proposition 3. (1) *Let $\mathcal{P}_\Gamma^H$ be the K-invariant distribution on the G-manifold $\Gamma\backslash G$ defined by*

$$< \mathcal{P}_\Gamma^H, f >= \int_{\Gamma \cap H \backslash H} \int_K f(hk)\, dk\, dh, \quad \forall f \in C_c^\infty(\Gamma\backslash G).$$

Fixing a Haar measure of G properly and regarding $\mathcal{G}_s^{[r]}(g) \in L^1(\Gamma\backslash G)$ as K-invariant distributions on $\Gamma\backslash G$ as usual, we have a system of differential equations among them:

$$(C_\mathfrak{g} + \rho_0^2 - s^2)\, \mathcal{G}_s^{[r]} = \mathcal{G}_s^{[r-1]} \quad (r \geqslant 1),$$
$$(C_\mathfrak{g} + \rho_0^2 - s^2)\, \mathcal{G}_s^{[0]} = \mathcal{P}_\Gamma^H.$$

(2) *There exists a constant $\tau = \tau(G, \sigma) \in [0, 2]$ such that $\mathcal{G}_s^{[r]}(g) \in L^p(\Gamma\backslash G)\, (\forall p \in [1, (1 - 2^{-1}\tau)^{-1}), \forall r \geqslant m - 1)$ for $\mathrm{Re}(s) > \rho_0(1 + \tau)$. If $\tau > 1$, then $\mathcal{G}_s^{[r]}(g)$ is $L^{2+\epsilon}$ for $\mathrm{Re}(s) > \rho_0(\tau + 1)$.*

REMARK: For the definition of the number τ see [6, p.460]. The condition $\tau > 1$ is not always true especially for G of small size; in that case the validity of $L^{2+\epsilon}$ condition for $\mathcal{G}_s^{[r]}(g)$ gets subtler (and more difficult to establish if true).

If Γ is neat, then the double coset space $X = \Gamma\backslash G/K$ acquires the structure of Riemannian manifold from the G-invariant metric on G/K defined by B. Let $\triangle_X$ be the corresponding Laplacian of X and Λ_X the set of eigenvalues of $\triangle_X$ on $L^2(X)$; it is know that Λ is a countable discrete subset of $[0, +\infty)$ without accumulation points and each eigenvalue $\lambda \in \Lambda_X$ has finite multiplicity. For $\lambda \in \Lambda$, fix an orthonormal basis $\mathcal{B}(\lambda)$ of the λ-eigenspace. As a corollary of Proposition 3, we have the estimation

$$\sum_{\varphi\in\mathcal{B}(\lambda)} |\mathcal{P}_\Gamma^H(\varphi)|^2 = O(\lambda^m).$$

Let D_X be the set of $s \in \mathbb{C}$ such that $-\lambda = s^2 - \rho_0^2\,(\exists\lambda \in \Lambda_X)$. Then the estimation above ensures the locally-uniform convergence for $s \in \mathbb{C} - D_X$ of the series

$$\mathcal{G}_{s.\mathrm{dis}}^{[r]}(g) := \sum_{\lambda\in\Lambda_X} \sum_{\varphi\in\mathcal{B}(\lambda)} \frac{\mathcal{P}_\Gamma^H(\bar{\varphi})}{(s^2 - \rho_0^2 + \lambda)^{r+1}}\,\varphi(g) \tag{2.1}$$

in the Hilbert space $L^2(X)$. If $\Gamma\backslash G$ is compact, then $\mathcal{G}_s^{[r]} = \mathcal{G}_{s,\mathrm{dis}}^{[r]}$ in $L^2(X)$. In general the difference $\mathcal{G}_{s,\mathrm{Eis}}^{[r]} := \mathcal{G}_s^{[r]} - \mathcal{G}_{s,\mathrm{dis}}^{[r]}$ is discribed by the Eisenstein wave packets. By the result of Wallach ([7]) combined with Proposition 3, we have

Proposition 4. *The series* (2.1) *and the expression of* $\mathcal{G}_{s,\mathrm{Eis}}^{[r]}(g)$ *by the Eisenstein wave packet converge locally uniformly with respect to the variable* $g \in X$ *if* r *is sufficiently large.*

3. Miscellaneous remarks and applications

- We can obtain a Green current for the modular cycle $H\cap\Gamma\backslash H/H\cap K \to \Gamma\backslash G/K$ by the vector-valued analogue of the construction above. For the unitary case $(G, H) = (U(n, 1), U(r) \times U(n - r, 1))$, see [6],[9] .
- When (G, H) is of the group case i.e., $G = G' \times G'$ and $\sigma(g_1, g_2) = (g_2, g_1)$ with G' a $\mathbb{R}$-rank one Lie group, our $\mathcal{G}_s^{[r]}(g)$ essentially equals to the Miatello-Wallach's series $\mathbf{P}_{s,r}$ as was shown by [1]. Besides the group case, rank one hyperbolic spaces is an interesting class of symmetric spaces satisfying our assumptions in section 2.
- 'The function $\mathcal{G}_s^{[r]}(g)$ is a relative version of the classical automorphic Green function' is an impressive phrase which well summalizes the definition of our $\mathcal{G}_s^{[r]}(g)$ (for $r = 0$) in a single sentence. In the classical case, it has long been known how to deduce the Selberg trace formula from the automorphic Green function (see [2], [4] for example). So it is natural to expect that our $\mathcal{G}_s^{[r]}$ can be used in some way to have some kind of summation formula which may be a spacial case of the relative trace formula of Jacquet ([5]). If R is $\mathbb{Q}$-anisotropic $\mathbb{Q}$-subgroup of G, then $R\cap\Gamma\backslash R$ is compact subset of $\Gamma\backslash G$. Let w be an automorphic form on $R\cap\Gamma\backslash R$ and consider the integral $\mathcal{P}_\Gamma^{R,w}(\varphi) = \int_{R\cap\Gamma\backslash R} \varphi(r)w(r)\,dr$ for automorphic form φ on $\Gamma\backslash G$. (The unipotent radical of a $\mathbb{Q}$-parabolic subgroup of G and its automorphic charac! ter is an interesting possible choise for (R, w).) By Proposition 4, the termwise integration of the spectral expansion of $\mathcal{G}_s^{[r]}(g)$ is permissible

at least for sufficiently large r. When $\Gamma\backslash G$ is compact,

$$\int_{R\cap\Gamma\backslash R} \mathcal{G}_s^{[r]}(r)\, w(r)\, dr = \sum_{\lambda\in\Lambda_X} \sum_{\varphi\in\mathcal{B}(\lambda)} \frac{\mathcal{P}_\Gamma^H(\bar{\varphi})\, \mathcal{P}_\Gamma^{R,w}(\varphi)}{(s^2-\rho_0^2+\lambda)^{r+1}}. \tag{3.1}$$

This encodes the products of period-integrals $\mathcal{P}_\Gamma^H(\bar{\varphi})\mathcal{P}_\Gamma^{R,w}(\varphi)$ for various wave-functions φ. On the other hand, if we put the defining series of $\mathcal{G}_s^{[r]}(g)$ in the left hand side of (3.1) and further assume that we can unfold the series and integrals in a proper way similarly to the deduction of the geometric side of the Selberg trace formula, we would obtain another expression of the integral (3.1) at least when $\mathrm{Re}(s) > \rho_0$. The resulting identity expressing the integral $\int_{R\cap\Gamma\backslash R}\mathcal{G}_s^{[r]}(r)w(r)\,dr$ two ways may be regarded as a form of relative trace formula ([5]), which would give some information of the quantity $\sum_{\varphi\in\mathcal{B}(\lambda)}\mathcal{P}_\Gamma^H(\bar{\varphi})\mathcal{P}_\Gamma^{R,w}(\varphi)$. Indeed, when $(G,H) = (U(n,1), U(n-1,1))$ and (R,w) is a pair of the unipotent radical of a proper $\mathbb{Q}$-parabolic subgroup of G and its automorphic character, we have an identity relating the quantity $\sum_{\varphi\in\mathcal{B}(\lambda)}\mathcal{P}_\Gamma^H(\bar{\varphi})\mathcal{P}_\Gamma^{R,w}(\varphi)$ to some average of Fourier coefficients of cusp forms on $\mathfrak{H}$ for some modular group; the identity has several interesting applications ([10]).

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