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Functional equations of Spherical functions

on p-adic homogeneous spaces

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§0 Introduction

Let $\mathbb{G}$ be a reductive linear algebraic group defined over k, and $\mathbb{X}$ be an affine algebraic variety defined over k which is $\mathbb{G}$-homogeneous, where and henceforth k stands for a non-archimedian local field of characteristic 0. The Hecke algebra $\mathcal{H}(G,K)$ of G with respect to K acts by convolution product on the space of $\mathcal{C}^\infty(K\backslash X)$ of K-invariant $\mathbb{C}$-valued functions on X, where K is a maximal compact open subgroup of $G=\mathbb{G}(k)$ and $X=\mathbb{X}(k)$.

A nonzero function in $\mathcal{C}^\infty(K\backslash X)$ is called *a spherical function on* X if it is a common $\mathcal{H}(G,K)$-eigen function.

Spherical functions on homogeneous spaces are an interesting object to investigate and a basic tool to study harmonic analysis on G-space X. They have been studied also as spherical vectors of distinguished models, Shalika functions and Whittaker-Shintani functions, and have a close relation to the theory of automorphic forms and representation theory. When $\mathbb{G}$ and $\mathbb{X}$ are defined over $\mathbb{Q}$, spherical functions appear in local factors of global objects, e.g. Rankin-Selberg convolutions and Eisenstein series (e.g. [CS], [Fl], [HS3], [Ja], [KMS], [Sf1]).

The theory of spherical functions also have applications to classical number theory, for example when X is the space of symmetric forms, alternating forms or hermitian forms, spherical functions can be considered as generating functions of local densities, and have been applied to obtain their explicit formulas (cf. [HS1], [HS2], [H1]-[H3]).

To obtain explicit expressions of spherical functions is one of basic problems. For the group cases, it has been done by I. G. Macdonald and afterwards by W. Casselman by a representation theoretical method (cf. [Ma], [Cas]). There are some results on homogeneous cases mainly for the case that the space of spherical functions attached to each Satake parameter is of dimension one (e.g. [CS], [KMS], [Of]). On the other hand, the author has given an expression of spherical functions of dimension not necessary one based on the data of the group G and functional equations of spherical functions ([H2, Proposition 1.9]). Hence the knowledge of functional equations is important to obtain explicit expressions of spherical functions.

We have investigated functional equations of spherical functions individually in a series of papers ([HS1], [H1], [H4]). Here we will show a unified method to obtain functional equations which is applicable to more general cases under the condition (AF) below, and explain that functional equations are reduced to those of p-adic local zeta functions of *small* prehomogeneous vector spaces. This method is a generalization of one in [H4, §3] used for the spherical homogeneous space Sp_2.

In order to state our main results, we prepare some notations.

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A full paper will be appeared in elsewhere. Because of the restriction of pages, the author deletes the proof here.

First, we introduce a notion of type (F) for a connected linear algebraic group $\mathbb{H}$ and an affine algebraic variety $\mathbb{Y}$ on which $\mathbb{H}$ acts, where everything is assumed to be defined over k. We denote by $\mathfrak{X}(\mathbb{H})$ the group of k-rational characters of $\mathbb{H}$, which is a free abelian group of finite rank. We set $\mathfrak{X}_0(\mathbb{H})$ for the subgroup consisting of characters corresponding to some relative $\mathbb{H}$-invariants on $\mathbb{Y}$, where a rational function f on $\mathbb{Y}$ defined over k is called *relative* $\mathbb{H}$*-invariant* if it satisfies, for some $\psi \in \mathfrak{X}(\mathbb{H})$,

$$f(g \cdot y) = \psi(g) f(y), \quad g \in \mathbb{H}.$$

When $f_i(y)$, $1 \leq i \leq n$, are relative $\mathbb{H}$-invariants on $\mathbb{Y}$ defined over k and the characters corresponding to them form a basis for $\mathfrak{X}_0(\mathbb{H})$, we say the set $\{ f_i(y) \mid 1 \leq i \leq n\}$ is *basic*, then every relative $\mathbb{H}$-invariant on $\mathbb{Y}$ defined over k is given in the following form:

$$c \cdot \prod_{i=1}^{n} f_i(y)^{e_i}, \qquad c \in k^{\times},\ e_i \in \mathbb{Z}.$$

We say $(\mathbb{H}, \mathbb{Y})$ is *of type* (F) if it satisfies the following conditions:

(F1) $\mathbb{Y}$ has only a finite number of $\mathbb{H}$-orbits. (Then $\mathbb{Y}$ has only one open $\mathbb{H}$-orbit $\mathbb{Y}^{op}$.)

(F2) For $y \in \mathbb{Y} \backslash \mathbb{Y}^{op}$, there exists some ψ in $\mathfrak{X}(\mathbb{H})$ whose restriction to the identity component of the stabilizer $\mathbb{H}_y$ is not trivial.

(F3) The index of $\mathfrak{X}_0(\mathbb{H})$ in $\mathfrak{X}(\mathbb{H})$ is finite.

(F4) A basic set of relative $\mathbb{H}$-invariants on $\mathbb{Y}$ can be taken from regular functions on $\mathbb{Y}$.

Hereafter, let $\mathbb{G}$ be a connected reductive linear algebraic group defined over k, K a maximal compact open subgroup of G, $\mathbb{B}$ a minimal parabolic subgroup of $\mathbb{G}$ defined over k satisfying $G = KB = BK$. The group $\mathbb{B}$ is not necessarily a Borel subgroup. For an algebraic set, we use the same ordinary letter for the set of k-rational points, e.g. $G = \mathbb{G}(k)$, $B = \mathbb{B}(k)$. We denote by $|\ |$ the absolute value on k normalized by $|\pi| = q^{-1}$, where π is a prime element of k and q is the cardinal number of the residue class field of k.

Let $(\mathbb{B}, \mathbb{X})$ be of type (F), $\{ f_i(x) \mid 1 \leq i \leq n\}$ a regular basic set of relative $\mathbb{B}$-invariants, and $\psi_i \in \mathfrak{X}_0(\mathbb{B})$ the character corresponding to $f_i(x)$ for each i, where $n = \mathrm{rank}(\mathfrak{X}(\mathbb{B}))$. The open $\mathbb{B}$-orbit $\mathbb{X}^{op}$ decomposes into a finite number of open B-orbits over k, which we write

$$\mathbb{X}^{op}(k) = \bigsqcup_{u \in J(X)} X_u.$$

For $x \in X$, $s \in \mathbb{C}^n$ and $u \in J(X)$, we consider

$$\omega_u(x; s) = \int_K |f(k \cdot x)|_u^{s+\varepsilon}\, dk, \tag{0.1}$$

where dk is the normalized Haar measure on K, $\varepsilon \in \mathbb{Q}^n$ determined from the modulus character δ of B by the relation

$$\prod_{i=1}^{n} |\psi_i(b)|^{\varepsilon_i} = \delta^{\frac{1}{2}}(b), \quad b \in B,$$

and

$$|f(x)|_u^{s+\varepsilon} = \begin{cases} \displaystyle\prod_{i=1}^{n} |f_i(x)|^{s_i+\varepsilon_i} & \text{if } x \in X_u, \\ 0 & \text{otherwise.} \end{cases} \tag{0.2}$$

The right hand side of (0.1) is absolutely convergent if $\mathrm{Re}(s_i) \geq -\varepsilon_i$, $1 \leq i \leq n$, analytically continued to a rational function of $q^{s_1}, \ldots, q^{s_n}$, and becomes an $\mathcal{H}(G, K)$-common eigenfunction on X (cf. [H2,

§1]). Hence $\omega_u(x;s)$ is a spherical functions on X, and for generic s they are linearly independent for $u \in J(X)$.

Let W be the relative Weyl group of G with respect to T, where $\mathbb{T}$ is a maximal k-split torus contained in $\mathbb{B}$. The group W acts on $s \in \mathbb{C}^n$ through the canonical action on $\mathfrak{X}(\mathbb{B})$ and the identification $\mathfrak{X}(\mathbb{B}) \otimes_{\mathbb{Z}} \mathbb{C} \cong \mathbb{C}^n$. If $(\mathbb{B}, \mathbb{X})$ is of type (F), there should be functional equations between $\omega_u(x;s)$'s with respect to the action of W. In this present paper, we will show the functional equation between s and $w_\alpha(s)$ for a simple root α in case there is a representation ρ like (0.3) below.

For a simple root α, let $\mathbb{P}$ be the standard parabolic subgroup $\mathbb{P}_{\{\alpha\}}$ in the sense of [Bo, 21.11]. We consider a k-rational representation $\rho : \mathbb{P} \longrightarrow R_{k'/k}(GL_2)$ satisfying

$$\begin{aligned} &\rho(\mathbb{P}) = R_{k'/k}(GL_2) \text{ or } R_{k'/k}(SL_2), \quad \rho(w_\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ &\rho^{-1}(\mathbb{B}_2) \subset \mathbb{B}, \quad \rho(\mathbb{P}(\mathcal{O})) \supset R_{k'/k}(SL_2)(\mathcal{O}), \end{aligned} \tag{0.3}$$

where k' is a finite unramified extension of k, $R_{k'/k}$ is the restriction functor of base field, $w_\alpha \in N_G(T)$ is a representative of the reflection in W attached to α, and $\mathbb{B}_2$ is the Borel subgroup of $\rho(\mathbb{P})$ consisting of upper triangular matrices.

Chevalley groups are typical examples which have ρ as above for $k = k'$ (cf. [Sf1, §4.1]).

Now we assume that

(AF) $(\mathbb{B}, \mathbb{X})$ is of type (F) and there is a k-rational representation ρ satisfying (0.3) for a simple root α.

For each $u \in J(X)$, set $J_u = \{\nu \in J(X) \mid P \cdot X_\nu = P \cdot X_u\}$. Denote by e the group index $[\mathfrak{X}(\mathbb{B}) : \mathfrak{X}_0(\mathbb{B})]$ and by d the extension degree of k'/k. Then, our main results are the following.

Theorem 1 *We have the following functional equation:*

$$\omega_u(x;s) = \frac{1 - q^{-2d - \sum_i e_i s_i}}{1 - q^{-2d - \sum_i e_i w_\alpha(s)_i}} \times \sum_{\nu \in J_u} \gamma_{u\nu}(s) \cdot \omega_\nu(x; w_\alpha(s)), \tag{0.4}$$

where $\gamma_{u\nu}(s)$'s are rational functions of $q^{\frac{s_1}{e}}, \ldots, q^{\frac{s_n}{e}}$ and $e_i \in \mathbb{N} \cup \{0\}$, $1 \le i \le n$ are determined explicitly ($e_i = \deg_v \widetilde{f}_i(x,v)$ below).

Fix an element $x_u \in X_u$ and denote by $\mathbb{P}_u$ the stabilizer of x_u in $\mathbb{P}$. The group $\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1)$ acts on $\mathbb{V} = R_{k'/k}(M_{21})$ by $(g,r) \cdot v = gvr^{-1}$, where we consider $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$ is realized in $(GL_{2d} \times GL_d, M_{2d,d})$. Let $v_0 \in \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in V = \mathbb{V}(k) \cong M_{21}(k')$. There are regular relative $(\mathbb{P} \times R_{k'/k}(GL_1))$-invariants $\left\{ \widetilde{f}_i(x,v) \;\middle|\; 1 \le i \le n \right\}$ on $\mathbb{X} \times \mathbb{V}$ satisfying $\widetilde{f}_i(x, v_0) = f_i(x)$ for each i (cf. §1).

The following theorem shows that the above functional equations are reduced to those for "small" prehomogeneous vector spaces.

Theorem 2 (i) *The space $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$ is a prehomogeneous vector space defined over k with open orbit $\rho(\mathbb{P}_u) v_0 R_{k'/k}(GL_1)$, which decomposes over k as*

$$\left(\rho(\mathbb{P}_u) v_0 R_{k'/k}(GL_1)\right)(k) = \bigsqcup_{\nu \in J_u} \rho(P_u p_\nu) v_0 k'^{\times},$$

where $p_\nu \in P$ satisfying $p_\nu^{-1} \cdot x_u \in X_\nu$.

(ii) *The zeta integral of the above prehomogeneous space has the following functional equation over k:*

$$\int_V \mathcal{F}_V(\phi)(v) \left| \widetilde{f}(x_u, v) \right|_u^{s+\varepsilon} dv = \sum_{\nu \in J_u} \gamma_{u\nu}(s) \int_V \phi(v) \left| \widetilde{f}(x_u, v) \right|_\nu^{w_\alpha(s)+\varepsilon} dv, \quad \phi \in \mathcal{S}(V).$$

Here ε and $\gamma_{u\nu}(s)$ are the same as in Theorem 1, dv is the normalized Haar measure on V,

$$\left|\widetilde{f}(x_u, v)\right|_{\nu}^{s} = \begin{cases} \prod_{i=1}^{n} \left|\widetilde{f}_i(x_u, v)\right|^{s_i} & \text{if } v \in \rho(P_u p_\nu) v_0 {k'}^{\times} \\ 0 & \text{otherwise,} \end{cases}$$

and the Fourier transform $\mathcal{F}_V(\phi)$ is defined by

$$\mathcal{F}_V(\phi)(v) = \int_V \eta({}^t v \sigma w)\phi(w) dw, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where η is an additive character on k' of conductor $\mathcal{O}_{k'}$.

(iii) *The identity component of $\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1)$ is isomorphic to $R_{k'/k}(GL_1 \times GL_1)$ over the algebraic closure of k.*

The above results explain how functional equations of spherical functions occur and how to calculate them. If there is a representation ρ as in (0.3) for each simple root, then we will obtain functional equations for $\omega_u(x; s)$ with respect to the whole Weyl group, which are reduced to those of p-adic local zeta functions of small prehomogeneous vector spaces isomorphic to $R_{k'/k}(GL_1 \times GL_1, M_{21})$ over the algebraic closure of k. Then, we could expect to have explicit expressions of spherical functions by using a method introduced in [H2, §1].

One would be able to consider in this line spherical functions on homogeneous spaces given by Chevalley groups and their involutions as typical cases, which would be discussed in forthcoming papers.

§1 Preliminaries

Set $\widetilde{\mathbb{X}} = \mathbb{X} \times \mathbb{V}$ and $\widetilde{\mathbb{P}} = \mathbb{P} \times R_{k'/k}(GL_1)$, where $\mathbb{V} = R_{k'/k}(M_{2,1})$ defined over k, and consider the following $\widetilde{\mathbb{P}}$-action on $\widetilde{\mathbb{X}}$:

$$(p, t) \cdot (x, v) = (p \cdot x, \rho(p) v t^{-1}), \quad (p, t) \in \widetilde{\mathbb{P}},\ (x, v) \in \widetilde{\mathbb{X}}. \tag{1.1}$$

Here we identify k' with its image by the regular representation in $M_d(k)$ and realize $R_{k'/k}(GL_2)$ (resp. $\mathbb{V}$) in $GL_{2d}(\overline{k})$ (resp. $M_{2d,d}(\overline{k})$), where $d = [k' : k]$ and $\overline{k}$ is the algebraic closure of k. We note here that we may identify as $\widetilde{P} = P \times GL_2(k')$ and $V = {k'}^2$.

We regard $\mathbb{B}$ as a subgroup of $\widetilde{\mathbb{P}}$ by the embedding

$$\mathbb{B} \longrightarrow \widetilde{\mathbb{P}},\ b \longmapsto (b, \rho(b)_1), \tag{1.2}$$

where $\rho(b)_1 \in R_{k'/k}(GL_1)$ is the upper left d by d block of $\rho(b) \in R_{k'/k}(GL_2)$. Then, one can identify $\mathbb{B}$ as the stabilizer subgroup of $\widetilde{\mathbb{P}}$ at $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in $V = \mathbb{V}(k)$.

Lemma 1.1 *We have the following isomorphism:*

$$\begin{array}{rcl} \mathfrak{X}(\widetilde{\mathbb{P}}) \cong \mathfrak{X}(\mathbb{P}) \times \mathfrak{X}(R_{k'/k}(GL_1)) & \xrightarrow{\sim} & \mathfrak{X}(\mathbb{B}) \\ (\psi_1, \psi_2) & \longmapsto & [p \longmapsto \psi_1(p)\psi_2(\rho(p)_1)] \end{array} \tag{1.3}$$

Proposition 1.2 (i) *The space $(\widetilde{\mathbb{P}}, \widetilde{\mathbb{X}})$ is of type (F).*

(ii) *The set of open B-orbits in X corresponds bijectively to the set of open $\widetilde{P}$-orbits in $\widetilde{X}$ by the map $B \cdot x \longmapsto \widetilde{P} \cdot (x, v_0)$.*

Now let $\left\{ \widetilde{f}_i(x,v) \,\middle|\, 1 \le i \le n \right\}$ be the basic set of relative $\widetilde{\mathbb{P}}$-invariants, which are regular on $\widetilde{\mathbb{X}}$ and satisfy $f_i(x) = \widetilde{f}_i(x, v_0)$ for each i. We denote by $\widetilde{\psi}_i$ the character corresponding to $\widetilde{f}_i(x,v)$ and $\psi_i = \widetilde{\psi}_i \mid_{\mathbb{B}}$ for each i, and by $\widetilde{X_u}$ the $\widetilde{P}$-orbit corresponding to X_u for each $u \in J(X)$.

Denote by $\mathcal{S}(X)$ and $\mathcal{S}(\widetilde{X})$ the spaces of Schwartz-Bruhat functions on X and $\widetilde{X}$, respectively. For $s \in \mathbb{C}^n$ and $u \in J(X)$, we consider the following integrals

$$\Omega_u(\phi; s) = \int_X \phi(x) \cdot |f(x)|_u^{s+\varepsilon}\, dx, \qquad (\phi \in \mathcal{S}(X)),$$

$$\widetilde{\Omega}_u(\widetilde{\phi}; s) = \int_{\widetilde{X}} \widetilde{\phi}(x,v) \cdot \left|\widetilde{f}(x,v)\right|_u^{s+\varepsilon} dxdv, \qquad (\widetilde{\phi} \in \mathcal{S}(\widetilde{X})),$$

where dx is a G-invariant measure on X, dv is a Haar measure on V, $\varepsilon \in \mathbb{Q}^n$ and $|f(x)|_u^s$ are the same as in the definition (0.1), and $\left|\widetilde{f}(x,v)\right|_u^s$ is defined similarly for $\widetilde{X_u}$. The above integrals are absolutely convergent for $\mathrm{Re}(s_i) \ge -\varepsilon_i$, $1 \le i \le n$, and analytically continued to rational functions of q^{s_i}, $1 \le i \le n$. It is easy to see that

$$\omega_u(x; s) = v(K \cdot x)^{-1} \cdot \Omega_u(ch_x; s), \qquad (x \in X), \tag{1.4}$$

where ch_x is the characteristic function of $K \cdot x$ in $\mathcal{S}(X)$ and $v(K \cdot x)$ is the volume with respect to the above measure dx.

We see the relation between $\Omega_u(\phi; s)$ and $\widetilde{\Omega}_u(\widetilde{\phi}; s)$ in the following.

Proposition 1.3 *Let $\widetilde{\phi} = \phi \otimes ch_{V(\mathfrak{p}^m)}$ where $\phi \in \mathcal{S}(K\backslash X)$, $ch_{V(\mathfrak{p}^m)}$ is the characteristic function of $V(\mathfrak{p}^m)$ in $\mathcal{S}(V)$ and $\mathfrak{p}^m = \pi^m \mathcal{O}$. Then for any $u \in J(X)$,*

$$\widetilde{\Omega}_u(\widetilde{\phi}; s) = c \cdot \frac{q^{-m(2d + \sum_i e_i(s_i + \varepsilon_i))}}{1 - q^{-2d - \sum_i e_i(s_i + \varepsilon_i)}} \times \Omega_u(\phi; s).$$

Here $e_i = \deg_v \widetilde{f}_i(x,v)$ for each i, c is a constant depending only on the normalization of measures, in particular it is independent of the choice of u.

§2 Functional equations

Take an additive character η on k' of conductor $\pi^l \mathcal{O}_{k'}$, and define the partial Fourier transform $\mathcal{F}(\widetilde{\phi})$ for $\widetilde{\phi} \in \mathcal{S}(\widetilde{X})$ by

$$\mathcal{F}(\widetilde{\phi})(x,v) = \int_V \eta({}^t v \sigma w) \widetilde{\phi}(x,w) dw, \qquad \sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.1}$$

We consider the following distributions on $\mathcal{S}(\widetilde{X})$

$$T_{u,s}(\widetilde{\phi}) = \widetilde{\Omega}_u(\widetilde{\phi}; s), \quad T^*_{u,s}(\widetilde{\phi}) = T_{u,s}(\mathcal{F}(\widetilde{\phi})),$$

and calculate their behaviour under the action of $\widetilde{P} = P \times GL_1(k')$, which is given for $(p,t) \in \widetilde{P}$ and $\widetilde{\phi} \in \mathcal{S}(\widetilde{X})$ by

$${}^{(p,t)}\widetilde{\phi}(x,v) = \widetilde{\phi}((p,t)^{-1} \cdot (x,v)) = \widetilde{\phi}(p^{-1} \cdot x, \rho(p)^{-1} vt), \quad (x,v) \in \widetilde{X}.$$

We consider $\delta \in \mathfrak{X}(\widetilde{\mathbb{P}})$ under the isomorphism $\mathfrak{X}(\mathbb{B}) \cong \mathfrak{X}(\widetilde{\mathbb{P}})$ by Lemma 1.1. We may identify $\mathfrak{X}(\mathbb{B}) \otimes_Z \mathbb{C} = \mathfrak{X}(\widetilde{\mathbb{P}}) \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^n$, on which the Weyl group W acts through the natural action on $\mathfrak{X}(\mathbb{B})$.

Lemma 2.1 *For $(p,t) \in \widetilde{P}$ and $\widetilde{\phi} \in \mathcal{S}(\widetilde{X})$, we have*

$$\begin{aligned}
\text{(i)}\quad T_{u,s}\left({}^{(p,t)}\widetilde{\phi}\right) &= (w_\alpha \delta^{\frac{1}{2}})(p,t)\cdot \left|\widetilde{\psi}(p,t)\right|^{s} \times T_{u,s}(\widetilde{\phi}),\\
\text{(ii)}\quad T^*_{u,s}\left({}^{(p,t)}\widetilde{\phi}\right) &= (w_\alpha \delta^{\frac{1}{2}})(p,t)\cdot \left|\widetilde{\psi}(p,t)\right|^{w_\alpha(s)} \times T^*_{u,s}(\widetilde{\phi}),
\end{aligned}$$

where,

$$\left|\widetilde{\psi}(p,t)\right|^{s} = \prod_{i=1}^{n} \left|\widetilde{\psi}_i(p,t)\right|^{s_i}.$$

If $(\mathbb{H}, \mathbb{Y})$ is of type (F), then essentially by (F1) and (F2), it satisfies also the following property (F5) (see [Sf2, Lemma 2.3, Corollary 2.4]). Let $\{\psi_i \mid 1 \le i \le n\}$ be the set of characters corresponding to a basic set of relative $\mathbb{H}$-invariants taken as regular.

(F5) There is a finite set (L) of linear congruences of type

$$\sum_{i=1}^{n} m_i s_i - \lambda \in \frac{2\pi\sqrt{-1}}{\log q}\mathbb{Z}, \qquad m_i \in \mathbb{Z},\ \lambda \in \mathbb{C}$$

which satisfies the following: If T is a nonzero distribution whose support is contained in $Y \backslash Y^{op}$ and satisfies

$$T({}^g\phi) = |\psi|^s(g)\cdot T(\phi), \qquad \phi \in \mathcal{S}(Y),\ g \in H,$$

then s satisfies a relation in (L).

By a result of Igusa on relative invariant distributions on homogeneous spaces[Ig, Prop. 7.2.1] and the property (F5), we have the following.

Proposition 2.2 *There exist rational functions $\gamma_{u\nu}(s)$ of $q^{\frac{s_1}{e}}, \ldots, q^{\frac{s_n}{e}}$ which satisfy the following identity:*

$$T^*_{u,s}(\widetilde{\phi}) = \sum_{\nu \in J_u} \gamma^\eta_{u\nu}(s) T_{\nu, w_\alpha(s)}(\widetilde{\phi}), \qquad \widetilde{\phi} \in \mathcal{S}(\widetilde{X}). \tag{2.2}$$

Hence we obtain

Theorem 2.3 *There exist rational functions $\gamma^\eta_{u\nu}(s)$ of $q^{\frac{s_1}{e}}, \ldots, q^{\frac{s_n}{e}}$, which satisfy the following functional equation :*

$$\widetilde{\Omega}_u(\mathcal{F}(\widetilde{\phi}); s) = \sum_{\nu \in J_u} \gamma^\eta_{u\nu}(s)\cdot \widetilde{\Omega}_\nu(\widetilde{\phi}; w_\alpha(s)), \qquad \widetilde{\phi} \in \mathcal{S}(\widetilde{X}).$$

Let normalize dv on V to be self dual with respect to the inner product $(v,w) \longmapsto \eta({}^t v \sigma w)$. Then

Corollary 2.4 *For any $\phi \in \mathcal{S}(K \backslash X)$, we have*

$$\Omega_u(\phi; s) = \frac{1 - q^{-2d - \sum_i e_i(s_i + \varepsilon_i)}}{1 - q^{-2d - \sum_i e_i(w_\alpha(s)_i + \varepsilon_i)}} \times \sum_{\nu \in J_u} \gamma_{u\nu}(s)\cdot \Omega_\nu(\phi; w_\alpha(s)),$$

where

$$\gamma_{u\nu}(s) = q^{l(d + \sum_i e_i(s_i + \varepsilon_i))}\cdot \gamma^\eta_{u\nu}(s),$$

which is independent of the choice of the character η on k'.

By Corollary 2.4 and the relation (1.4), we get

Theorem 2.5 *For any $x \in X$, we have*

$$\omega_u(x;s) = \frac{1-q^{-2d-\sum_i e_i(s_i+\varepsilon_i)}}{1-q^{-2d-\sum_i e_i(w_\alpha(s)_i+\varepsilon_i)}} \times \sum_{\nu \in J_u} \gamma_{u\nu}(s)\cdot \omega_\nu(x; w_\alpha(s)).$$

Remark 2.6 The set $J(X)$ can be often naturally identified with a subgroup of the finite abelian group

$$\left(k^\times\right)^n / \prod_{i=1}^{n} \psi_i(B), \tag{2.3}$$

where $\prod_{i=1}^n \psi_i(B)$ is regarded as a subgroup of $T \cong (k^\times)^n$. Then, it is natural to consider *whole* zeta distributions and spherical functions with character in the following. Let $\mathcal{U}$ be $J(X)$ or its subgroup containing J_u which is canonically identified with a subgroup of (2.3). Taking a character χ of $\mathcal{U}$, we set

$$\begin{aligned} &\omega(x;\chi;s) = \sum_{u\in\mathcal{U}} \chi(u)\omega_u(x;s), \quad \Omega(\phi;\chi;s) = \sum_{u\in\mathcal{U}} \chi(u)\Omega_u(\phi;s), \\ &\widetilde{\Omega}(\widetilde{\phi};\chi;s) = \sum_{u\in\mathcal{U}} \chi(u)\widetilde{\Omega}_u(\widetilde{\phi};s). \end{aligned} \tag{2.4}$$

Then we have the following formula instead of Theorem 2.5, we have

$$\omega(x;\chi;s) = \frac{1-q^{-2d-\sum_i e_i(s_i+\varepsilon_i)}}{1-q^{-2d-\sum_i e_i(w_\alpha(s)_i+\varepsilon_i)}} \times \sum_{\xi\in\widehat{\mathcal{U}}} A_{\chi\xi}(s)\omega(x;\xi;w_\alpha(s)), \tag{2.5}$$

where

$$A_{\chi\xi}(s) = \frac{1}{\sharp(\mathcal{U})} \sum_{u,\nu\in\mathcal{U}} \chi(u)\overline{\xi}(\nu)\gamma_{u\nu}(s), \quad \gamma_{u\nu}(s) = 0 \text{ unless } \nu \in J_u,$$

and so

$$\gamma_{u\nu}(s) = \frac{1}{\sharp(\mathcal{U})^2} \sum_{\chi,\xi\in\widehat{\mathcal{U}}} \overline{\chi}(u)\xi(\nu)A_{\chi\xi}(s).$$

For example, when X is the space of nondegenerate symmetric forms of size n, $J(X) \cong \left(k^\times/k^{\times 2}\right)^n$, and when X is the space of nondegenerate hermitian forms of size n over a quadratic extension k' of k, the group $\left(k'^\times/N_{k'/k}(k'^\times)\right)^n$ appears.

§3 Small prehomogeneous vector spaces

In this section we look at the $\left(\rho(\mathbb{P}_x)\times R_{k'/k}(GL_1)\right)$-space $\mathbb{V} = R_{k'/k}(M_{21})$ for $x \in X^{op}$. We recall that $\mathbb{P}_u$ is the stabilizer of $\mathbb{P}$ at fixed $x_u \in X_u$ for each $u \in J(X)$.

Lemma 3.1 (i) *For any $u, \nu \in J(X)$, take $p_\nu \in \mathbb{B}$ satisfying $p_\nu \cdot x_\nu = x_u$. Then the map*

$$\mathbb{P}_\nu \times R_{k'/k}(GL_1)\times\mathbb{V} \longrightarrow \mathbb{P}_u \times R_{k'/k}(GL_1)\times\mathbb{V}, \quad (p,r,v) \longmapsto (p_\nu p p_\nu^{-1}, r, \rho(p_\nu)v)$$

gives an isomorphism of prehomogeneous vector spaces $(\rho(\mathbb{P}_\nu)\times R_{k'/k}(GL_1),\mathbb{V})$ and $(\rho(\mathbb{P}_u)\times R_{k'/k}(GL_1),\mathbb{V})$. If $\nu \in J_u$, we may take $p_\nu \in P$ and then the above isomorphism is defined over k.

(ii) *The set of k-rational points of the open orbit in* $(\rho(\mathbb{B}_u)\times R_{k'/k}(GL_1),\mathbb{V})$ *decomposes as*

$$\left(\rho(\mathbb{P}_u)v_0R_{k'/k}(GL_1)\right)(k)=\bigsqcup_{\nu\in J_u}V_\nu,\qquad V_\nu=\rho(P_up_\nu)v_0k'^{\times},\tag{3.1}$$

where $p_\nu\in P$ *satisfying* $p_\nu^{-1}\cdot x_u\in X_\nu$.

For $\widetilde{\phi}=\phi_1\otimes\phi_2$ with $\phi_1\in\mathcal{S}(X)$ and $\phi_2\in\mathcal{S}(V)$, we have

$$\mathcal{F}(\widetilde{\phi})=\phi_1\otimes\mathcal{F}_V(\phi_2),\quad \mathcal{F}_V(\phi_2)(v)=\int_V\eta({}^tv\sigma w)\phi_2(w)dw.$$

By Theorem 2.3, we obtain

Theorem 3.2 *The prehomogeneous vector space* $(\mathbb{P}_u\times GL_1,\mathbb{V})$ *has the following functional equation:*

$$\int_V\mathcal{F}_V(\phi)(v)\left|\widetilde{f}(x_u,v)\right|_u^{s+\varepsilon}dv\;=\;\sum_{\nu\in J_u}\gamma^\eta_{u\nu}(s)\int_V\phi(v)\left|\widetilde{f}(x_u,v)\right|_\nu^{w_\alpha(s)+\varepsilon}dv\qquad({}^\forall\phi\in\mathcal{S}(V)),$$

where the gamma factors $\gamma^\eta_{u\nu}(s)$ *are the same of those for* $\widetilde{\Omega}_u(\widetilde{\phi};s^*)$ *in Theorem 2.4.*

Remark 3.3 (1) If we take the character η to be of conductor $\mathcal{O}'$ and normalize dv as $vol(V(\mathcal{O}))=1$, then each $\gamma^\eta_{u\nu}(s)$ coincides with $\gamma_{u\nu}(s)$ in Theorem 2 in the introduction.

(2) The identity in Theorem 3.2 can be rewritten as follows:

$$\begin{aligned}&\int_V\mathcal{F}_V(\phi)(v)\prod_{i\in I_+}\left|\widetilde{f}_i(x_u,v)\right|_u^{s_i+\varepsilon_i}dv\\&=\prod_{i\in I_0}|f_i(x_u)|^{w_\alpha(s)_i-s_i}\cdot\sum_{\nu\in J_u}\gamma^\eta_{u\nu}(s)\int_V\phi(v)\prod_{i\in I_+}\left|\widetilde{f}_i(x_u,v)\right|_\nu^{w_\alpha(s)_i+\varepsilon_i}dv\qquad({}^\forall\phi\in\mathcal{S}(V)),\end{aligned}$$

where $I_0=\left\{i\;\middle|\;\deg_v\widetilde{f}_i(x,v)=0\right\}$ and $I_+=\left\{i\;\middle|\;\deg_v\widetilde{f}_i(x,v)>0\right\}$.

Remark 3.4 Here we consider the similar situation as in Remark 2.6. Let $u_0\in J$ and assume that the index set $J_0=J_{u_0}$ can be canonically identified as a subgroup of (2.3). For simplicity, we write x_0 instead of x_{u_0}. For each character $\chi\in\widehat{J_0}$ and $\phi\in\mathcal{S}(V)$, set

$$\Omega_V(\phi;\chi;s)=\sum_{\nu\in J_0}\chi(\nu)\Omega_{V,u}(\phi;s),\qquad \Omega_{V,u}(\phi;s)=\int_V\phi(v)\left|\widetilde{f}(x_0,v)\right|_u^s dv.$$

Then we have by Theorem 3.2

$$\Omega_V(\mathcal{F}_V(\phi);\chi;s)\;=\;\sum_{\xi\in\widehat{J_0}}A'_{\chi\xi}(s)\Omega_V(\phi;\xi;w_\alpha(s)),\quad A'_{\chi\xi}(s)=\frac{1}{\sharp(J_0)}\sum_{u,\nu\in J_0}\chi(u)\overline{\xi}(\nu)\gamma^\eta_{u\nu}(s),$$

and so

$$\gamma^\eta_{u\nu}(s)=\frac{1}{\sharp(J_0)^2}\sum_{\chi,\xi\in\widehat{J_0}}\overline{\chi}(u)\xi(\nu)A'_{\chi\xi}(s),$$

and $A'_{\chi\xi}(s)$ coincides with $A_{\chi\xi}(s)$ if $\mathcal{U}=J_0$ in Remark 2.6.

The existence of the functional equations as above gives the following.

Theorem 3.5 *For the prehomogeneous vector space* $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$, *the identity component of* $\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1)$ *is isomorphic to* $R_{k'/k}(GL_1 \times GL_1)$ *over the algebraic closure* $\overline{k}$ *of* k.

Remark 3.6 By Theorems 3.2 and 3.5, the calculation of the gamma factors $\gamma_{u\nu}(s)$ in §2 is reduced to that for the small prehomogeneous vector spaces $(\rho(\mathbb{P}_u) \times R_{k'/k}(GL_1), \mathbb{V})$, for which the connected component of the groups are isomorphic to $R_{k'/k}(GL_1 \times GL_1)$ over $\overline{k}$.

The set of isomorphism classes of k-forms of $GL_1 \times GL_1$ corresponds bijectively to $\mathrm{Hom}(Gal(\overline{k}/k), GL_2(\mathbb{Z}))$. (cf. [PR, §2.2.4]).

§4 Examples

In the following examples, minimal parabolic subgroups are nothing but Borel subgroups. For Examples 4.1 and 4.2, $(\mathbb{B}, \mathbb{X})$ satisfies the assumption (AF) for each simple root, and explicit formulas of spherical functions have been calculated based on [H2, Proposition 1.9], where the necessary condition to apply it is essentially that $(\mathbb{B}, \mathbb{X})$ is of type (F). For Example 4.3, we give functional equations with respect to the whole Weyl group.

For a matrix x, we denote by $d_i(x)$ the determinant of the upper left i by i block of x.

4.1. $Sp_2 \times (Sp_1)^2$-space Sp_2 (cf. [H4])

Let $\mathbb{G} = Sp_2 \times (Sp_1)^2$, $\mathbb{X} = Sp_2$, where

$$Sp_2 = \left\{ x \in SL_4 \mid {}^t x J x = J \right\}, \qquad J = \begin{pmatrix} & 1_2 \\ -1_2 & \end{pmatrix} \in SL_4,$$

where $(Sp_1)^2 = (SL_2)^2$ is embedded into Sp_2 by

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) \longmapsto \left(\begin{array}{cc|cc} a & & b & \\ & e & & f \\ \hline c & & d & \\ & g & & h \end{array} \right),$$

and the action is given by

$$g \cdot x = g_1 x {}^t g_2, \qquad g = (g_1, g_2) \in \mathbb{G},\ x \in \mathbb{X}.$$

We take the Borel subgroup $\mathbb{B} = \mathbb{B}_1 \times \mathbb{B}_2$ of $\mathbb{G}$ as

$$\mathbb{B}_1 = \left(\begin{array}{cc|cc} * & * & \multicolumn{2}{c}{\multirow{2}{*}{$*$}} \\ 0 & * & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & * & 0 \\ & & * & * \end{array} \right) \subset Sp_2, \quad \mathbb{B}_2 = \left(\begin{array}{cc|cc} * & & 0 & \\ & * & & 0 \\ \hline * & & * & \\ & * & & * \end{array} \right) \subset (Sp_1)^2.$$

Then, a set of regular basic relative $\mathbb{B}$-invariants on $\mathbb{X}$ and corresponding characters are given by

$$f_i(x) = \begin{cases} x_{31} & \text{if } i = 1 \\ x_{32} & \text{if } i = 2 \\ x_{31}x_{42} - x_{32}x_{41} & \text{if } i = 3 \\ x_{31}x_{43} - x_{41}x_{33} & \text{if } i = 4, \end{cases} \qquad \psi_i(\mathbf{b}) = \begin{cases} b_1 b_3 & \text{if } i = 1 \\ b_1 b_4 & \text{if } i = 2 \\ b_1 b_2 b_3 b_4 & \text{if } i = 3 \\ b_1 b_2 & \text{if } i = 4, \end{cases}$$

where $x = (x_{ij}) \in \mathbb{X}$ and $\mathbf{b} = \left(\left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline & & b_1 & \\ & & & b_2 \end{array}\right), \left(\begin{array}{cc|cc} b_3 & & & \\ & b_4 & & \\ \hline & & & \\ & & & \end{array}\right) \right) \in \mathbb{B}$, and

$$\mathbb{X}^{op} = \{x \in \mathbb{X} \mid f_i(x) \neq 0,\ 1 \leq i \leq 4\}, \quad \mathbb{X}^{op}(k) = \bigsqcup_{u \in J(X)} X_u, \quad J(X) = k^\times / k^{\times 2}$$

$$X_u = \left\{ x \in X \;\middle|\; \prod_{i=1}^{4} f_i(x) \equiv u \pmod{(k^\times)^2} \right\} \ni x_u = \left(\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \hline -1 & 1 & -u & u \\ 0 & -1 & u & 0 \end{array}\right).$$

For the simple root α attached to

$$w_\alpha = \left(\begin{array}{cc|cc} & 1 & & \\ -1 & & & \\ \hline & & & 1 \\ & & -1 & \end{array}\right),$$

set

$$\mathbb{P} = \mathbb{P}_\alpha = \left\{ (\mathbf{b_1}, \mathbf{b_2}) \in \mathbb{G} \;\middle|\; \mathbf{b_1} = \left(\begin{array}{c|c} S & * \\ \hline 0 & {}^tS^{-1} \end{array}\right) \in Sp_2,\ \mathbf{b_2} \in \mathbb{B}_2 \right\},$$

and define

$$\rho : \mathbb{P} \longrightarrow GL_2, \ (\mathbf{b}_1, \mathbf{b}_2) \longmapsto S.$$

The above $\mathbb{P}$ coincides with $\mathbb{P}_1$ in [H4], $(\mathbb{B}, \mathbb{X})$ satisfied the condition (AF), and we see the following.

1. The relative $\widetilde{\mathbb{P}}$-invariants satisfying $\widetilde{f}_i(x, v_0) = f_i(x)$ are given by

$$\widetilde{f}_i(x,v) = \begin{cases} x_{31}v_1 + x_{41}v_2 & \text{if } i = 1 \\ x_{32}v_1 + x_{42}v_2 & \text{if } i = 2 \\ f_3(x) & \text{if } i = 3 \\ f_4(x) & \text{if } i = 4, \end{cases} \qquad x = (x_{ij}) \in \mathbb{X},\ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{V}.$$

2. The small prehomogeneous vector space are isomorphic to $(GL_1 \times GL_1, \mathbb{V})$ over k.

3. If we change the variable s into z by

$$z_1 = s_1 + s_2 + s_3 + s_4, \quad z_2 = s_3 + s_4, \quad z_3 = s_1 + s_3, \quad z_4 = s_2 + s_3,$$

we have $w_\alpha(z) = (z_2, z_1, z_3, z_4)$.

For $\chi \in \widehat{k^\times / k^{\times 2}}$ and $\varepsilon = (-\frac{1}{2}, \ldots, -\frac{1}{2})$, set

$$\omega(x; \chi; s) = \int_K \chi(f(k \cdot x)) \, |f(k \cdot x)|^{s+\varepsilon} \, dk,$$
$$\Omega_V(\phi; \chi; s) = \int_V \chi(\widetilde{f}(x_1, v)) \left| \widetilde{f}(x_1, v) \right|^{s+\varepsilon},$$

where $\widetilde{f}(x,v) = \prod_{i=1}^4 \widetilde{f}_i(x,v)$. By using Tate's formula ([Ta, §2]) we have the functional equation for $\Omega_V(\phi; \chi; s)$, hence we obtain the functional equation for $\omega(x; \chi; s)$:

$$\omega(x; \chi; s) = \gamma(\chi; s) \cdot \omega(x; \chi; w_\alpha(s)),$$

where $\gamma(\chi; s)$ can be calculated explicitly, and if k is odd residual characteristic it is given by

$$\gamma(\chi; s) = \frac{1-q^{-s_1-s_2-1}}{1-q^{s_1+s_2-1}} \times \begin{cases} \prod_{i=1,2} \dfrac{1-\chi(\pi)q^{s_i-\frac{1}{2}}}{1-\chi(\pi)q^{-s_i-\frac{1}{2}}} & \text{if } \chi(\mathcal{O}^\times) = 1 \\ q^{s_1+s_2} & \text{if } \chi(\mathcal{O}^\times) \neq 1. \end{cases}$$

In a similar way we can obtain the functional equations for other simple roots, and using these data, the explicit formula of $\omega(x; \chi; s)$ has been obtained when k has an odd residual characteristic. For details see [H4], where z is the same as before though we shift the variable s by ε here.

4.2. The space of unramified hermitian forms (cf. [H2, §2])

Let k'/k be an unramified quadratic extension, $*$ be the involution on k' with fixed field k, and consider the space of hermitian matrices $X = \{x \in GL_n(k') \mid x^* = x\}$. Here we denote by $y^* = (y_{ji}^*) \in M_{nm}(k')$ for $y = (y_{ij}) \in M_{mn}(k')$. The action of $G = GL_n(k')$ on X is given by $g \cdot x = gxg^*$ for $g \in G$ and $x \in X$.

We realize $\mathbb{G} = R_{k'/k}(GL_n)$ and $\mathbb{X}$ defined over k such that $G = \mathbb{G}(k)$ and $X = \mathbb{X}(k)$ as follows, by taking $u \in \mathcal{O}$ for which $k' = k(\sqrt{u})$ and $\mathcal{O}' = \mathcal{O}[\sqrt{u}]$,

$$\begin{aligned} \mathbb{G} &= \left\{ (g_{ij})_{1\le i,j\le n} \in GL_{2n}(\overline{k}) \;\middle|\; g_{ij} = \begin{pmatrix} a_{ij} & b_{ij}u \\ b_{ij} & a_{ij} \end{pmatrix} \in M_2(\overline{k}), \quad {}^\forall i,j \right\}, \\ \mathbb{X} &= \left\{ (x_{ij})_{1\le i,j\le n} \in GL_{2n}(\overline{k}) \;\middle|\; x_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x_{ji} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad {}^\forall i,j \right\}. \end{aligned}$$

Take the Borel group $\mathbb{B}$ as

$$\mathbb{B} = \left\{ (b_{ij}) \in \mathbb{G} \mid b_{ij} = 0 (\text{ in } M_2(\overline{k})) \text{ unless } i \ge j \right\},$$

then $B = \mathbb{B}(k)$ consists of lower triangular matrices in G.

The set $\{ f_i(x) = d_{2i}(x) \mid 1 \le i \le n\}$ is a set of basic relative $\mathbb{B}$-invariants on $\mathbb{X}$, $\psi_i(b) = d_{2i}(b)$ is the character on $\mathbb{B}$ corresponding to $f_i(x)$, $\mathbb{X}^{op} = \{x \in \mathbb{X} \mid f_i(x) \neq 0,\ 1 \le i \le n\}$, and

$$\begin{aligned} &X^{op} = \bigsqcup_{u \in J(X)} X_u, \quad J(X) = \{0,\ 1\}^n \\ &X_u = \{x \in X \mid v_\pi(f_i(x)) \equiv u_1 + \cdots + u_i \pmod 2,\ 1 \le i \le n\}, \\ &x_u = Diag(\pi^{u_1}, \ldots, \pi^{u_n}) \in X_u, \end{aligned}$$

where $v_\pi(\ )$ be the additive valuation on k.

For the simple root corresponding to the transposition $(\alpha, \alpha+1)$, $1 \le \alpha \le n-1$, set

$$\mathbb{P} = \mathbb{P}_\alpha = \left\{ (p_{ij}) \in \mathbb{G} \mid p_{ij} = 0 (\in M_2(\overline{k})) \text{ unless } i \ge j \text{ or } (i,j) = (\alpha, \alpha+1) \right\},$$

and define

$$\rho : \mathbb{P} \longrightarrow R_{k'/k}(GL_2),\ (p_{ij})_{1 \le i,j \le n} \longmapsto \begin{pmatrix} p_{\alpha+1,\alpha+1} & -p_{\alpha+1,\alpha} \\ -p_{\alpha,\alpha+1} & p_{\alpha,\alpha} \end{pmatrix}.$$

Then $(\mathbb{B}, \mathbb{X})$ satisfies the condition (AF), and we see the following.

1. The relative $\widetilde{\mathbb{P}}$-invariants satisfying $\widetilde{f_i}(x, v_0) = f_i(x)$ are given by $\widetilde{f_i}(x, v) = f_i(x)$ unless $i = \alpha$, and

$$\widetilde{f_\alpha}(x, v) = f_{\alpha+1}(x)^t(\sigma v) M_\alpha(x) (\sigma v)^*.$$

Here σ is the same as before, and $M_\alpha(x)$ is the lower right 2 by 2 block of the inverse of the upper right $(\alpha+1)$ by $(\alpha+1)$ block of x.

2. For any $u \in J(X)$, $J_u = \{\nu \in J(X) \mid \nu_\alpha + \nu_{\alpha+1} \equiv u_\alpha + u_{\alpha+1} \pmod 2\}$, and

$$\widetilde{f_\alpha}(x_u, v) = \pi^{u_1+\cdots+u_{\alpha-1}} \left(\pi^{u_\alpha} N_{k'/k}(v_1) + \pi^{u_{\alpha+1}} N_{k'/k}(v_2)\right).$$

3. The small prehomogeneous vector spaces are isomorphic to $(H_0 \times GL_1(k'), V)$ or $(H_1 \times GL_1(k'), V)$ over k, where $H_i = \left\{ g \in GL_2(k') \,\middle|\, g \cdot \begin{pmatrix} 1 & \\ & \pi^i \end{pmatrix} = \begin{pmatrix} 1 & \\ & \pi^i \end{pmatrix} \right\}$, $i = 0, 1$.

Let χ_π be the character on $k^\times$ given by $\chi_\pi(\pi) = -1$ and $\chi_\pi(\mathcal{O}^\times) = 1$, set $f(x) = \prod_{i=1}^n f_i(x)$, and modify the spherical function as follows.

$$\omega(x; s) = \int_K \chi_\pi(f(k \cdot x)) \, |f(k \cdot x)|^{s+\varepsilon} \, dk, \qquad \varepsilon = (-1, \ldots, -1, \frac{n-1}{2}).$$

If we change the variable s into z by

$$s_i = -z_i + z_{i+1}, \quad (1 \leq i \leq n-1), \qquad s_n = -z_n,$$

then W acts on z by the permutation of indices, and we have

$$\omega(x; z) = \frac{q^{z_{\alpha+1}} - q^{z_\alpha - 1}}{q^{z_\alpha} - q^{z_{\alpha+1}-1}} \times \omega(x; w_\alpha(z)),$$

moreover we see that

$$\prod_{1 \leq i < j \leq n} \frac{q^{z_j} + q^{z_i}}{q^{z_j} - q^{z_i - 1}} \times \omega(x; z)$$

is S_n-invariant and holomorphic. Further using these facts, the explicit formula of $\omega(x; z)$ has been obtained ([H2, Theorem 1]), where the variable z is the same as before.

Remark For the case of symmetric forms, the situation is similar to this case, where $k' = k$ and $J(X) \cong \left(k^\times / k^{\times 2}\right)^n$. But functional equations are much more complicated (cf. [H1-III, 2-adic] and (4.7) below), and any explicit formula of spherical functions is not known for general n.

4.3. The space isomorphic to $SO(2n)/S(O(n) \times O(n))$

For symmetric matrix A of size m and $v \in M_{mn}$, we use the symbol $A[v] = {}^t v A v$, which is symmetric of size n, and define $O(A) = \{g \in GL_m \mid A[g] = A\}$ and $SO(A) = O(A) \cap SL_m$. Set

$$H_n = \frac{1}{2} \begin{pmatrix} 0_n & 1_n \\ 1_n & 0_n \end{pmatrix}, \quad \mathbb{G} = SO(H_n), \quad G = \mathbb{G}(k), \quad K = \mathbb{G}(\mathcal{O}).$$

Take a non-degenerate integral symmetric matrix T of size n, and set

$$\mathbb{Y} = \mathbb{Y}_T = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in M_{2n,n} \,\middle|\, H_n[x] = T \right\} \ni x_T = \begin{pmatrix} T \\ 1 \end{pmatrix},$$
$$\mathbb{X} = \mathbb{Y}/\mathbb{H}, \qquad \mathbb{H} = O(T),$$

which are $\mathbb{G}$-homogeneous by the left multiplication. Since $\mathbb{Y}_T$ is isomorphic to $\mathbb{Y}_{T[h]}$ for $h \in GL_n(k)$, we may assume that T is diagonal.

Since the stabilizer subgroup of $\mathbb{G}$ at $x_T\mathbb{H} \in \mathbb{X}$ is isomorphic to $S(O(T) \times O(T))$, $\mathbb{X}$ is isomorphic to $SO(2n)/S(O(n) \times O(n))$ over $\bar{k}$. Spherical functions on this space with respect to the Siegel parabolic subgroup have a close relation to Siegel's singular series, which we will discuss in another paper([SH]).

Let take the Borel subgroup of $\mathbb{G}$ by

$$\mathbb{B} = \left\{ \begin{pmatrix} b & * \\ 0 & {}^t b^{-1} \end{pmatrix} \in \mathbb{G} \;\middle|\; b \text{ is upper triangular of size } n \right\}.$$

For simplicity of notation, we write the element $x\mathbb{H}$ of $\mathbb{X}$ (resp. xH of X) by its representative in $\mathbb{Y}$ (resp. Y). For any element $x \in \mathbb{Y}$ we denote by x_2 its lower n by n block.

For $x \in \mathbb{Y}$, set

$$f_i(x) = d_i(x_2 T^{-1}\, {}^t x_2), \quad \psi_i(b) = (b_1 \cdots b_i)^{-2}, \quad 1 \le i \le n,$$

where b_i is the i-th diagonal component of $b \in \mathbb{B}$ for each i. Then, $\{ f_i(x) \mid 1 \le i \le n\}$ is a basic set of regular relative $\mathbb{B}$-invariants on $\mathbb{X}$, $f_i(x)$ corresponds to ψ_i for each i, and $\varepsilon = (-\frac{1}{2}, \ldots, -\frac{1}{2}, 0) \in \mathbb{Q}^n$. Further we see

$$\begin{aligned} &\mathbb{X}^{op} = \{ x\mathbb{H} \in \mathbb{X} \mid f_i(x) \neq 0,\ 1 \le i \le n\}, \\ &X^{op} = \bigsqcup_{u \in J(X)} X_u, \qquad J(X) = \left(k^\times / k^{\times 2}\right)^{n-1}, \\ &X_u = \left\{ xH \in X \;\middle|\; f_i(x) \equiv u_1 \cdots u_i \pmod{k^{\times 2}},\ 1 \le i \le n-1 \right\}. \end{aligned}$$

For the simple root τ attached to

$$w_\tau = \left(\begin{array}{cc|cc} 1_{n-2} & & & \\ & & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \\ \hline & & 1_{n-2} & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & \end{array} \right),$$

set

$$\mathbb{P} = \mathbb{P}_\tau = \mathbb{B} \cup \mathbb{B} w_\tau \mathbb{B},$$

then

$$\mathbb{P} = \left\{ \left(\begin{array}{ccc|ccc} p & & & & & \\ & a & 0 & & 0 & b \\ & 0 & a' & & b' & 0 \\ \hline & & & {}^t p^{-1} & & \\ & 0 & c' & & d' & 0 \\ & c & 0 & & 0 & d \end{array} \right) \times u \;\middle|\; \begin{array}{l} p \in GL_{n-2},\ \text{upper triangular}, \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2,\ \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \frac{1}{ad-bc} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ u \in \mathbb{B},\ u_{ii} = 1\ (1 \le i \le n), \\ u_{ij} = 0\ (1 \le i < j \le n-2) \end{array} \right\}.$$

We define

$$\rho : \mathbb{P} \longrightarrow GL_2, \quad p = (p_{ij}) \longmapsto \begin{pmatrix} p_{n-1,n-1} & p_{n-1,2n} \\ p_{2n,n-1} & p_{2n,2n} \end{pmatrix},$$

then it is clear that ρ satisfied the condition (0.3), and we see the following.

1. For $x \in \mathbb{Y}$, denote by z_1 (resp. z_2) its $(2n-1)$-th row vector (resp. n-th row vector), and set

$$D(x) = f_{n-2}(x) \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} T^{-1} \left({}^t z_1\, {}^t z_2\right),$$

which is well-defined for $x\mathbb{H} \in \mathbb{X}$. The relative $\widetilde{\mathbb{P}}$-invariants satisfying $\widetilde{f}_i(x, v_0) = f_i(x)$ are given by

$$\widetilde{f}_i(x,v) = \begin{cases} f_i(x) & \text{if } 1 \le i \le n-2 \\ {}^t v D(x) v & \text{if } i = n-1 \\ f_{n-1}^{-2}(x) f_n(x) \cdot ({}^t v D(x) v)^2 & \text{if } i = n, \end{cases}$$

and

$$\begin{aligned} \det D(x) &= f_{n-2}(x) f_{n-1}^2(x) f_n^{-1}(x) \\ &\equiv (\det T) u_1 \cdots u_{n-2} \pmod{k^{\times 2}} \quad \text{on } X_u. \end{aligned}$$

2. The small prehomogeneous vector spaces are isomorphic to $(O(2) \times GL_1, \mathbb{V})$ over $\overline{k}$.
3. If we change the variable s into z by

$$s_i = -\frac{1}{2}(z_i - z_{i+1}), \quad (1 \le i \le n-1), \qquad s_n = -\frac{1}{2} z_n, \tag{4.1}$$

we see

$$\begin{aligned} w_\tau(s) = (s_1, \dots, s_{n-3}, s_{n-2} + s_{n-1} + s_n, s_{n-1}, -s_{n-1} - s_n) \\ \longleftrightarrow w_\tau(z) = (z_1, \dots, z_{n-2}, -z_n, -z_{n-1}). \end{aligned}$$

Hereafter, we assume that k has an odd residual characteristic or 2 is a prime element in k. We consider a modified zeta integral as follows. For $u \in J(X)$, a character χ of $k^\times / k^{\times 2}$, $\phi \in \mathcal{S}(K \backslash X)$ and $s \in \mathbb{C}^n$

$$\Omega'_u(\phi; \chi; s) = \int_{X_u \times V(\mathcal{O})} \phi(x) \cdot \chi(\widetilde{f_{n-1}}(x,v)) \left| \widetilde{f}(x,v) \right|^{s+\varepsilon} dx dv.$$

Then, we have

$$\begin{aligned} &\Omega'_u(\phi; \chi; s) \\ &= \int_{X_u} \phi(x) \cdot \prod_{i=1}^{n-2} |f_i(x)|^{s_i - \frac{1}{2}} \cdot |f_{n-1}(x)|^{-2s_n} |f_n(x)|^{s_n} dx \\ &\quad \times \int_{V(\mathcal{O})} \chi({}^t v D(x) v) \left| {}^t v D(x) v \right|^{s_{n-1} + 2s_n - \frac{1}{2}} dv \\ &= \frac{c_1}{1 - q^{-2s_{n-1} - 2s_n - 1}} \int_{X_u} \phi(x) \cdot \prod_{i=1}^{n-2} |f_i(x)|^{s_i - \frac{1}{2}} \cdot |f_{n-1}(x)|^{-2s_n} |f_n(x)|^{s_n} \\ &\quad \times \zeta^{(2)}(D(x); \chi; s_{n-1} + 2s_n) dx, \end{aligned}$$

where $c_1 = (1 - q^{-2}) \displaystyle\int_{V(\mathcal{O})} dv$. Here, for a nondegenerate symmetric matrix y of size 2, a character χ of $k^\times / k^{\times 2}$ and $t \in \mathbb{C}$

$$\zeta^{(2)}(y; \chi; t) = \int_{GL_2(\mathcal{O})} \chi(d_1(k \cdot y)) \left| d_1(k \cdot y) \right|^{t - \frac{1}{2}} dk$$

is a spherical function on the space $Sym_2^{nd}(k) = \{ y \in GL_2(k) \mid {}^t y = y \}$ and satisfies a functional equation of the form

$$\zeta^{(2)}(x; \chi; t) = |\det x|^t \cdot \gamma_{\det x}(\chi, t) \times \zeta^{(2)}(x; \chi; -t), \tag{4.2}$$

where $\gamma_u(\chi,t)$ depends only on $u \in k^\times/k^{\times 2}$ and is calculated explicitly(cf. [H1-II, 2-adic], (4.7) below). Hence we obtain

$$\Omega'_u(\phi;\chi;s) \quad = \quad \frac{1-q^{2s_{n-1}+4s_n-1}}{1-q^{-2s_{n-1}-4s_n-1}} \cdot \gamma_{u_o}(\chi;s_{n-1}+2s_n) \times \Omega'_u(\phi;\chi;w_\tau(s)), \tag{4.3}$$

where $u_o = (\det T)u_1\cdots u_{n-2} \in k^\times/k^{\times 2}$. On the other hand, in a similar way to the proof of Proposition 1.4, we get

$$\sum_{\xi\in k^\times/k^{\times 2}} \Omega'_{u*\xi}(\phi;\chi;s) = c_1 \cdot \frac{\chi(u_1\cdots u_{n-1})}{1-q^{-2s_{n-1}-4s_n-1}} \times \sum_{\xi\in k^\times/k^{\times 2}} \chi(\xi)\cdot\Omega_{u*\xi}(\phi;s), \tag{4.4}$$

where $u*\xi = (u_1,\dots,u_{n-2},u_{n-1}\xi) \in J(X)$. By (4.3) and (4.4), we obtain for $u \in J(X)$ and $\phi \in \mathcal{S}(K\backslash X)$

$$\begin{aligned}\Omega_u(\phi;s) &= \frac{1}{[k^\times:k^{\times 2}]}\sum_{\chi\in\widehat{k^\times/k^{\times 2}}}\sum_{\xi\in k^\times/k^{\times 2}} \chi(\xi)\cdot\Omega_{u*\xi}(\phi;s)\\ &= \sum_{\xi}\left(\frac{1}{[k^\times:k^{\times 2}]}\sum_{\chi}\chi(\xi)\gamma_{u_o}(\chi;s_{n-1}+2s_n)\right)\times\Omega_{u*\xi}(\phi;w_\tau(s)).\end{aligned} \tag{4.5}$$

For each $w_\alpha \in W$ corresponding to the transposition $(\alpha,\alpha+1)$ with $1\le\alpha\le n-1$, we obtain the functional equation in a similar way: for $u \in J(X)$ and $\phi \in \mathcal{S}(K\backslash X)$

$$\Omega_u(\phi;s) \quad = \quad \sum_{\xi\in k^\times/k^{\times 2}}\left(\frac{1}{[k^\times:k^{\times 2}]}\sum_{\chi\in\widehat{k^\times/k^{\times 2}}}\chi(\xi)\gamma_{u_o}(\chi;s_\alpha)\right)\times\Omega_{u*_\alpha\xi}(\phi;w_\alpha(s)), \tag{4.6}$$

where

$$\begin{aligned} u_o &= \begin{cases} u_\alpha u_{\alpha+1} & \text{if } \alpha\le n-2\\ u_{n-1} & \text{if } \alpha = n-1,\end{cases}\\ u*_\alpha\xi &= \begin{cases} (u_1,\dots,u_{\alpha-1},\xi,u_\alpha u_{\alpha+1}\xi,u_{\alpha+2},\dots,u_{n-1}) & \text{if } \alpha\le n-2\\ (u_1,\dots,u_{n-2},\xi) & \text{if } \alpha=n-1.\end{cases}\end{aligned}$$

Since the Weyl group W is generated by w_τ and w_α, $1\le\alpha\le n-1$, using (4.5) and (4.6), we have the following functional equation of $\omega_u(x;s)=\omega(x;z)$, where the relation of s and z is given by (4.1), so w_α acts on z as the transposition of z_α and $z_{\alpha+1}$.

Theorem 4.1 *For $x\in X$, $u\in J(X)$ and $\sigma\in W$, we have*

$$\omega_u(x;\sigma(z)) = \sum_{\nu\in J(X)}\Gamma_{u\nu}(\chi;z)\cdot\omega_\nu(x;z).$$

Here $\Gamma_{u\nu}(\chi;z)$'s are rational functions of $q^{\frac{z_1}{2}},\dots,q^{\frac{z_n}{2}}$, satisfy cocycle relations

$$\Gamma_{u\nu}(\sigma\sigma';z) = \sum_{\mu\in J(X)}\Gamma_{u\mu}(\sigma,\sigma'(z))\cdot\Gamma_{\mu\nu}(\sigma';z), \quad \sigma,\sigma'\in W,$$

and are given for w_τ and w_α by

$$\Gamma_{u\nu}(w_\tau; z) = \begin{cases} \frac{1}{[k^\times : k^{\times 2}]}\chi(u_{n-1}\nu_{n-1}) \cdot \gamma_{u_\tau}(\chi; \frac{z_{n-1}+z_n}{2}) & \text{if } u_i = \nu_i,\ 1 \le i \le n-2 \\ 0 & \text{otherwise}, \end{cases}$$

$$\Gamma_{u\nu}(w_\alpha; z) = \begin{cases} \frac{1}{[k^\times : k^{\times 2}]}\chi(\nu_\alpha) \cdot \gamma_{u_{<\alpha>}}(\chi; \frac{z_\alpha - z_{\alpha+1}}{2}) & \begin{array}{l} \text{if } u_i = \nu_i, i \ne \alpha, \alpha+1, \\ \text{and } u_\alpha u_{\alpha+1} = \nu_\alpha \nu_{\alpha+1} \text{ if } \alpha \le n-2 \end{array} \\ 0 & \text{otherwise}, \end{cases}$$

$$u_\tau = (\det T) u_1 \cdots u_{n-2}, \quad u_{<\alpha>} = \begin{cases} u_\alpha u_{\alpha+1} & \text{if } \alpha \le n-2 \\ u_{n-1} & \text{if } \alpha = n-1 \end{cases} \quad (\in k^\times / k^{\times 2}).$$

For convenience, we note here the value of $\gamma_u(\chi, t)$ for the case of odd characteristic (cf. [H1-II §4] or [H-III §4]): for $u \in k^\times/k^{\times 2}$ and $\chi \in \widehat{k^\times/k^{\times 2}}$

$$\gamma_u(\chi, t) = \begin{cases} \chi(u) \cdot q^{\frac{1}{2}} \cdot \dfrac{1 + \chi(u) q^{-t-\frac{1}{2}}}{1 + \chi(u) q^{-t+\frac{1}{2}}} & \text{if } v_\pi(u) \equiv 1 \pmod 2 \\ 1 & \text{if } \chi(\mathcal{O}^\times) = 1,\ v_\pi(u) \equiv 0 \pmod 2,\ \left(\frac{-u}{\mathfrak{p}}\right) = -1 \\ -\dfrac{(1 + \chi(\pi) q^{-t-\frac{1}{2}})(1 - \chi(\pi) q^{-t+\frac{1}{2}})}{(1 - \chi(\pi) q^{-t-\frac{1}{2}})(1 + \chi(\pi) q^{-t+\frac{1}{2}})} & \text{if } \chi(\mathcal{O}^\times) = 1,\ v_\pi(u) \equiv 0 \pmod 2,\ \left(\frac{-u}{\mathfrak{p}}\right) = 1 \\ -\dfrac{q(1 - q^{-2t-1})}{1 - q^{-2t+1}} & \text{if } \chi(\mathcal{O}^\times) \ne 1,\ v_\pi(u) \equiv 0 \pmod 2. \end{cases} \tag{4.7}$$

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