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Nonvanishing of the Central Value of the Rankin-Selberg L-functions

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1 Introduction

This is a written version of the two lectures I gave at RIMS, January, 2005, which contains mainly the results from my joint work with David Ginzburg and Stephen Rallis. I would like to thank Professor Masaaki Furusawa for his warm invitation and for the wonderful conference, and thank Professor Horish Saito for his hospitality. The work is partly supported by NSF grant DMS-0400414.

Let us start with the simplest L-function, which is the well-known Riemann zeta function

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}.$$

It is known that $\zeta(s)$ converges absolutely for the real part of s greater than one, has a meromorphic continuation to the whole complex plane $\mathbb{C}$, and satisfies a functional equation which relates s to $1-s$. Any information about $\zeta(s)$ in the vertical strip $0 \leq Re(s) \leq 1$ will be sensitive, mysterious, important, etc.

More generally, we consider automorphic L-functions attached to automorphic representations.

Let G be a reductive algebraic group defined over a number field k. Let ${}^LG$ be the Langlands dual group of G. Let π be an irreducible, unitary, cuspidal, automorphic representation of $G(\mathbb{A})$, where $\mathbb{A}$ is the ring of adeles of k and ρ be a finite-dimensional complex representation of ${}^LG$. According to Langlands, the automorphic L-function attached to the pair (π, ρ) is given by the following Euler product

$$L(s, \pi, \rho) = \prod_v L(s, \pi_v, \rho)$$

if we write $\pi = \otimes_v \pi_v$.

Theorem 1.1 (Langlands). *For any given pair (π, ρ) given as above, the automorphic L-function $L(s, \pi, \rho)$ converges absolutely for the real part of s large.*

Conjecture 1.2 (Langlands). *For any given pair (π, ρ) given as above, the automorphic L-function $L(s, \pi, \rho)$ has meromorphic continuation to the whole complex plane $\mathbb{C}$ and satisfies a functional equation which related to s to $1-s$.*

It follows that there is a positive real number S_0, such that $L(s, \pi, \rho)$ will be sensitive, mysterious, and important when s lies in the vertical strip

$$\frac{1}{2} - s_0 \leq Re(s) \leq \frac{1}{2} + s_0.$$

Remark 1.3. (1) *The Langlands conjecture has been verified for many cases, by either the Langlands-Shahidi method or by the Rankin-Selberg method. See [GSh88] and [Bmp05] for detailed discussions.*

(2) *If π is generic, i.e. has a nonzero Whittaker-Fourier coefficient, then it follows from the generalized Ramanujan conjecture that $s_0 = 1$. It was proved in [Sh88] that one can have $s_0 = 2$ for generic cuspidal automorphic π. See [Sh88] for a detailed discussion.*

(3) *It seems that $s = \frac{1}{2}$ is the most sensitive point for $L(s, \pi, \rho)$.*

We will discuss in more detail about the nonvanishing of $L(\frac{1}{2}, \pi, \rho)$ in terms of the representation theory of automorphic forms. In other words, I want to find how much one will be able to tell about the structure of the representation π in terms of the nonvanishing of $L(\frac{1}{2}, \pi, \rho)$.

1.1 Residual representations

From Langlands' theory of Eisenstein series ([L76] and [MW95]), the non-vanishing condition of certain automorphic L-functions is equivalent to the condition for the existence of a residue of certain Eisenstein series.

For simplicity of discussion, we may assume that G is quasi-split or even split over k. Let $P = MN$ be a standard maximal parabolic k-subgroup of G. Let $K = \prod_v K_v$ be a standard maximal compact subgroup of $G(\mathbb{A})$ such that

$$G(\mathbb{A}) = P(\mathbb{A})K$$

is the Iwasawa decomposition of $G(\mathbb{A})$.

Let σ be an irreducible cuspidal automorphic representation of $M(\mathbb{A})$. Consider V_σ-valued smooth functions ϕ_σ over $G(\mathbb{A})$ satisfying following properties:

(1) for a fixed $g \in G(\mathbb{A})$,

$$k \mapsto \phi_\sigma(gk)$$

generates a finite-dimensional representations of K;

(2) for a fixed $g \in G(\mathbb{A})$,

$$m \mapsto \phi_\sigma(mg)$$

is a smooth vector in V_σ and the function ϕ_σ satisfies

$$\phi_\sigma(ng) = \phi_\sigma(g)$$

for all $n \in N(\mathbb{A})$.

Define

$$\Phi(g; s, \phi_\sigma) := \phi_\sigma(g) \exp < s + \rho_P, H_P(g) >$$

where ρ_P is the half of the sum of all positive roots with respect to the center of M, the parameter s is normalized as in [Sh88], and H_P is the Harish-Chandra map with respect to P. Then we define Eisenstein series of $G(\mathbb{A})$

attached to the cuspidal datum (P, σ) as

$$E(g; s, \phi_\sigma) := \sum_{\gamma \in P(k)\backslash G(k)} \Phi(\gamma g; s, \phi_\sigma). \tag{1.1}$$

Theorem 1.4 (Langlands ([L76] and [MW95]). *For any given cuspidal datum (P, σ), the Eisenstein series $E(g; s, \phi_\sigma)$ converges absolutely for the real part of s large, has meromorphic continuation to the whole complex plane $\mathbb{C}$ (with finitely many poles at the positive real line after suitable normalization), and satisfies a functional equation relating the value s to the value $-s$.*

Remark 1.5. *It is expected that the structure of the residual representations of $E(g; s, \phi_\sigma)$ at a real number $s_0 > 0$ should have direct relations with the structure of the cuspidal datum (P, σ). However, this is not completely known.*

From the Langlands theory of the constant terms of Eisenstein series, the poles of Eisenstein series can be detected in terms of explicit calculation of the constant terms of the Eisenstein series. The constant term of the Eisenstein series $E(g; s, \phi_\sigma)$ along a standard parabolic subgroup P' is always zero unless $P' = P$ ([MW95], II.1.7). In this case the constant term can be expressed as

$$\begin{aligned} E_P(g; s, \phi_\sigma) &= \int_{N(k)\backslash N(\mathbb{A})} E(ug; s, \phi_\sigma) dn \\ &= \Phi(g; s, \phi_\sigma) + \mathcal{M}(w_0, s)(\Phi(\cdot; s, \phi_\sigma))(g) \end{aligned} \tag{1.2}$$

where w_0 the longest Weyl element in the representatives of the double coset decomposition $W_M\backslash W_G/W_M$ of the Weyl groups.

It follows from the Langlands theory of Eisenstein series that $E(g; s, \phi_\sigma)$ has a pole at $s = s_0$ if and only if the term $\mathcal{M}(w_0, s)(\Phi(\cdot; s, \phi_\sigma))$ above has a pole at $s = s_0$ for some holomorphic (or standard) section $\Phi(g; s, \phi_\sigma)$ as defined before. Following the standard argument, if we consider factorizable sections

$$\Phi(\cdot; s, \phi_\sigma) = \otimes_v \Phi_v(\cdot; s, \phi_{\sigma_v}),$$

then we have

$$\mathcal{M}(w_0, s)(\Phi(\cdot; s, \phi_\sigma)) = \prod_v \mathcal{M}_v(w_0, s)(\Phi_v(\cdot; s, \phi_{\sigma_v})).$$

By the Langlands-Shahidi theory ([L71], [Sh88]) we have

$$\mathcal{M}(w_0, s) = \frac{\prod_l L(ls, \sigma, r_l)}{\prod_l L(ls+1, \sigma, r_l)} \cdot \prod_v \mathcal{N}_v(w_0, s), \tag{1.3}$$

where $\mathcal{N}_v(w_0, s)$ is the normalized intertwining operator

$$\mathcal{N}_v(w_0, s) = \frac{1}{r(s, \pi_v, \sigma_v, w_0)} \cdot \mathcal{M}_v(w_0, s).$$

Here the function $r(s, \pi_v, \sigma_v, w_0)$ is equal to

$$\frac{\prod_l L(ls, \sigma_v, r_l)}{\prod_l L(ls+1, \sigma_v, r_l) \prod_l \epsilon(ls, \sigma_v, r_l, \psi_v)}.$$

By a conjecture of Shahidi [Sh90], it is expected that $\mathcal{N}_v(w_0, s)(\Phi_v(\cdot; s, \phi_{\sigma_v}))$ is holomorphic and nonzero for $Re(s) \geq 0$ if σ_v is tempered. It has been checked for many cases when σ_v is tempered and generic, and when G is a classical group ([CKPSS04]).

By assuming Shahidi's conjecture on $\mathcal{N}_v(w_0, s)(\Phi_v(\cdot; s, \phi_{\sigma_v}))$, we know that the existence of a pole at $s = s_0$ of $E(g; s, \phi_\sigma)$ is equivalent to the existence of a pole at $s = s_0$ of the product of L-functions

$$\prod_l L(ls, \sigma, r_l).$$

Following from Shahidi ([Sh88]), if σ is generic, then the L-function $L(s, \sigma, r_l)$ is holomorphic for $Re(s) > 1$. If we write

$$\prod_l L(ls, \sigma, r_l) = L(s, \sigma, r_1) L(2s, \sigma, r_2) \prod_{l \geq 3} L(ls, \sigma, r_l).$$

It follows that $\prod_{l \geq 3} L(ls, \sigma, r_l)$ is holomorphic for $Re(s) \geq \frac{1}{2}$. Then the existence of a pole at $s = s_0$ of the product of L-functions

$$\prod_l L(ls, \sigma, r_l)$$

is equivalent to either that $L(s, \sigma, r_1)$ has a pole at $s = 1$ (Case (1))or that $L(s, \sigma, r_1)$ does not vanish at $s = \frac{1}{2}$ and $L(s, \sigma, r_2)$ has a pole at $s = 1$ (Case $(\frac{1}{2})$).

Remark 1.6. *One expects that Case* (1) *and Case* $(\frac{1}{2})$ *should not occur at the same time, but it is possible for both not to occur. For classical groups and generic* σ, *this follows from the explicit Langlands functorial lifting from classical groups to the general linear group. In particular, the structures of* σ *related to these two cases are essentially different.*

1.2 Model-comparison

In the previous subsection, the existence of the residual representation

$$\mathcal{E}_{s_0}(g;\phi_\sigma) := \mathrm{Res}_{s=s_0} E(g;s,\phi_\sigma) \tag{1.4}$$

is characterized in terms of the existence of a pole at $s = s_0$ of the product of L-functions

$$L(s,\sigma,r_1)L(2s,\sigma,r_2) \tag{1.5}$$

assuming that σ is generic. The question in this subsection is to ask how to describe the structure of the residual representation $\mathcal{E}_{s_0}(g;\phi_\sigma)$ in terms of the structure of the cuspidal support σ, or vise versa. From a representation-theoretic point of view, structures of representations are usually characterized by means of certain invariants attached to the representations. For automorphic representations, it is natural to consider period integrals in order to distinguish automorphic representation from each other. By reciprocity, that an automorphic representation has a certain nonzero period is equivalent to that the automorphic representation has a certain model. For example, it has a Whittaker model if the representation is generic.

When consider two automorphic representations which are related via a certain kind of transfers, say, Langlands functoriality, theta correspondence, or endoscopy lifting, it is natural to expect the two automorphic representations should share certain compatible models. This should be viewed as a version of the Langlands functoriality principle. In reality, one may expect an explicit formula which relates a certain model for one representation to a certain model for the other representation. This is what we called *Model-comparison identity*. Such an idea is not really new. One can find such model-comparison identities in the constructive theory of automorphic forms, such as the Langlands theory of constant terms of Eisenstein series, the Langlands-Shahidi method and the Rankin-Selberg method to study automorphic L-functions, explicit theta correspondences, the descent method (backward liftings) of automorphic forms, relative trace formula method, etc.

We consider in the following the relation between the cuspidal datum (P, σ) and the residual representation $\mathcal{E}_{s_0}(g; \phi_\sigma)$ of $G(\mathbb{A})$. It is the simplest case of Langlands functoriality from (M, σ) to $(G, \mathcal{E}_{s_0}(g; \phi_\sigma))$. The problem is: for a given (M, σ) and the value s_0, to find a suitable spherical subgroup H of G, an automorphic representation τ of $M \cap H$, and an automorphic representation $\mathcal{E}_0(h; \phi_\tau)$ of $H(\mathbb{A})$, such that there exists an identity which relates the period (called the outer period) of $\mathcal{E}_{s_0}(g; \phi_\sigma)$

$$\int_{H(k)\backslash H(\mathbb{A})} \mathcal{E}_{s_0}(h; \phi_\sigma)\mathcal{E}_0(h; \phi_\tau)dh \tag{1.6}$$

and the period (called the inner period) of σ

$$\int_{M\cap H(k)\backslash M\cap H(\mathbb{A})} \phi_\sigma(x)\phi_\tau(x)dx. \tag{1.7}$$

A Theorem one wishes to prove is

Theorem 1.7. *The nonvanishing of outer period (1.6) is equivalent to the nonvanishing of inner period (1.7).*

Remark 1.8. *Since the Langlands functoriality from σ to $\mathcal{E}_{s_0}(g; \phi_\sigma)$ is nothing but the 'parabolic induction', it is natural to find the data $(H, \tau, \mathcal{E}_0(h; \phi_\tau))$ through the Frobenius reciprocity law. More precisely, one has to study the orbital decomposition*

$$P\backslash G/H$$

and possible quasi-invariant distributions attached to each of the orbits. Note that the Rankin-Selberg method for automorphic L-functions uses the Zariski open orbit! However, in the model-comparison study, one pays more attention to the other lower dimensional orbits, the closed orbits in particular.

Of course, one has to regularize the period integral to deal with the convergence problem. One can apply the Arthur's truncation method to one of the residual automorphic forms in the integrand. Other methods of regularization may also apply, see [LR04] for instance.

Remark 1.9. *Some very interesting cases have been treated by this approach and have some striking consequences.*

(1) The first case was studied by Jacquet and Rallis in [JR92]. In this case, $G = \mathrm{GL}_{2n}$, $M = \mathrm{GL}_n \times \mathrm{GL}_n$, $H = \mathrm{Sp}_{2n}$, and $M \cap H = \mathrm{GL}_n^\Delta$, the

diagonal embedding into M. The representation τ is one-dimensional and the representation $\mathcal{E}_0(\cdot;\phi_\tau)$ is the trivial representation of $H(\mathbb{A})$.

(2) The following three cases was uniformly treated by the author in [J98a], which was for the first time to connect the model-comparison method with the Langlands-Shahidi method and to connect the (inner) period to the special value of certain L-functions without help of the Rankin-Selberg integral formula for automorphic L-functions.

- *(2i) $G = G_2$ (k-split), $M = \mathrm{GL}_2$ (attached to the short root), $H = \mathrm{SL}_3$ (naturally embedded in G_2), and $M \cap H$ is GL_1 (one-dimensional torus).*
- *(2ii) $G = B_3$ (k-split), $M = A_1 \times B_1$, $H = G_2$ (naturally embedded into B_3, and $M \cap H$ is A_1^Δ (diagonally embedded into M).*
- *(2iii) $G = D_4$ (k-split), $M = A_1 \times A_1 \times A_1$, $H = G_2$ (embedded into D_4 via triality), and $M \cap H$ is A_1^Δ (diagonally embedded into M). This last case gives alternative proof of a conjecture of Jacquet for the split period case.*

(3) The case (F_4, D_4) was considered in [GJ01].
(4) The case $(\mathrm{Sp}_{4n}, \mathrm{Sp}_{2n} \times \mathrm{Sp}_{2n})$ was studied in [GRS99].

Remark 1.10. *From the above arguments, it follows that the nonvanishing of the inner period in (1.7) implies the existence of the pole at s_0 of the product of L-functions in (1.5). The converse to this statement is known as a version of the conjecture of Gross-Prasad type ([GP92], [GP94], [GJR04], and [GJR]). We will discuss this issue in the next section.*

2 Rankin-Selberg L-functions

In this section, we discuss in more detail how to apply the general approach described in the previous section to the case of the Rankin-Selberg L-functions $L(s, \pi_1 \times \pi_2)$, where π_1, π_2 are irreducible cuspidal automorphic representations of $\mathrm{GL}_n(\mathbb{A})$, $\mathrm{GL}_m(\mathbb{A})$, respectively. This automorphic L-function has been studied by many people in [JPSS83], [Sh84], [MW89], and [CPS04], and the following properties have been proved among others:

(1) $L(s, \pi_1 \times \pi_2)$ converges for $Re(s) > 1$ and has meromorphic continuation to the whole complex plane $\mathbb{C}$;

(2) $L(s, \pi_1 \times \pi_2)$ has a possible simple pole at $s = 1$, and it has a simple pole at $s = 1$ if and only if $\pi_2 \cong \pi_1^\vee$;

(3) the functional equation of $L(s, \pi_1 \times \pi_2)$ relates the value s to $1 - s$, and if both π_1 and π_2 are self-dual, then $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is a real number.

We are interested in a characterization of the nonvanishing of the central value $L(\frac{1}{2}, \pi_1 \times \pi_2)$. We recall first the global integral from [CPS04]. Assume that $n > m$. Define a unipotent subgroup $Y_{n,m}$ to be

$$Y_{n,m} := \{y = \begin{pmatrix} I_{m+1} & * \\ 0 & n \end{pmatrix} \mid n \in N_{n-m-1}\}$$

where N_{n-m-1} is the standard maximal upper triangular unipotent subgroup of GL_{n-m-1}. Define a character of $Y_{n,m}$ by

$$\psi_{n,m}(y) := \psi_0(n_{1,2} + \cdots + n_{n-m-2,n-m-1})$$

where ψ_0 is a given nontrivial additive character. For an automorphic form φ_{π_1} in the space of π_1, we define the $\psi_{n,m}$-Fourier coefficient of φ_{π_1} by

$$\mathcal{F}^{\psi_{n,m}}(\varphi_{\pi_1})(g) := \int_{Y_{n,m}(k)\backslash Y_{n,m}(\mathbb{A})} \varphi_{\pi_1}(yg)\psi_{n,m}^{-1}(y)dy.$$

It is clear that $\mathcal{F}^{\psi_{n,m}}(\varphi_{\pi_1})(g)$ is automorphic over $\mathrm{GL}_m(\mathbb{A})$ via the embedding

$$x \in \mathrm{GL}_m \mapsto \begin{pmatrix} x & 0 \\ 0 & I_{n-m} \end{pmatrix} \subset \mathrm{GL}_n.$$

From [CPS04], one knows

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathcal{F}^{\psi_{n,m}}(\varphi_{\pi_1})(x)\varphi_{\pi_2}(x)|\det x|^{s-\frac{1}{2}}dx = \mathcal{Z}_S(s, \varphi_{\pi_1}, \varphi_{\pi_2}) \cdot L^S(s, \pi_1 \times \pi_2)$$

where S is the finite subset of local places, including the archimedean places, where one of π_1, π_2, and ψ is ramified. After establishing the complete local theory of the Rankin-Selberg convolution, one expects that the central value $L(\frac{1}{2}, \pi_1 \times \pi_2)$ does not vanish if and only if the following period

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathcal{F}^{\psi_{n,m}}(\varphi_{\pi_1})(x)\varphi_{\pi_2}(x)dx \tag{2.1}$$

does not vanish.

When both π_1 and π_2 are self-dual, the Langlands principle of functoriality asserts that π_1 and π_2 are lifted from classical groups. In such a situation, the central value $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is expected to have connection with period integral over classical groups. Such periods may have more interesting arithmetic meanings.

2.1 A classification

In the following we assume that both π_1 and π_2 are self-dual. We first classify the tensor product $\pi_1 \otimes \pi_2$ in terms of the Langlands principle of functoriality.

Let π be an irreducible, unitary, self-dual, cuspidal, automorphic representation of $GL_n(\mathbb{A})$. Then $L(s, \pi \otimes \pi^\vee)$ has a simple pole at $s = 1$. Since the tensor product L-function $L(s, \pi \otimes \pi^\vee)$ can be expressed as a product of the exterior square L-function and the symmetric square L-function:

$$L(s, \pi \otimes \pi^\vee) = L(s, \pi, \Lambda^2) L(s, \pi, S^2),$$

one and only of the L-functions $L(s, \pi, \Lambda^2)$ and $L(s, \pi, S^2)$ has a simple pole at $s = 1$.

We call π *symplectic* (or *orthogonal*) if the exterior square L-function $L(s, \pi, \Lambda^2)$ (or the symmetric square L-function $L(s, \pi, S^2)$, resp.) has the pole at $s = 1$. This gives a classification of irreducible self-dual cuspidal automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$.

Theorem 2.1 (Jiang-Soudry [JS03]). *If $L(s, \pi, \Lambda^2)$ has a (simple) pole at $s = 1$, then each local component π_v of π is symplectic, i.e. the local Langlands parameter for π_v factorizes through $\mathrm{Sp}_n(\mathbb{C})$, the dual group of SO_{n+1}. Note that in this case n must be even.*

Remark 2.2. *This was a conjecture of Prasad and Ramakrishnan in [PR96]. For other classical groups, it is our work in progress.*

Now we apply the above classification to the tensor product $\pi_1 \otimes \pi_2$, which can be classified as follows:

- **Case (O).** π_1 and π_2 are either both orthogonal or both symplectic.
- **Case (S).** One of the π_1 and π_2 is orthogonal and the other is symplectic.

These two cases are *essentially different* when we consider problems related to $L(\frac{1}{2}, \pi_1 \times \pi_2)$, since $\pi_1 \otimes \pi_2$ is orthogonal (or symplectic) (conjecturally) in **Case (O)** (or in **Case (S)**), resp.)

In **Case (O)**, we understand the possible pole of $L(s, \pi_1 \times \pi_2)$ at $s = 1$ and the 'functorial' relation between π_1 and π_2. But, it is mysterious to find a 'reasonable' period relating to $L(\frac{1}{2}, \pi_1 \times \pi_2)$ since $\frac{1}{2}$ may not be critical in motivic sense. However, in **Case (S)**, $\pi_1 \otimes \pi_2$ is **symplectic**. it was proved in [LR03] and [Lp03] that

$$L(\frac{1}{2}, \pi_1 \times \pi_2) \geq 0$$

based on the functorial relations between generic cuspidal representations of classical groups and GL ([CKPSS04], [GRS01], [K02], and [JS03]), and based on the spectral theory of automorphic forms. It is a very interesting question to characterize the nonvanishing of the central value $L(\frac{1}{2}, \pi_1 \times \pi_2)$ in **Case (S)**.

Assume π_1 is *symplectic*. This implies $n = 2r$ is even ([K00]). Then π_2 must be *orthogonal*. This leads two different cases:

- **Case (S1)** if $m = 2l$; and
- **Case (S2)** if $m = 2l + 1$.

By means of the automorphic descent construction of Ginzburg, Rallis and Soudry, we find, in **Case (S1)**, periods defined over **orthogonal** groups, which is of the generalized Gelfand-Graev type, and in n **Case (S2)**, periods defined over *symplectic groups* or *metaplectic groups*, which is of the Fourier-Jacobi type.

To define the periods of the Fourier-Jacobi type (**Case (S2)**) we have to introduce Fourier-Jacobi coefficients of automorphic forms. For simplicity, I discuss only in detail the periods of the generalized Gelfand-Graev type (**Case (S1)**).

Theorem 2.3 (Ginzburg-Rallis-Soudry [GRS01], [S02]). *If π_1 is symplectic as defined above, there exists an irreducible generic cuspidal automorphic representation σ of $\mathrm{SO}_{2r+1}(\mathbb{A})$ $(n = 2r)$ s.t. π_1 is a weak Langlands functorial lift from σ. If π_2 is orthogonal and $m = 2l$ as above, there exists an irreducible generic cuspidal automorphic representation τ of $\mathrm{SO}_{2l}(\mathbb{A})$ s.t. π_2 is a weak Langlands functorial lift from τ. If π_2 is orthogonal and*

$m = 2l+1$ *as above, there exists an irreducible generic cuspidal automorphic representation* τ *of* $\mathrm{Sp}_{2l}(\mathbb{A})$ *s.t.* π_2 *is a weak Langlands functorial lift from* τ.

It is proved in [CKPSS04] ([JS03], [K02] for SO_{2n+1}) that these weak liftings are all strong Langlands functorial liftings.

2.2 Periods

We may assume that $r \geq l$. We denote that $G_r := \mathrm{SO}_{2r+1}$ and $H_l := \mathrm{SO}_{2l}$. We introduce a standard unipotent subgroup

$$V_{r,l} := \{v(n,x,z) = \begin{pmatrix} n & x & z \\ & I_{2l+1} & x^* \\ & & n^* \end{pmatrix}\} \subset G_r$$

where $n \in N_{r-l}$, which is the standard upper triangular unipotent subgroup of GL_{r-l}. Define a character $\psi_{r,l}$ of $V_{r,l}(\mathbb{A})$ by

$$\psi_{r,l}(v) := \psi_0(n_{1,2} + \cdots + n_{r-l-1,r-l})\psi_0(x_{r-l,l+1}).$$

It is clear that $\psi_{r,l}$ is trivial on $V_{r,l}(k)$. Denote the normalizer of $V_{r,l}$ in G_r by

$$M := N_{G_r}(V_{r,l}) \cong GL_1^{r-l} \times SO_{2l+1}.$$

Then the connected component of the stabilizer of $\psi_{r,l}$ in M is isomorphic to H_l.

When $r \leq l$. similar notations can be introduced, and we omit the details.

For $\varphi_\sigma \in V_\sigma$ and $\varphi_\tau \in V_\tau$, define $\psi_{r,l}$-Fourier coefficient of φ_σ to be

$$\mathcal{F}^{\psi_{r,l}}(\varphi_\sigma)(h) := \int_{V_{r,l}(k)\backslash V_{r,l}(\mathbb{A})} \varphi_\sigma(vh)\psi_{r,l}(v)^{-1}dv,$$

and the period is defined as follows:

$$\mathcal{P}_{r,l}(\varphi_\sigma, \varphi_\tau, \psi_{r,l}) := \int_{H_l(k)\backslash H_l(\mathbb{A})} \mathcal{F}^{\psi_{r,l}}(\varphi_\sigma)(h)\varphi_\tau(h)dh. \tag{2.2}$$

Remark 2.4. *This period is also called a period of generalized Gelfand-Graev type. One can also relate the above definition with the unipotent orbit corresponding to the partition* $[2l+1, 1^{2(r-l)}]$ *of* $2r+1$. *See [JS05] for detailed discussion, for instance.*

One of the main results related to the central value

$$L(\frac{1}{2}, \sigma \times \tau) = L(\frac{1}{2}, \pi_1 \times \pi_2)$$

is the following theorem.

Theorem 2.5 ([GJR04],[GJR]). *If the period $\mathcal{P}_{r,l}(\varphi_\sigma, \varphi_\tau, \psi_{r,l})$ is not identically zero on the space $V_\sigma \otimes V_\tau$, then $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is nonzero.*

This theorem for **Case (S1)** is proved in [GJR] and for **Case (S2)** is proved in [GJR04]. The proof follows the general argument sketched in §1. The key point is to consider the following two Eisenstein series:

(1) the Eisenstein series $E(g; s, \phi_{\pi_1 \otimes \tau})$ on SO_{4r+2l} attached to the cuspidal data $\pi_1 \otimes \tau$ of $\mathrm{GL}_{2r} \times \mathrm{SO}_{2l}$;

(2) the Eisenstein series $E(g; s, \phi_{\pi_1 \otimes \sigma})$ on SO_{6r+1} attached to the cuspidal data $\pi_1 \otimes \sigma$ of $\mathrm{GL}_{2r} \times \mathrm{SO}_{2r+1}$.

Proposition 2.6 ([GJR04],[GJR]). *The Eisenstein series $E(g; s, \phi_{\pi_1 \otimes \tau})$ has a possible simple pole at $s = \frac{1}{2}$. The residue, denoted by $\mathcal{E}_{\frac{1}{2}}(g; \phi_{\pi_1 \otimes \tau})$, is nonzero if and only if $L(\frac{1}{2}, \pi_1 \times \tau)$ is nonzero. The Eisenstein series $E(g; s, \phi_{\pi_1 \times \sigma})$ has a simple pole at $s = 1$, and the residue is denoted by $\mathcal{E}_1(g; \phi_{\pi_1 \times \sigma})$.*

Then the 'outer' period is given by

$$\mathcal{P}_{3r,2r+l}(\mathcal{E}_1(\cdot; \phi_{\pi_1 \times \sigma}), \mathcal{E}_{\frac{1}{2}}(\cdot; \phi_{\pi_1 \otimes \tau}), \psi_{3r,2r+l}).$$

The 'specialization' of Theorem 1.7 to this case is the following theorem.

Theorem 2.7 ([GJR04],[GJR]). *The nonvanishing of outer period*

$$\mathcal{P}_{3r,2r+l}(\mathcal{E}_1(\cdot; \phi_{\pi_1 \times \sigma}), \mathcal{E}_{\frac{1}{2}}(\cdot; \phi_{\pi_1 \otimes \tau}), \psi_{3r,2r+l})$$

is equivalent to the nonvanishing of the inner period $\mathcal{P}_{r,l}(\varphi_\sigma, \varphi_\tau, \psi_{r,l})$.

Hence, following the argument in §1, we prove Theorem 2.1. Conversely, one expects to show that the nonvanishing of the central value $L(\frac{1}{2}, \pi_1 \times \pi_2)$ implies the nonvanishing of a certain period of type (1.7). This is an analogue of the Gross-Prasad conjecture ([GP92] and [GP94]). We proved some special cases as stated below.

Theorem 2.8 ([GJR04],[GJR]). *Assume that $r \geq l$ with $l \leq 1$. If $L(\frac{1}{2}, \pi_1 \times \pi_2)$ is nonzero, then there are σ' and τ' s.t. the period $\mathcal{P}_{r,l}(\varphi_{\sigma'}, \varphi_{\tau'}, \psi_{r,l})$ is not identically zero on the space $V_{\sigma'} \otimes V_{\tau'}$, where σ' is an irreducible cuspidal automorphic representation of SO'_{2r+1} (an inner form of SO_{2r+1}) which is nearly equivalent to σ, and τ' is an irreducible cuspidal automorphic representation of SO'_{2l} (an inner form of SO_{2l}) which is nearly equivalent to τ. Note that the pair $(\mathrm{SO}'_{2r+1}, \mathrm{SO}'_{2l})$ forms a relevant pair.*

Remark 2.9. *A relevant pair of two nondegenerate quadratic spaces can be defined as follows. Let $(V, (\cdot,\cdot)_V)$ and $(W, (\cdot,\cdot)_W)$ be two nondegenerate quadratic spaces over k. The pair (V, W) is called a* **relevant pair** *if the following holds*

(1) $|\dim_k V - \dim_k W|$ *is odd;*

(2) the quadratic space $V \oplus W$ with quadratic form

$$(\cdot,\cdot)_{V\oplus W} := (\cdot,\cdot)_V - (\cdot,\cdot)_W$$

is split; i.e. has the Witt index $[\frac{\dim_k V + \dim_k W}{2}]$.

The pair $(SO(V), SO(W))$ is called a relevant pair if (V, W) is a relevant pair. It is clear that for a relevant pair $(SO(V), SO(W))$, one can define the periods for automorphic forms on $SO(V)$ and $SO(W)$ as before.

Remark 2.10. *We proved the above theorem for general l under the assumption of the existence of a certain nontrivial Fourier coefficient of the residue $E_{\frac{1}{2}}(g, \phi_{\pi_2 \otimes \sigma})$. More precisely we assume that if the residue is nonzero, it has a nonzero Fourier coefficient associated to the unipotent orbit of $SO_{4l+2r+1}(\mathbb{C})$ corresponding to the partition $[(2l + 2r + 1)1^{2l}]$. We conjecture that the residue $E_{\frac{1}{2}}(g, \phi_{\pi_2 \otimes \sigma})$, if nonzero, has a nonzero Fourier coefficient associated to the unipotent orbit of $SO_{4l+2r+1}(\mathbb{C})$ corresponding to the partition $[(2l + 2r + 1)(2l - 1)(1)]$. More details can be found in [GJR04].*

Remark 2.11. *The most significant lower rank case of the Gross-Prasad conjecture is the case when the relevant pair is $(\mathrm{SO}_4, \mathrm{SO}_3)$. In this case, it is a conjecture of Jacquet. Let π_1, π_2, and π_3 be irreducible cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$ with the product of the central characters being trivial, i.e.*

$$\omega_{\pi_1} \cdot \omega_{\pi_2} \cdot \omega_{\pi_3} = 1.$$

The conjecture of Jacquet asserts that the central value of the triple product L-function

$$L(\frac{1}{2}, \pi_1 \times \pi_2 \times \pi_3)$$

is nonzero if and only if there is a unique quaternion algebra $\mathcal{D}$ *over* k *such that the trilinear period integral*

$$\int_{Z_{\mathcal{D}^\times}(\mathbb{A})\mathcal{D}^\times(k)\backslash\mathcal{D}^\times(\mathbb{A})} \varphi_{\pi_1^{\mathcal{D}}}(x)\varphi_{\pi_2^{\mathcal{D}}}(x)\varphi_{\pi_3^{\mathcal{D}}}(x)dx$$

is nonzero for a certain choice of $\varphi_{\pi_i^{\mathcal{D}}}$, $i- = 1, 2, 3$, *where* $\pi_1^{\mathcal{D}}$ *is the image of* π_i *under the Jacquet-Langlands correspondence.*

The conjecture of Jacquet was completely proved in [HK04]. In [J98a], [J98b], and [J01], a different approach has been applied to the split period case of Jacquet's conjecture.

Remark 2.12. *More recently, Boecherer, Furusawa, and Schultze-Pillot proved some special case for the relevant pair* $(\mathrm{SO}_5, \mathrm{SO}_4)$ *([BFSP04]), and Ichino and Ikeda proved some special case for the relevant pair* $(\mathrm{SO}_6, \mathrm{SO}_5)$ *([II04]).*

Remark 2.13. *The analogue for unitary groups is the work in progress of Ginzburg and Jiang.*

Remark 2.14. *In [J05], a systematic account of general periods of automorphic forms and related basic problems and topics has be found. Some useful references have been provided below. We hope they are useful to the interested readers.*

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