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AN EXPLICIT DIMENSION FORMULA FOR THE SPACES OF VECTOR VALUED SIEGEL CUSP FORMS OF DEGREE TWO

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1. Introduction

In this paper, we give an explicit dimension formula for the spaces of vector valued Siegel cusp forms of degree two with respect to the principal congruence subgroups of $Sp(2;\mathbb{Z})$, and certain arithmetic subgroups of non-split $\mathbb{Q}$-forms of $Sp(2;\mathbb{R})$. As for the principal congruence subgroups of $Sp(2;\mathbb{Z})$, Tsushima already gave the dimension formula by the Riemann-Roch theorem in [17], but we give an alternative proof by the Selberg trace formula and the theory of prehomogeneous vector spaces. As for the case of non-split $\mathbb{Q}$-forms, our result is new.

Our calculation is a generalization of the calculations of Morita [13], Shintani [14] and Arakawa [1]. In this short note, we explain only the points for the generalizations (we wrote the detail proof in [18]). In order to generalize their methods, first we must show the convergence of some infinite series. Then we need to calculate explicitly an integral of a certain function, which is related to the Fourier transformation of the trace of the irreducible rational representations. The integral is well-known in the scalar valued case, but the integral is unknown and nontrivial in the vector value case.

One of our motivation is as follows. Ibukiyama gave a conjecture for the Shimura correspondence between vector valued Siegel cusp forms of degree two of integral weight and half integral weight (cf. [10]). There, it is essential to take vector valued forms. In order to prove this conjecture, we must show the equality of the traces of Hecke operators. As the first step, we treat the traces of the trivial actions, which are the dimensions of the spaces.

The plan of this paper is as follows. In Section 2, we state our main results. In Section 3, we review the Godement formula. In Section 4, we explain the calculations of the vanishing part. In Section 4, we explain the calculations of the non-vanishing part. In Appendix A, for the reader's convenience, we copy the dimension formula for $Sp(2;\mathbb{Z})$ which was given by Tsushima [17]. In Appendix B, we give the dimension formula for the full modular groups of non-split $\mathbb{Q}$-forms, which was obtained by our recent calculation.

2. Main results

We define the spaces of Siegel cusp forms of degree two. Let $\rho_{k,j} : GL(2;\mathbb{C}) \to GL(j+1;\mathbb{C})$ be the irreducible rational representation of the signature $(j+k, k)$

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$(j, k \in \mathbb{Z}_{>0})$, i.e. $\rho_{k,j} = \det^k \otimes Sym_j$ where Sym_j is the symmetric j-tensor representation of $GL(2;\mathbb{C})$. Let $\mathfrak{H}_2$ be the Siegel upper half space of degree two, i.e. $\mathfrak{H}_2 = \{Z \in M(2;\mathbb{C})\,;\, {}^tZ = Z,\ \mathrm{Im}(Z) > 0\}$. The real symplectic group $Sp(2;\mathbb{R})$ acts on $\mathfrak{H}_2$ as $Z \mapsto g\cdot Z := (AZ+B)(CZ+D)^{-1}$ for $Z \in \mathfrak{H}_2$, $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\mathbb{R})$. Let Γ be an arithmetic subgroup of $Sp(2;\mathbb{R})$. Let $S_{k,j}(\Gamma)$ be the space of vector valued Siegel cusp forms of weight $\rho_{k,j}$, i.e. the space of holomorphic functions $f : \mathfrak{H}_2 \to \mathbb{C}^{j+1}$ satisfying (i) $f(\gamma\cdot Z) = \rho_{k,j}(CZ+D)f(Z)$ for all $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma$, (ii) $|\rho_{k,j}(\mathrm{Im}(Z))^{1/2})f(Z)|_{\mathbb{C}^{j+1}}$ is bounded on $\mathfrak{H}_2$.

One of our main results is as follows. The following result was already given by Tsushima [17]. We put $\Gamma(N) = \{\gamma \in Sp(2;\mathbb{Z});\ \gamma \equiv I_4 \,(\mathrm{mod}\, N)\}$.

Theorem 2.1. *If $k \geq 5$ and $N \geq 3$, then*

$$\begin{aligned}\dim_{\mathbb{C}} S_{k,j}(\Gamma(N)) &= [\Gamma(1) : \Gamma(N)] \\ &\quad \times \{2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \\ &\quad -2^{-6}3^{-2}(j+1)(j+2k-3)N^{-2} + 2^{-5}3^{-1}(j+1)N^{-3}\},\end{aligned}$$

where $[\Gamma(1) : \Gamma(N)] = N^{10}\prod_{p:\mathrm{prime},\ p|N}(1-p^{-2})(1-p^{-4})$.

We shall give the other main result. Let $\mathbf{B}$ be an indefinite division quaternion algebra over $\mathbb{Q}$, $\mathfrak{O}$ a maximal order of $\mathbf{B}$, $a \mapsto \bar{a}$ $(a \in \mathbf{B})$ the canonical involution of $\mathbf{B}$. Put

$$G_{\mathbb{Q}} = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M(2;\mathbf{B})\,;\, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\},$$

$$\Gamma^*(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_{\mathbb{Q}}\,;\, a-1, b, c, d-1 \in N\mathfrak{O}\right\}.$$

The following result is new.

Theorem 2.2. *If $k \geq 5$ and $N \geq 3$, then*

$$\begin{aligned}&\dim S_{k,j}(\Gamma^*(N)) = [\Gamma^*(1) : \Gamma^*(N)] \\ &\times \left\{2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3)\prod_{p|D(\mathbf{B})}(p-1)(p^2+1)\right. \\ &\left. +2^{-4}3^{-1}(j+1)N^{-3}\prod_{p|D(\mathbf{B})}(p-1)\right\},\end{aligned}$$

where $D(\mathbf{B})$ is the product of prime numbers which ramify in $\mathbf{B}$ over $\mathbb{Q}$, p is prime, and $[\Gamma^*(1) : \Gamma^*(N)] = N^{10}\times\prod_{p|N,\, p\nmid D(\mathbf{B})}(1-p^{-2})(1-p^{-4})\times\prod_{p|N,\ p|D(\mathbf{B})}(1-p^{-2})(1+p^{-1})$.

As for the scalar valued case, these dimension formulas were already known. The dimension formula of the scalar valued case for $\Gamma(N)$ $(N \geq 3,\ j=0)$ was calculated by Christian [3], Morita [13] and Yamazaki [19] independently. The dimension formula of the scalar valued case for $\Gamma^*(N)$ ($N \geq 3$, $j = 0$) was calculated by Arakawa [1] and Yamaguchi independently. Christian, Morita and Arakawa used the Selberg trace formula. Yamazaki and Yamaguchi used the Riemann-Roch theorem.

Numerical examples.

(i) $\dim_{\mathbb{C}} S_{k,j}(\Gamma(3))$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13
0	15	76	200	405	709	1130	1686	2395	3275	4344
1	224	440	800	1340	2096	3104	4400	6020	8000	10376
2	165	519	1116	2010	3255	4905	7014	9636	12825	16635
3	336	940	1904	3300	5200	7676	10800	14644	19280	24780
4	595	1530	2960	4975	7665	11120	15430	20685	26975	34390

($*$) Our theorem is not valid for $k = 4$. As for $j = 0$, $k = 4$, Yamazaki calculated it by the Riemann-Roch theorem in [19]. We formally put $k = 4$ in the formula of our theorem. We expect that the dimension of $S_{k,j}(\Gamma(3))$ is given by putting $k = 4$ in the formula (cf. [7] and [8]). We also expect it for other arithmetic subgroups. For $j = 0$, $k = 1, 2, 3$, Gunji proved $\dim_{\mathbb{C}} S_{k,0}(\Gamma(3)) = 0$ in [5].

(ii) $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(3))$, $D(\mathbf{B}) = 2 \times 3$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13
0	1323	3510	7398	13473	22221	34128	49680	69363	93663	123066
1	4104	9936	19656	34236	54648	81864	116856	160596	214056	278208
2	8829	20007	37746	63504	98739	144909	203472	275886	363609	468099
3	15984	34452	62640	102492	155952	224964	311472	417420	544752	695412
4	26055	54000	95310	152415	227745	323730	442800	587385	759915	962820

3. Godement Formula

In this section, we explain the Godement formula and the calculations of dimension formulas. We set

$$H_\gamma^{k,j}(Z) = \mathrm{tr}\left[\rho_{k,j}(CZ+D)^{-1}\rho_{k,j}\left(\frac{\gamma \cdot Z - \bar{Z}}{2\sqrt{-1}}\right)^{-1}\rho_{k,j}(Y)\right],$$

$$Z = \begin{pmatrix} z_1 & z_{12} \\ z_{12} & z_2 \end{pmatrix}, X = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix}, Y = \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix},$$

$$dZ = \det(Y)^{-3} dX dY, \quad dX = dx_1 dx_{12} dx_2, \quad dY = dy_1 dy_{12} dy_2,$$

for $Z = X + \sqrt{-1}Y \in \mathfrak{H}_2$, $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2;\mathbb{R})$. Godement gave the following formula (cf. [5, Expose 10, Théorème 8]).

Theorem 3.1 (Godement). *If $k \geq 5$, then*

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma) = \frac{c_{k,j}}{\sharp(Z(\Gamma))}\int_{\Gamma\backslash\mathfrak{H}_2}\sum_{\gamma\in\Gamma} H_\gamma^{k,j}(Z)dZ,$$

where $c_{k,j} = 2^{-6}\pi^{-3}(k-2)(j+k-1)(j+2k-3)$. $Z(\Gamma)$ is the center of Γ, $\sharp(Z(\Gamma))$ is the order of $Z(\Gamma)$.

We shall remark on the constant $c_{k,j}$. In [5], the constant $c_{k,j}$ was calculated for the only scalar valued case ($j = 0$), not for the vector valued case. In [12], we easily see that the constant $c_{k,j}$ is equal to (formal degree)/$(j+1) \times$ constant, where the constant is independent of the signature $(k+j, k)$. Furthermore, from [6], we have an explicit form of the formal degree. Hence we get explicitly the constant $c_{k,j}$.

We give the corollary of Theorem 3.1. We can easily show the following corollary from the equality $H^{k,j}_{g^{-1}\gamma g}(Z) = H^{k,j}_{\gamma}(g \cdot Z)$ $(g \in Sp(2;\mathbb{R}))$ and the normality of $\Gamma^{(*)}(N)$ in $\Gamma^{(*)}(1)$.

Corollary 3.2. *If $k \geq 5$ and $N \geq 3$, then*

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma^{(*)}(N)) = 2^{-1} c_{k,j} [\Gamma^{(*)}(1) : \Gamma^{(*)}(N)] \int_{F^{(*)}} \sum_{\gamma \in \Gamma^{(*)}(N)} H^{k,j}_{\gamma}(Z) dZ,$$

where the notation $\Gamma^{(*)}(N)$ means that $\Gamma(N)$ or $\Gamma^*(N)$, and $F^{(*)}$ is the fundamental domain of $\Gamma^{(*)}(1)$ in $\mathfrak{H}_2$.

For a subset S of Γ, we put

$$I(S) = \frac{c_{k,j}}{\sharp(Z(\Gamma))} \int_{\Gamma \backslash \mathfrak{H}_2} \sum_{\gamma \in S} H^{k,j}_{\gamma}(Z) dZ.$$

We call this value $I(S)$ the contribution of S to the dimension formula. We put

$$\begin{aligned} \Pi_r &= \left\{ \gamma \in \Gamma(N)\,;\, \gamma \text{ is } \Gamma(1)\text{-conjugate to } \begin{pmatrix} I_2 & u \\ 0 & I_2 \end{pmatrix},\ \mathrm{rank}(u) = r,\ {}^t u = u \right\}, \\ \Pi_0^* &= \{I_2\}, \\ \Pi_2^* &= \left\{ \gamma \in \Gamma^*(N)\,;\, \gamma \text{ is } \Gamma^*(1)\text{-conjugate to } \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ u \neq 0,\ \mathrm{tr}(u) = 0 \right\}. \end{aligned}$$

In Section 3, in case of $\Gamma(N)$ and $\Gamma^*(N)$ $(N \geq 3)$, we prove vanishing of the contributions other than these Π_r and Π_r^*. Hence for $N \geq 3$, we have

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma(N)) = I(\Pi_0) + I(\Pi_1) + I(\Pi_2), \quad \dim_{\mathbb{C}} S_{k,j}(\Gamma^*(N)) = I(\Pi_0^*) + I(\Pi_2^*).$$

In Section 4, we calculate explicitly the contributions of Π_r and Π_r^*. So we get our main results.

4. Vanishing Part

In this section, we explain the point of calculation of the vanishing for the contributions of the elements other than Π_r and Π_r^*.

We calculate the vanishing part by the Selberg trace formula. However it is well-known that we can not exchange directly the integral and the infinite sum of $H^{k,j}_{\gamma}(Z)$, because $\sum_{\gamma \in \Gamma^{(*)}(N)} \int_{F^{(*)}} |H^{k,j}_{\gamma}(Z)| dZ$ is not convergent. Hence we need some calculation techniques as Morita [13].

If X is a positive definite symmetric matrix over $\mathbb{R}$, then we write $X > 0$. Let $\Omega_2 = \{X \in M(2;\mathbb{R})\,;\, {}^t X = X,\ X > 0\}$. If $X - Y > 0$ $(X, Y \in \Omega_2)$, then we write $X > Y$. We take an arbitrary constant $\mu (> 0)$, and set $\mathfrak{H}_2(\mu) = \{X + \sqrt{-1}Y \in \mathfrak{H}_2 : Y > \mu I_2\}$. We proved the following lemma in [18].

Lemma 4.1. *Let $\gamma \in \Gamma$ and $Z \in \mathfrak{H}_2(\mu)$. Then there exists a constant C, which depends only on j and μ, such that*

$$|H_\gamma^{k,j}(Z)| < C_{j,\mu} \times |H_\gamma^{k,0}(Z)|.$$

The constant $C_{j,\mu}$ is independent of γ and Z.

By Lemma 4.1, we can reduce the problems of absolute convergences of vector valued case to those of the scalar valued case. Therefore we can generalize Morita's method [13] to the vector valued case, and calculate the vanishing part. Let $\gamma \in \Gamma^{(*)}(N)$ $(N \geq 3)$ and $\gamma \notin \Pi_r^{(*)}$. From the results of [1], [13] and the above lemma, we can express the contribution of the $\Gamma^{(*)}(1)$-conjugacy classes of γ as

$$\lim_{s \to +0} \int_{F_{\gamma,s}^{(*)}} H_\gamma^{k,j}(Z)dZ,$$

where $F_{\gamma,s}^{(*)}$ is the certain domain satisfying $\lim_{s\to+0} F_{\gamma,s}^{(*)} = F_\gamma^{(*)}$ (see, [1][13]) and $F_\gamma^{(*)}$ is the fundamental domain of the centralizer of γ. Furthermore we see that the integral

$$\int_{F_{\gamma,s}^{(*)}} H_\gamma^{k,0}(Z)dZ = \sum_{m,l\in\mathbb{Z}_{\geq 0},\, l+5\leq m\leq j+k} \int_{D_s} \left\{ \int_{-\infty}^{\infty} (f_1(P)p + f_2(P))^{-m} f_{l,m}(P)p^l dp \right\} dP,$$

where $F_{\gamma,s}^{(*)} \cong (-\infty,\infty) \times D_s$, $dZ = dpdP$, f_1, f_2, $f_{l,m}$ are polynomials of P, and $f_1(P)p + f_2(P) \neq 0$ $({}^\forall(p,P) \in (-\infty,\infty) \times D_s)$ (cf. [1], [3], [13]). From the partial integration and $\int_{-\infty}^{\infty}(ap+b)^{-m}dp = [a^{-1}(-m+1)^{-1}(ap+b)^{-m+1}]_{-\infty}^{\infty} = 0$, we see that the contribution is zero.

Theorem 4.2. *Let $\gamma \in \Gamma^{(*)}(N)$ $(N \geq 3)$ and $\gamma \notin \Pi_r^{(*)}$. The contribution of $\Gamma^{(*)}(1)$-conjugacy classes of γ to the dimension formula is zero.*

5. Non-vanishing Part

In this section, we calculate explicitly the contributions of Π_r and Π_r^*.

5.1. Contribution of Π_0 and Π_0^*.

From Corollary 3.2 and $H_{I_4}^{k,j}(Z) = j+1$, we get

$$I(\Pi_0^{(*)}) = 2^{-1}c_{k,j}[\Gamma^{(*)}(1) : \Gamma^{(*)}(N)](j+1)\int_{F^{(*)}} dZ.$$

The volume of the fundamental domain for $\Gamma(1)$ (resp. $\Gamma^*(1)$) was given explicitly by Siegel [16] (resp. Arakawa [1]). So we get the contributions

$$\begin{aligned} I(\Pi_0) &= [\Gamma(1) : \Gamma(N)] \times 2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3), \\ I(\Pi_0^*) &= [\Gamma^*(1) : \Gamma^*(N)] \times 2^{-8}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \\ &\quad \times \prod_{p|D(\mathbf{B})} (p-1)(p^2+1). \end{aligned}$$

5.2. **Fourier transformation.** We assume that r is equal to 1 or 2. We put $V_r = \{x \in M(r;\mathbb{R})\,;\, {}^t x = x\}$, $\Omega_r = \{x \in V_r\,;\, x > 0\}$. For $x \in V_r$, we put

$$f_r^*(x) = \mathrm{tr}\left[\rho_{k,j}\begin{pmatrix} 1-\sqrt{-1}x & 0 \\ 0 & 1\end{pmatrix}^{-1}\right] \; (r=1), \quad \mathrm{tr}\left[\rho_{k,j}(I_2 - \sqrt{-1}x)^{-1}\right] \; (r=2).$$

For $x \in \Omega_1$, we set

$$f_1(x) = \sum_{i=0}^{j} (2\pi)^{k+i}\Gamma(k+j)^{-1} x^{k+i-1}\exp(-2\pi x),$$

where $\Gamma(s)$ is the Gamma function. For $x \notin \Omega_1$, we set $f_1(x) = 0$. The spherical polynomial $\Phi_m(x)$ for $m = (m_1, m_2) \in \mathbb{Z}_{\geq 0}\,(m_1 \geq m_2)$ is defined by $\Phi_m(x) = \int_{SO(2;\mathbb{R})} \Delta_m({}^t gxg)dg$, where $\Delta_m(x) = x_1^{m_1-m_2}\det(x)^{m_2}$ and dg is the Haar measure on $SO(2;\mathbb{R})$ normalized by $\int_{SO(2;\mathbb{R})} dg = 1$. Since $\mathrm{tr}(\rho_{k,j}(x))$ is invariant for the action $x \mapsto {}^t gxg$ $(g \in SO(2;\mathbb{R}))$, we see that $\mathrm{tr}(\rho_{k,j}(x)) = \sum_{m_1+m_2=2k+j,\, m_2 \geq k} a_m\Phi_m(x)$ $(a_m \in \mathbb{R})$ (cf. [4]). For $x \in \Omega_2$, we set

$$f_2(x) = \sum_{m_1+m_2=2k+j,\, m_2\geq k} \frac{(2\pi)^{-(1/2)+m_1+m_2}a_m}{\Gamma(m_1)\Gamma(m_2-2^{-1})}\Phi_m(x)\det(x)^{-3/2}\exp(-2\pi\mathrm{tr}(x)).$$

For $x \notin \Omega_2$, we set $f_2(x) = 0$. We denote by dx the Lebesgue measure on V_r. As for the scalar valued case $(j = 0)$, the following lemma is due to Shintani [14] and Siegel [15].

Lemma 5.1. (i) *If* $-1 < \mathrm{Re}(s) < k - r$, *then the integral* $\int_{V_r} f_r^*(x)|\det(x)|^s dx$ *is absolutely convergent.*
(ii) *If* $k > (r-1)/2$, *then we get* $\int_{V_r} f_r(x)\exp(2\pi i\mathrm{tr}(xy))dx = f_r^*(y)$. *This integral is absolutely convergent.*

We need the following lemma to calculate the contributions. From [5, Expose 6, Théorème 6], we easily get the following lemma.

Lemma 5.2. *Suppose* $k > 2$. *Then we have*

$$f_2(x) = \begin{cases} 2^{-5+2k+j}c_{k,j}^{-1}\mathrm{tr}(\rho_{k,j}(x)H_{k,j}^{-1})\det(x)^{-3/2}\exp(-2\pi\mathrm{tr}(x)) & (x \in \Omega_2) \\ 0 & (x \notin \Omega_2)\end{cases}$$

where $H_{k,j} = \int_{\Omega_2}\rho_{k,j}(x)\exp(-\pi\mathrm{tr}(x))\det(x)^{-3}dx$.

5.3. **Zeta integrals.** We define the zeta integral $Z(P_r, L_r, s)$ by

$$Z(P_r, L_r, s) = \int_{G_+/D} \det(g)^{2s} \sum_{x \in L_r - L_r\cap\{x\in V_r : \det(x)=0\}} P_r({}^t gxg)dg$$

where P_r is a function on V_r, $G_+ = \{g \in GL(r;\mathbb{R}); \det(g) > 0\}$, $D = SL(r;\mathbb{Z})$ or $\mathfrak{O}^\times$ (the unit group with norm 1 of $\mathfrak{O}$), L_r is a D-invariant lattice of V_r, dg is the Haar measure on G_+ defined by $\det(g)^{-r}\prod_{1\leq i,j\leq r} dg_{ij}$.

For the case $D = SL(r;\mathbb{Z})$, we set $L_r = \{x \in M(r;\mathbb{Z});\, {}^t x = x\}$ or its dual lattice. For the case $D = \mathfrak{O}^\times$, we set $L_2 = \{x \in \mathfrak{O}\,;\, \mathrm{tr}(x) = 0\}$ or its dual lattice. In

[18], we have proved the convergence, the functional equation and the meromorphic continuous of the zeta integral. The following proposition is a generalization of the results of [14] and [1] of the scalar valued case.

Proposition 5.3. (i) *The integral* $Z(f_r, L_r, s)$ *is absolutely convergent if* $\mathrm{Re}(s) > (r+1)/2$ *and* $\mathrm{Re}(k+s) > r$. *The integral* $Z(f_r, L_r, s)$ *is a meromorphic function of* s *on* $\mathbb{C}$.
(ii) *The case* $D = SL(r;\mathbb{Z})$. *If* $\mathrm{Re}(s) > (r-1)/2$ *and*

$$\begin{cases} k > 1, \ \mathrm{Re}(s) < k & \text{for } r = 1 \\ k > 4, \ 2\mathrm{Re}(s) < k & \text{for } r = 2 \end{cases},$$

then the integral $Z(f_r^*, L_r^*, s)$ *is absolutely convergent. The integral* $Z(f_r^*, L_r^*, s)$ *is a meromorphic function of* s *on* $\mathbb{C}$.
(ii)' *The case* $D = \mathfrak{O}^\times$. *If* $0 < \mathrm{Re}(s) < k - 1/2$, *then the integral* $Z(f_r^*, L_r^*, s)$ *is absolutely convergent. The integral* $Z(f_r^*, L_r^*, s)$ *is a meromorphic function of* s *on* $\mathbb{C}$.
(iii) *We have the functional equation*

$$Z(f_r(x), L_r, s) = \mathrm{vol}(L_r)^{-1} Z(f_r^*(x), L_r^*, (r+1)/2 - s).$$

5.4. **Contributions of Π_1, Π_2 and Π_2^*.**

Theorem 5.4. *If* $k \geq 5$, *then we obtain*

$$\begin{aligned} I(\Pi_1) &= [\Gamma(1):\Gamma(N)] \times (-1)2^{-6}3^{-2}(j+1)(j+2k-3)N^{-2}, \\ I(\Pi_2) &= [\Gamma(1):\Gamma(N)] \times 2^{-5}3^{-1}(j+1)N^{-3}, \\ I(\Pi_2^*) &= [\Gamma^*(1):\Gamma^*(N)] \times 2^{-4}3^{-1}(j+1)N^{-3} \prod_{p|D(\mathbf{B})}(p-1). \end{aligned}$$

Proof. We put $L_r = \{x \in M(r;Z);\, {}^t x = x\}$ in case of $\Gamma(N)$, $L_r = \{x \in \mathfrak{O};\, \mathrm{tr}(x) = 0\}$ in case of $\Gamma^*(N)$. By the method of [14, Section 3, Chapter 2] and Proposition 5.3 (ii), we get

$$I(\Pi_r^{(*)}) = c_{k,j} \times c^{(*)}(r) \times [\Gamma^{(*)}(1):\Gamma^{(*)}(N)] \times Z(f_r^*, L_r, 2 - 2^{-1}(r-1)),$$

where $c(1) = 2 \times 3^{-1}N^{-2}\pi$, $c(2) = 2^3N^{-3}\pi^{-1}$, $c^*(2) = 2^3N^{-3}\pi^{-1}D(\mathbf{B})$. By the functional equation, we get

$$Z(f_r^*, L_r, 2 - 2^{-1}(r-1)) = \mathrm{vol}(L_r^*) \times Z(f_r, L_r^*, r-2),$$

where $\mathrm{vol}(L_r^*) = 1$ for Π_1, 2^{-1} for Π_2, $2^{-1}D(\mathbf{B})^{-1}$ for Π_2^*. Furthermore we have

$$Z(f_r, 2N^{-1}L_r^*, r-2) = \xi_r^{(*)}(r-2) \times P_r.$$

Here $2 \times \xi_1(s)$ is the Riemann zeta function, $\xi_2^{(*)}(s)$ is zeta functions of quadratic forms, $\xi_2(s)$ is defined in [14], $\xi_2^*(s)$ is defined in [1], and we set the integrals

$$P_1 = \int_{\Omega_1} f_1(x)x^{-2}dx, \quad P_2 = \int_{\Omega_2} f_2(x)\det(x)^{-3/2}dx.$$

We know $\xi_1(-1) = -1/24$. The special value $\xi_2(0) = 2^{-5}3^{-1}\pi$ was given in [14]. The special value $\xi_2^*(0) = 2^{-4}3^{-1}\pi\prod_{p|D(\mathbf{B})}(p-1)$ was given in [1]. By simple calculation, we get $P_1 = (2\pi)^2(j+1)(k-2)^{-1}(j+k-1)^{-1}$. By Lemma 5.2, we calculate

$$\begin{aligned} P_2 &= 2^{-5+2k+j}c_{k,j}^{-1}\int_{\Omega_2}\mathrm{tr}(\rho_{k,j}(x)H_{k,j}^{-1})\exp(-2\pi\mathrm{tr}(x))\det(x)^{-3}dx \\ &= 2^{-2}c_{k,j}^{-1}\int_{\Omega_2}\mathrm{tr}(\rho_{k,j}(x)H_{k,j}^{-1})\exp(-\pi\mathrm{tr}(x))\det(x)^{-3}dx \\ &= 2^{-2}c_{k,j}^{-1}\mathrm{tr}\left\{\left(\int_{\Omega_2}\rho_{k,j}(x)\exp(-\pi\mathrm{tr}(x))\det(x)^{-3}dx\right)H_{k,j}^{-1}\right\} \\ &= 2^{-2}c_{k,j}^{-1}\mathrm{tr}(H_{k,j}H_{k,j}^{-1}) = 2^{-2}c_{k,j}^{-1}(j+1). \end{aligned}$$

So we get explicitly the contributions of Π_1, Π_2 and Π_2^*. □

Appendix A. Dimension formula for $\Gamma(1)$

The following dimension formula is due to [17]. For the scalar valued case ($j=0$), the dimension was also calculated in [11] and [7]. Let i, ρ, ω and σ be $\sqrt{-1}$, $e^{2\pi i/3}$, $e^{2\pi i/5}$ and $e^{\pi i/6}$ respectively. We denote $\mathrm{tr}_{\mathbb{Q}[\alpha]/\mathbb{Q}}$ by tr_α for an algebraic number α. We remark on $\dim_{\mathbb{C}} S_{k,j}(Sp(2;\mathbb{Z})) = 0$ if j is odd.

Theorem A.1 (R. Tsushima). *$k \geq 5$, $j>0$ or $k\geq 4$, $j=0$. j is even.*

$$\begin{aligned} &\dim_{\mathbb{C}} S_{k,j}(Sp(2;\mathbb{Z})) = \\ &2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) - 2^{-5}3^{-2}(j+1)(j+2k-3) \\ &+2^{-4}3^{-1}(j+1) \\ &+(-1)^k(2^{-7}3^{-2}7(k-2)(j+k-1) - 2^{-4}3^{-1}(j+2k-3) + 2^{-5}3) \\ &+(-1)^{j/2}(2^{-7}3^{-1}5(j+2k-3) - 2^{-3}) + (-1)^k(-1)^{j/2}2^{-7}(j+1) \\ &+\mathrm{tr}_i(i)^k(2^{-6}3^{-1}(i)(j+k-1) - 2^{-4}(i)) + \mathrm{tr}_i(-1)^k(i)^{j/2}2^{-5}(i+1) \\ &+\mathrm{tr}_i(i)^k(-1)^{j/2}(2^{-6}3^{-1}(k-2) - 2^{-4}) + \mathrm{tr}_i(-i)^k(i)^{j/2}2^{-5}(i+1) \\ &+\mathrm{tr}_\rho(-1)^k(\rho)^{j/2}3^{-3}(\rho+1) + \mathrm{tr}_\rho(\rho)^k(\rho)^{j/2}2^{-2}3^{-4}(2\rho+1)(j+1) \\ &-\mathrm{tr}_\rho(\rho)^k(-\rho)^{j/2}2^{-2}3^{-2}(2\rho+1) + \mathrm{tr}_\rho(-\rho)^k(\rho)^{j/2}3^{-3} \\ &+\mathrm{tr}_\rho(\rho)^{j/2}(2^{-1}3^{-4}(1-\rho)(j+2k-3) - 2^{-1}3^{-2}(1-\rho)) \\ &+\mathrm{tr}_\rho(\rho)^k(2^{-3}3^{-4}(2+\rho)(j+k-1) - 2^{-2}3^{-3}(6+5\rho)) \\ &-\mathrm{tr}_\rho(-\rho)^k(2^{-3}3^{-3}(2+\rho)(j+k-1) - 2^{-2}3^{-2}(2+\rho)) \\ &+\mathrm{tr}_\rho(\rho)^k(\rho)^j(2^{-3}3^{-4}(1-\rho)(k-2) + 2^{-2}3^{-3}(-5+\rho)) \\ &+\mathrm{tr}_\rho(-\rho)^k(\rho)^j(2^{-3}3^{-3}(1-\rho)(k-2) - 2^{-2}3^{-2}(1-\rho)) \\ &+\mathrm{tr}_\omega(\omega)^k(\omega^4)^{j/2}5^{-2} - \mathrm{tr}_\omega(\omega)^k(\omega^3)^{j/2}5^{-2}\omega^2 \\ &+\mathrm{tr}_\sigma(\sigma^7)^k(-1)^{j/2}2^{-3}3^{-2}(\sigma^2+1) - \mathrm{tr}_\sigma(\sigma^7)^k(\sigma^8)^{j/2}2^{-3}3^{-2}(\sigma+\sigma^3) \end{aligned}$$

Numerical example of $\dim_{\mathbb{C}} S_{k,j}(Sp(2;\mathbb{Z}))$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	2	0	3
2	0	0	0	0	0	0	0	0	0	0	1	0	2	0	2	0	3
4	0	0	0	0	0	0	1	0	1	0	2	1	3	1	4	2	6
6	0	0	0	0	1	0	1	1	2	1	3	2	5	3	7	4	9
8	0	0	0	0	1	1	2	1	3	2	5	4	7	5	9	7	13
10	0	0	0	0	0	1	2	1	3	2	5	5	8	6	11	9	15

Appendix B. Dimension formula for $\Gamma^*(1)$

In order to get the dimension formula for $\Gamma^*(1)$, we need to calculate explicitly the contributions of elliptic elements and quasi-unipotent elements. Because $\Gamma^{(*)}(N)$ $(N \geq 3)$ have no such elements (cf. [13], [1] and [6]). We can get easily the contributions of elliptic elements by the results of [12], [8] and [9]. So we have only to calculate explicitly the orbital integrals of quasi-unipotent elements (we calculated it explicitly in [18]). For the scalar valued case, the orbital integrals were calculated in [7].

For the scalar valued case ($j = 0$), the following dimension formula was given by Hashimoto [8]. We generalized it to the vector valued case in [18]. We also remark on $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1)) = 0$ if j is odd.

Theorem B.1. $k \geq 5$. *j is even.*

$$\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1)) = \sum_{i=1}^{12} H_i$$

where H_i is the total contribution of elements $\Gamma^*(1)$ with principal polynomial $f_i(\pm x)$. and are as follows:

$$H_1 = H_1^e + H_1^u, \qquad H_1^u = 2^{-3}3^{-1}(j+1) \prod_{p|D(\mathbf{B})} (p-1),$$

$$H_1^e = 2^{-7}3^{-3}5^{-1}(j+1)(k-2)(j+k-1)(j+2k-3) \times \prod_{p|D(\mathbf{B})} (p-1)(p^2+1).$$

$$H_2 = 2^{-7}3^{-2}(-1)^k(j+k-1)(k-2) \prod_{p|D(\mathbf{B})} (p-1)^2 \times \begin{cases} 7 & \text{if } \ 2 \nmid D(\mathbf{B}) \\ 13 & \text{if } \ 2|D(\mathbf{B}) \end{cases}.$$

$$H_3 = 2^{-5}3^{-1}\left\{(j+k-1)\sin(k-2)\frac{\pi}{2} - (k-2)\sin(j+k-1)\frac{\pi}{2}\right\} \prod_{p|D(\mathbf{B})} (p-1)\left(1 - \left(\frac{-1}{p}\right)\right).$$

$$\begin{aligned} H_4 \;&=\; 2^{-3}3^{-3}\left\{(j+k-1)\sin(k-2)\frac{2\pi}{3} - (k-2)\sin(j+k-1)\frac{2\pi}{3}\right\} \\ &\quad \times \left(\sin\frac{2\pi}{3}\right)^{-1} \times \prod_{p|D(\mathbf{B})} (p-1)\left(1 - \left(\frac{-3}{p}\right)\right). \end{aligned}$$

$$H_5 = 2^{-3}3^{-2}\left\{(j+k-1)\sin(k-2)\frac{\pi}{3} - (k-2)\sin(j+k-1)\frac{\pi}{3}\right\} \times \left(\sin\frac{\pi}{3}\right)^{-1} \times \prod_{p|D(\mathbf{B})}(p-1)\left(1-\left(\frac{-3}{p}\right)\right).$$

$$H_6 = H_6^{pe} + H_6^{e}, \qquad H_6^{pe} = -2^{-3}(-1)^{j/2}\prod_{p|D(\mathbf{B})}\left(1-\left(\frac{-1}{p}\right)\right),$$

$$H_6^e = 2^{-7}3^{-1}(-1)^{j/2+k}(j+1)\sum_{D_0|2D(\mathbf{B})}\prod_{q|D_0}(q-1)\times\prod_{p|2D(\mathbf{B})/D_0}\left(1-\left(\frac{-1}{p}\right)\right)\times A$$
$$+2^{-7}3^{-1}(-1)^{j/2}(j+2k-3)\sum_{D_e|2D(\mathbf{B})}\prod_{q|D_e}(q-1)\times\prod_{p|2D(\mathbf{B})/D_e}\left(1-\left(\frac{-1}{p}\right)\right)\times B,$$

where

$$A\,(\text{resp. } B) = \begin{cases} 3 & \text{if } \ 2\nmid D(\mathbf{B}),\ 2|D^* \\ 5 & \text{if } \ 2|D(\mathbf{B}),\ 2|D^*;\ \text{or } 2\nmid D(\mathbf{B}),\ 2\nmid D^* \\ 11 & \text{if } \ 2|D(\mathbf{B}),\ 2\nmid D^* \end{cases}$$

and $D^* = D_0$ (resp. D_e) runs through the set of divisors of $2D(\mathbf{B})$ which are the product of odd (resp. even) number of distinct primes.

$$H_7 = H_7^{pe} + H_7^{e},$$

$$H_7^{pe} = -2^{-1}3^{-1}\left(\sin(j+1)\frac{2\pi}{3}\right)\left(\sin\frac{2\pi}{3}\right)^{-1}\prod_{p|D(\mathbf{B})}\left(1-\left(\frac{-3}{p}\right)\right),$$

$$H_7^e = 2^{-3}3^{-3}(j+1)\left(\sin(j+2k)\frac{2\pi}{3}\right)\left(\sin\frac{2\pi}{3}\right)^{-1} \times \sum_{D_0|3D(\mathbf{B})}\prod_{q|D_0}(q-1)\times\prod_{p|3D(\mathbf{B})/D_0}\left(1-\left(\frac{-3}{p}\right)\right)\times A$$
$$+2^{-3}3^{-3}(j+2k-3)\left(\sin(j+1)\frac{2\pi}{3}\right)\left(\sin\frac{2\pi}{3}\right)^{-1} \times \sum_{D_e|3D(\mathbf{B})}\prod_{q|D_e}(q-1)\times\prod_{p|3D(\mathbf{B})/D_e}\left(1-\left(\frac{-3}{p}\right)\right)\times B,$$

where

$$A\,(\text{resp. } B) = \begin{cases} 1 & \text{if } \ 3|D^* \\ 4 & \text{if } \ 3\nmid D(\mathbf{B}),\ 3\nmid D^* \\ 16 & \text{if } \ 3|D(\mathbf{B}),\ 3\nmid D^* \end{cases}$$

and $D^* = D_0$ (resp. D_e) runs through the set of divisors of $3D(\mathbf{B})$ which are the product of odd (resp. even) number of distinct primes. We set

$$C(\mu,\nu) = c(\mu,\nu) + c(-\mu,\nu) + c(\mu,-\nu) + c(-\mu,-\nu),$$

where

$$c(\mu,\nu) = \frac{e^{\sqrt{-1}(k-2)\mu}e^{\sqrt{-1}(j+k-1)\nu} - e^{\sqrt{-1}(j+k-1)\mu}e^{\sqrt{-1}(k-2)\nu}}{(1-e^{2\sqrt{-1}\mu})(1-e^{2\sqrt{-1}\nu})(1-e^{\sqrt{-1}(\mu+\nu)})(e^{-\sqrt{-1}\mu}-e^{-\sqrt{-1}\nu})e^{-\sqrt{-1}(\mu+\nu)}}.$$

$$H_8 = 2^{-2}3^{-1}C\left(\frac{\pi}{2},\frac{2\pi}{3}\right)\prod_{p|D(\mathbf{B})}\left(1-\left(\frac{-1}{p}\right)\right)\left(1-\left(\frac{-3}{p}\right)\right).$$

$$H_9 = 2^{-1}3^{-2}C\left(\frac{2\pi}{3},\frac{\pi}{3}\right)\prod_{p|D(\mathbf{B}),p\neq 2}\left(1-\left(\frac{-3}{p}\right)\right)^2\times\begin{cases} 2 & \text{if } \ 2\nmid D(\mathbf{B}) \\ 5 & \text{if } \ 2|D(\mathbf{B}) \end{cases}.$$

$$H_{10} = 2^{-1}5^{-1}C\left(\frac{2\pi}{5},\frac{4\pi}{5}\right)\prod_{p|D(\mathbf{B})} 2\times \prod_{p\in D(-1;5)} 2\times \begin{cases} 0 & \text{if } \cup_{i=1}^{3} D(i;5)\neq\emptyset \\ 1 & \text{if } \cup_{i=1}^{3} D(i;5)=\emptyset,\ 5\not| D(\mathbf{B}) \\ 2 & \text{if } \cup_{i=1}^{3} D(i;5)=\emptyset,\ 5|D(\mathbf{B}) \end{cases},$$

where we set $D(i;j) = \{p|D(\mathbf{B});\, p\equiv i \pmod j\}$.

$$H_{11} = 2^{-3}C\left(\frac{\pi}{4},\frac{3\pi}{4}\right)\prod_{p|D(\mathbf{B}),p\neq 2} 2\times \prod_{p\in D(-1;8)} 2\times \begin{cases} 0 & \text{if } D(1;8)\neq\emptyset \\ 1 & \text{if } D(1;8)=\emptyset \end{cases}.$$

$$H_{12} = 0, \quad \text{if} \quad D(1;12)\neq\emptyset,$$

if the otherwise

$$\begin{aligned} H_{12} &= 2^{-2}3^{-1}\prod_{p|D(\mathbf{B})} 2\times \prod_{p\in D(-1;12)} 2\times \left(c\left(\frac{\pi}{6},\frac{5\pi}{6}\right)+c\left(-\frac{\pi}{6},-\frac{5\pi}{6}\right)\right)\times A \\ &+2^{-2}3^{-1}\prod_{p|D(\mathbf{B})} 2\times \prod_{p\in D(-1;12)} 2\times \left(c\left(\frac{\pi}{6},-\frac{5\pi}{6}\right)+c\left(-\frac{\pi}{6},\frac{5\pi}{6}\right)\right)\times B, \end{aligned}$$

where

(i) if $2\not| D(\mathbf{B})$, $3\not| D(\mathbf{B})$,

$$A\,(\text{resp. } B) = \begin{cases} 1/2 & \text{if } D(-1;12)\neq\emptyset \\ 0 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)} \\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)}, \end{cases}$$

(ii) if $2\not| D(\mathbf{B})$, $3|D(\mathbf{B})$,

$$A\,(\text{resp. } B) = \begin{cases} 3/4 & \text{if } D(-1;12)\neq\emptyset \\ 1/2 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)} \\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)}, \end{cases}$$

(iii) if $2|D(\mathbf{B})$, $3\not| D(\mathbf{B})$,

$$A\,(\text{resp. } B) = \begin{cases} 3/4 & \text{if } D(-1;12)\neq\emptyset \\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)} \\ 1/2 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)}. \end{cases}$$

(iv) if $6|D(\mathbf{B})$,

$$A\,(\text{resp. } B) = \begin{cases} 9/8 & \text{if } D(-1;12)\neq\emptyset \\ 5/4 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is even (resp. odd)} \\ 1 & \text{if } D(-1;12)=\emptyset,\ \sharp D(5;12) \text{ is odd (resp. even)}. \end{cases}$$

$$\left(\frac{-1}{p}\right) = \begin{cases} 0, & p=2, \\ 1, & p\equiv 1 \pmod 4, \\ -1, & p\equiv 3 \pmod 4, \end{cases} \qquad \left(\frac{-3}{p}\right) = \begin{cases} 0, & p=3, \\ 1, & p\equiv 1 \pmod 3, \\ -1, & p\equiv 2 \pmod 3. \end{cases}$$

Principal polynomials (see, [9]) :

$f_1(x) = (x-1)^4$, $f_1(-x)$, $f_2 = (x-1)^2(x+1)^2$, $f_3(x) = (x-1)^2(x^2+1)$, $f_3(-x)$, $f_4(x) = (x-1)^2(x^2+x+1)$, $f_4(-x)$, $f_5(x) = (x-1)^2(x^2-x+1)$, $f_5(-x)$, $f_6(x) = (x^2+1)^2$, $f_7(x) = (x^2+x+1)^2$, $f_8(x) = (x^2+1)(x^2+x+1)$, $f_8(-x)$, $f_9(x) = (x^2+x+1)(x^2-x+1)$, $f_{10}(x) = (x^4+x^3+x^2+x+1)$, $f_{10}(-x)$, $f_{11}(x) = (x^4+1)$, $f_{12}(x) = (x^4-x^2+1)$.

Numerical examples of $\dim_{\mathbb{C}} S_{k,j}(\Gamma^*(1))$.

(i) $D(\mathbf{B}) = 2 \times 3$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16
0	2	0	4	2	8	5	15	10	25	15	34	26	53
2	2	2	5	7	15	17	33	34	53	58	91	96	138
4	4	6	14	19	35	42	67	77	114	126	179	200	264
6	9	17	30	40	65	82	118	145	195	224	299	341	432
8	19	27	49	67	106	131	188	223	298	346	448	514	642

(ii) $D(\mathbf{B}) = 2 \times 5$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16
0	4	2	13	5	26	19	56	41	98	70	149	123	232
2	9	12	28	39	82	99	170	185	285	316	470	513	714
4	23	33	76	99	180	227	346	408	587	675	926	1051	1364
6	46	83	150	203	330	423	607	742	1004	1173	1534	1771	2228
8	88	141	246	347	532	684	955	1157	1522	1805	2302	2669	3298

(iii) $D(\mathbf{B}) = 3 \times 5$.

$j \diagdown k$	4*	5	6	7	8	9	10	11	12	13	14	15	16
0	9	8	34	29	86	85	183	178	331	318	536	531	828
2	30	52	117	170	311	405	640	775	1120	1324	1821	2100	2759
4	84	149	298	431	703	934	1357	1694	2316	2789	3644	4283	5387
6	174	323	574	834	1281	1702	2373	2985	3936	4757	6044	7136	8787
8	330	575	979	1416	2091	2756	3752	4681	6044	7305	9117	10746	13053

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