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On equidistribution properties of Hecke eigenforms

Wenzhi Luo

Ohio State University

1 Introduction.

This is a brief survey of some of the recent progress on equidistribution properties of Hecke eigenforms on arithmetic surfaces. For simplicity let $\Gamma=SL(2,\mathbf{Z})$ be the full modular group and $X_{\Gamma}=\Gamma\backslash\mathbf{H}$ be the corresponding modular surface, where $\mathbf{H}$ denotes the upper half plane. On X_{Γ} we have the (normalized) invariant measure

$$d\mu=\frac{1}{\text{area}(X_{\Gamma})}\frac{dxdy}{y^{2}},$$

associated to the Poincaré metric $y^{-2}(dx^{2}+dy^{2})$. For integer $k\geq 8$, denote by $S_{2k}(\Gamma)$ the space of holomorphiic cusp forms of weight $2k$ with respect to Γ. It is well-known that $S_{2k}(\Gamma)$ is a finite-dimensional Hilbert space with respect to the Petersson scalar product. Define $J_{k}=\dim_{\mathbf{C}}S_{2k}(\Gamma)$, and recall that by the Riemann-Roch theorem we have

$$J_{k}\sim k/6$$

as $k\rightarrow\infty$. Let $\{f_{k,j}\}_{1\leq j\leq J_{k}}$ be the orthonormal basis of Hecke eigenforms in $S_{2k}(\Gamma)$. Each $f_{k,j}$ gives rise to a new probablity measure on $S_{2k}(\Gamma)$:

$$d\mu_{k,j}=y^{2k}|f_{k,j}|^{2}d\mu.$$

As an analogue of the quantum unique ergodicity conjecture for the Maass-Hecke eigenforms [Sar], one can formulate the following equidistribution conjecture for the holomorphic Hecke eigenforms:

Conjecture. For any compact region $A \subset X_\Gamma$, we have

$$\lim_{k\to\infty} \int_A d\mu_{k,j} = \int_A d\mu \tag{1}$$

This is equivalent to

$$\lim_{k\to\infty} \int_{X_\Gamma} \phi d\mu_{k,j} = \int_{X_\Gamma} \phi d\mu \tag{2}$$

for any Schwarz function $\phi \in \mathcal{S}(X_\Gamma)$.

Let's first examine the depth and the arithmetic implication of this conjecture. Take ϕ to be an even Maass-Hecke eigenform of eigenvalue $\lambda_\phi = \frac{1}{4} + t_\phi^2$, then

$$\int_{X_\Gamma} \phi d\mu = 0 \ .$$

On the other hand, Harris-Kudla [HK] and Watson [Wa] proved, by means of the theta correspondence and Siegel-Weil formula, that

$$\left| \int_{X_\Gamma} \phi d\mu_{k,j} \right|^2 = \frac{\Lambda(1/2, \mathrm{sym}^2(f_{k,j}) \otimes \phi)\Lambda(1/2,\phi)}{\Lambda(1, \mathrm{sym}^2(f_{k,j}))^2 \Lambda(1, \mathrm{sym}^2(\phi))}, \tag{3}$$

where

$$\Lambda(s,\ \phi) = \pi^{-s}\Gamma\left(\frac{s+it_\phi}{2}\right)\Gamma\left(\frac{s-it_\phi}{2}\right)L(s,\ \phi),$$

$$\Lambda(s, \mathrm{sym}^2(\phi)) = \pi^{-3s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+2it_\phi}{2}\right)\Gamma\left(\frac{s-2it_\phi}{2}\right)L(s, \mathrm{sym}^2(\phi)),$$

$$\Lambda(s,\ f_{k,j}) = (2\pi)^{-s}\Gamma(s+(2k-1)/2)L(s,\ f),$$

$$\Lambda(s,\ \mathrm{sym}^2(f_{k,j})) = \pi^{-3s/2}\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+2k-1}{2}\right)\Gamma\left(\frac{s+2k}{2}\right)L(s,\ \mathrm{sym}^2(f_{k,j})),$$

$$\Lambda(s, \mathrm{sym}^2(f_{k,j}) \otimes \phi) = \pi^{-3s}\Gamma\left(\frac{s+2k-1+it_\phi}{2}\right)\Gamma\left(\frac{s+2k-1-it_\phi}{2}\right)$$
$$\times\Gamma\left(\frac{s+2k+it_\phi}{2}\right)\Gamma\left(\frac{s+2k-it_\phi}{2}\right)\Gamma\left(\frac{s+1+it_\phi}{2}\right)\Gamma\left(\frac{s+1-it_\phi}{2}\right)L(s,\ \mathrm{sym}^2(f_{k,j})\otimes\phi).$$

Thus, (2) boils down to the subconvexity bound

$$L(1/2,\ \mathrm{sym}^2(f_{k,j}) \otimes \phi) = o(k)$$

as $k \to \infty$, while currently the best bound we know is only

$$L(1/2,\ \mathrm{sym}^2(f_{k,j}) \otimes \phi) = O_{\phi,\,\epsilon}(k^{1+\epsilon}),$$

which follows from the Phragmén-Lindelöf convexity principle. The Generlized Riemann Hypothesis would implies

$$L(1/2,\ \mathrm{sym}^2(f_{k,j}) \otimes \phi) = O_{\phi,\,\epsilon}(k^{\epsilon}),$$

which in turn predicts the rate of convergence for the equdistribution

$$\int_{X_\Gamma} \phi d\mu_{k,j} = O_{\phi,\,\epsilon}(k^{-1/2+\epsilon}). \tag{4}$$

2 A theorem of Shiffman-Zelditch

Shiffman-Zelditch [SZ] proved the following theorem:

Theorem (Shiffman-Zelditch): There exists a full density subsequence of $\{f_{k,j}\}_{1\le j\le J_k,\, k\ge 8}$ such that (1) holds, i.e. there exist a subset $\Lambda_k \subset \{1, \cdots, J_k\}$ satisfying

$$\lim_{k\to\infty} \frac{\#\Lambda_k}{J_k} = 1\ ,$$

such that for any compact region $A \subset X_\Gamma$, we have

$$\lim_{k\to\infty, j\in\Lambda_k} \int_A d\mu_{k,j} = \int_A d\mu\ . \tag{5}$$

Moreover using the potential theory, they showed that the zeros of the sequence $f_{k,j}$, $j \in \Lambda_k$ are also equidistributed:

$$\lim_{k\to\infty, j\in\Lambda_k} \frac{\#\{z \in A,\ f_{k,j}(z) = 0\}}{J_k} = \int_A d\mu\ . \tag{6}$$

Note by the Riemann-Roch theorem, we have

$$\#\{z \in X_\Gamma,\ f_{k,j}(z) = 0\} \sim J_k, \quad \text{as } k \to \infty\,.$$

If we define $L = T^*(X_\Gamma)$, the cotangent bundle to X_Γ, and denote by X_Γ^* the compactification of X_Γ, then we have the interpretation of $S_{2k}(\Gamma)$ as the space of holomorphic sections of $2k$-th tensor power of L with vanishing condition at the cusp:

$$S_{2k}(\Gamma) = H^0_{cusp}(X_\Gamma^*,\ L^{2k}).$$

In this context, the theorem of Shiffman-Zelditch has an anologue for holomorphic sections of tensor powers of any ample Hermitian line bundle L on a compact Kähler manifold X of higher dimension. For details see [SZ].

3 Bergman kernel

Define a new probablity measure on X_Γ by

$$d\mu_k = \frac{\sum_{j=1}^{J_k} y^{2k}|f_{k,j}(z)|^2}{J_k} d\mu,$$

which is an average of the measures $d\mu_{k,j}$, $1 \le j \le J_k$. As a first step towards the conjecture (1), we proved the following theorem [L]:

Theorem. For any measurable subset A on the modular surface X, we have

$$\lim_{k\to\infty} \int_A d\mu_k = \int_A d\mu. \tag{7}$$

In fact, for any $\epsilon > 0$,

$$\int_A d\mu_k = \int_A d\mu + O_\epsilon(k^{-1/2+\epsilon}) \tag{8}$$

holds uniformly for all A on X.

The Hecke operator $T_k(m)$ $(m \ge 1)$ acts on cusp form $f \in S_{2k}(\Gamma)$ by

$$T_k(m)f = \frac{1}{n} \sum_{ad=m} a^{2k} \sum_{0\le b<d} f\left(\frac{az+b}{d}\right).$$

$T_k(m)$ can be represented by the holomorphic automorphic kernel $C_k^{-1}m^{2k-1}h_{k,m}(z,\ z')$ (C_k defined in (12) below),

$$h_{k,m}(z,z') = \sum_{ad-bc=m} (czz' + dz' + az + b)^{-2k}, \tag{9}$$

where the sum is taken over all integer matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant m, in the sense that

$$\left\langle f,\ C_k^{-1}m^{2k-1}h_{k,m}(\cdot\ , -\overline{z'}\right\rangle_k = (T_k(m)f)(z'), \tag{10}$$

where $<,>_k$ is the (normalized) Petersson inner product on $S_{2k}(\Gamma)$. This kernel was first studied by Petersson, and later used by Zagier to give a new proof the Eichler-Selberg trace formula for the Hecke operatos.

The series in (9) is absolutely convergent and $h_{k,m}(z,\ z')$ as a function of each variable z or z' is a cusp form in $S_{2k}(\Gamma)$, and we have the identity

$$C_k^{-1}m^{2k-1}h_{k,m}(z,\ z') = \sum_{j=1}^{J_k} \lambda_{j,k}(m) f_{j,k}(z) f_{j,k}(z'), \tag{11}$$

where $\lambda_{j,k}(m)$ is the Hecke eigenvalue of $f_{j,k}$ under $T_k(m)$ and

$$C_k = \frac{3(-1)^k}{2^{(2k-3)}(2k-1)}. \tag{12}$$

In particular, for $m = 1$ and $z' = z$, we obtain

$$C_k^{-1}h_{k,1}(z,\ -\overline{z}) = \sum_{j=1}^{J_k} |f_{j,k}(z)|^2. \tag{13}$$

Let χ_A denote the characteristic function of A on X. One can extend it (with the same notation) to $\mathbf{H}$ as a Γ-invariant function. We have

$$\begin{aligned} \int_A d\mu_k &= \frac{1}{J_k}\int_X \chi_A(z)(\sum_{j=1}^{J_k} y^{2k}|f_{j,k}(z)|^2)d\mu \\ &= \frac{1}{J_kC_k}\int_X \chi_A(z)h_1(z,-\overline{z})y^{2k}d\mu \\ &= \frac{1}{J_kC_k}\int_X \chi_A(z)\left(\sum_{ad-bc=1} \frac{y^{2k}}{(c|z|^2+d\overline{z}-az-b)^{2k}}\right)d\mu. \end{aligned} \tag{14}$$

Since replacing z by γz ($\gamma \in \Gamma$) in each term of the sum in (14) amounts to replacing

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{by} \quad \gamma^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\gamma,$$

we may decompose the sum into Γ-invarint pieces with $a+d=t$, $t \in \mathbf{Z}$. Thus,

$$\int_A d\mu_k = \sum_{t=-\infty}^{\infty} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{ad-bc=1,\, a+d=t} \frac{y^{2k}}{(c|z|^2 + d\bar{z} - az - b)^{2k}} \right) d\mu. \tag{15}$$

There is a bijection between the integral matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with determinant 1 and trace t, and the set of integral binary quadratic forms g with discriminant $\text{disc}(g) = t^2 - 4$. The bijection is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto g(u,\, v) = cu^2 + (d-a)uv - bv^2. \tag{16}$$

$$g(u,\, v) = \alpha u^2 + \beta uv + \gamma v^2 \mapsto \begin{pmatrix} (t-\beta)/2 & -\gamma \\ \alpha & (t+\beta)/2 \end{pmatrix}. \tag{17}$$

For $g(u,\, v) = \alpha u^2 + \beta uv + \gamma v^2$ and $z = x + iy$, set

$$R_g(z,\, t) = \frac{y^{2k}}{(\alpha(x^2+y^2) + \beta x + \gamma - ity)^{2k}}, \tag{18}$$

then for $\gamma \in \Gamma$ we have

$$R_{\gamma^T g \gamma}(z,\, t) = R_g(\gamma z,\, t), \tag{19}$$

and (15) can be written as

$$\int_A d\mu_k = \sum_{t=-\infty}^{\infty} \frac{1}{J_k C_k} \int_X \chi_A(z) \left(\sum_{\text{disc}(g)=t^2-4} R_g(z,\, t) \right) d\mu, \tag{20}$$

where the sum is taken over all forms of discriminant $t^2 - 4$.

For each discriminant $D = t^2 - 4$ and a quadratic form g of discriminant D, we let γ_g to denote the isotropy group of elements leaving g fixed, and observe that

$$\sum_{\text{disc}(g)=D} R_g(z,\, t) = \sum_{\text{disc}(g)=D,\, \text{mod}\, \Gamma} \; \sum_{\gamma \in \Gamma_g \backslash \Gamma} R_{\gamma^T g \gamma}(z,\, t) = \sum_{\text{disc}(g)=D,\, \text{mod}\, \Gamma} \; \sum_{\gamma \in \Gamma_g \backslash \Gamma} R_g(\gamma z,\, t), \tag{21}$$

where mod Γ means the sum is taken over a set of representatives for classes of quadratic forms with discriminant D. For $D \neq 0$, recall the class number $h(D)$ is finite, and thus we obtain

$$\int_X \chi_A(z) \left(\sum_{\text{disc}(g)=D} R_g(z,\, t) \right) d\mu = \sum_{\text{disc}(g)=D,\, \text{mod}\, \Gamma} \int_{X_g} \chi_A(z) R_g(z,\, t) d\mu, \tag{22}$$

where

$$X_g = \bigcup_{\gamma \in \Gamma_g \backslash \Gamma} \gamma X, \tag{23}$$

is a fundamnetal domain for the action of Γ_g on $\mathbf{H}$ with X identified with a fundamental domain of Γ.

Denote by I_{elliptic}, $I_{\text{hyperbolic}}$ and $I_{\text{parabolic}}$ the corresponding contributions from those terms with $D = t^2 - 4 < 0$, $D = t^2 - 4 > 0$, and $D = t^2 - 4 - 0$ respectively. Then we compute (see [L])

$$I_{\text{elliptic}} = O(k^{-1/2+\epsilon}),\;\; I_{\text{hyperbolic}} = O((4/5)^k),\;\; I_{\text{parabolic}} = \int_A d\mu + O(k^{-1/2+\epsilon}) \,,$$

and conclude that

$$\int_A d\mu_k = I_{\text{elliptic}} + I_{\text{hyperbolic}} + I_{\text{parabolic}} = \int_A d\mu + O_\epsilon(k^{-1/2+\epsilon}).$$

4 Asymptotics for the co-variance

For $\phi \in \mathcal{S}(X_\Gamma)$, define

$$\mu_{k,j}(\phi) = \int_{X_\Gamma} \phi d\mu_{k,j}, \;\; \mu(\phi) = \int_{X_\Gamma} \phi d\mu \,.$$

In [LS], we computed asymptotically the variance for the equidistribution. For $\phi \in \mathcal{S}(X_\Gamma)$, we showed

$$\sum_{k \le K} \sum_{j=1}^{J_k} |\mu_{f_{k,j}}(\phi) - \mu(\phi)|^2 \sim C_\phi K \,,$$

as $K \to \infty$. Actually we studied a smoothed version of the above sum.

Theorem (Luo-Sarnak). Fix $u \in C_0^\infty(0,\infty)$. There is a non-negative Hermitian form B defined on $\mathcal{S}(X_\Gamma)$ such that for $\phi, \psi \in \mathcal{S}(X_\Gamma)$ and any $\epsilon > 0$,

$$\begin{aligned} &\sum_{k\geq 1} u\left(\frac{2k-1}{K}\right) \sum_{j=1}^{J_k} L(1, \mathrm{sym}^2(f_{k,j}))(\mu_{f_{k,j}}(\phi) - \mu(\phi))\overline{(\mu_{f_{k,j}}(\psi) - \mu(\psi))} \\ = \;& B(\phi,\ \psi)\left(\int_0^\infty u(t)dt\right) K + O_{\epsilon,\psi}(K^{1/2+\epsilon}), \end{aligned}$$

as $K \to \infty$. The Laplacian Δ and the Hecke operators T_n are self-adjoint with respect to B, i.e.

$$B(\Delta\phi, \psi) = B(\phi, \Delta\psi),$$

and

$$B(T_n\phi, \psi) = B(\psi, T_n\psi).$$

Moreover for two Maass-Hecke cusp forms ϕ and ψ, normalized so that their first Fourier coefficients are 1, we have the relation

$$B(\phi,\ \psi) = <\phi, \psi> L(1/2, \phi)\ ,$$

where $< \cdot\ ,\ \cdot >$ is the Petersson scalar product.

Since (see [I2])

$$k^{-\epsilon} \ll_\epsilon L(1, \mathrm{sym}^2(f_{k,j})) \ll_\epsilon k^\epsilon,$$

for any $\epsilon > 0$, our theorem indicates that on the average, $(\mu_{f_{k,j}}(\phi) - \mu(\phi))$ has the size of $k^{-1/2}$. Also as a by-product, we obtain a new proof of the fact that $L(1/2, \phi) \geq 0$ for Maass-Hecke cusp forms ϕ.

Our proof uses Poincaré series and trace formula, involving subtle analysis of the sum of the Salié sums and the Neumann series for J-Bessel functions of large orders. For details see [LS].

5 Linnik problem, an analogy

There is a striking analogy between the equidistributions problem for Hecke eigenforms and the Linnik problem for integer points on the spheres and the Heegner points on X_Γ. For n square-free and satisfying $-n \not\equiv 1 \pmod 8$, consider

$$V_n = \{m/|m| \in S^2,\ m \in \mathbf{Z}^3,\ |m|^2 = n\},$$

where $|m|$ is the usual Euclidean norm. A result of Gauss provides an exact formula for the number of integer points lying on the sphere $x_1^2 + x_2^2 + x_3^2 = n$,

$$\#V_n = \frac{24h(d_n)}{w_n}\left[1 - \left(\frac{d_n}{2}\right)\right],$$

where d_n, $h(d_n)$ and w_n are the discriminant, the class number and the number of units of $\mathbf{Q}(\sqrt{-n})$ respectively. Recall by Siegel's theorem,

$$n^{1/2-\epsilon} \ll_\epsilon h(-n) \ll_\epsilon n^{1/2+\epsilon}.$$

To establish that the points in V_n are equidistributed as $n \to \infty$, by Weyl's criterion, we need to show that the Weyl sum

$$W_u(n) = \frac{1}{\#V_n} \sum_{x \in V_n} u(x) = o(1), \tag{24}$$

for any spherical harmonics $u(x)$ of degree $l \geq 1$. Now $a(n) = n^{l/2}\#V_n W_u(n)$ is the n-th Fourier coefficient of the theta series

$$\theta(z,\ u) = \sum_{m \in \mathbf{Z}^3} u(m) e(z|m|^2),$$

which is a holomorphic cusp form for $\Gamma_0(4)$ of half-integral weight $l + 3/2$. Without loss of generality we may assume $\theta(z,\ u)$ is a Hecke eigenform. By Waldspurger's formula, we have

$$|a(n)|^2 = cn^{l+\frac{1}{2}} L(1/2,\ f \otimes \chi_{d_n}), \tag{25}$$

where f is the Shimura lift of $\theta(z,\ u)$ to $S_{2l+2}(\Gamma)$, χ_{d_n} is the quadratic character associated to the field $\mathbf{Q}(\sqrt{-n})$, and c is a constant depending only on $\theta(z,\ u)$ and f. Thus the resolution of Linnik's problem would result from any improvement of the convexity bound for the central L-values of the quadratic twists of f,

$$L(1/2,\ f \otimes \chi_{d_n}) = O_\epsilon(n^{1/2+\epsilon}).$$

Linnik's problem was solved by Duke [D] in 1988 based on Iwaniec's subconvexity bound [I1] for $L(1/2,\ f \otimes \chi_{d_n})$,

$$L(1/2,\ f \otimes \chi_{d_n}) = O_\epsilon(n^{3/7+\epsilon})\ .$$

Similarly Duke [D] showed that the Heegner points on X_Γ are also equidistributed.

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