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Existence of solutions with prescribed numbers of zeros of boundary value problems for ordinary differential equations with the one-dimensional p-Laplacian

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1. Introduction

In this paper we consider the existence and multiplicity of solutions for the boundary value problem

$$(1.1)\qquad (|u'|^{p-2}u')' + \lambda a(x) f(u) = 0, \quad 0 < x < 1,$$

$$(1.2)\qquad u(0) = u(1) = 0,$$

where $p > 1$ and $\lambda > 0$ is a parameter. In (1.1) we assume that a satisfies

$$(1.3)\qquad a \in C^1[0,1], \quad a(x) > 0 \quad \text{for } 0 \le x \le 1,$$

and that f satisfies the following conditions (H1) and (H2):

(H1) $f \in C(\mathbf{R})$, $sf(s) > 0$ for $s \neq 0$, and f is locally Lipschitz continuous on $\mathbf{R} \setminus \{0\}$;

(H2) there exist limits f_0 and f_∞ with $f_0, f_\infty \in [0, \infty]$ such that

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{|s|^{p-2}s}.$$

Define f_* and f^* by

$$f_* = \inf_{s\in\mathbf{R}\setminus\{0\}} \frac{f(s)}{|s|^{p-2}s} \quad \text{and} \quad f^* = \sup_{s\in\mathbf{R}\setminus\{0\}} \frac{f(s)}{|s|^{p-2}s},$$

respectively. Then it follows that $f_0, f_\infty \in [f_*, f^*]$. We note that $f(0) = 0$ by (H1). The case where $f(s) = |s|^{q-2}s$ with $q > 1$ is a typical case satisfying (H1) and (H2). In this case, $f_0 = 0$ and $f_\infty = \infty$ if $q > p$ and $f_0 = \infty$ and $f_\infty = 0$ if $q < p$.

By a solution u of (1.1) we mean a function $u \in C^1[0,1]$ with $|u'|^{p-2}u' \in C^1[0,1]$ which satisfies (1.1) at all points in $(0,1)$.

Problems of the form (1.1)–(1.2) describe some nonlinear phenomena in mathematical sciences and have been studied in recent years by many authors (see [1, 9, 13, 17, 18, 20, 22–25] and references therein). This paper is motivated by the recent works of Agarwal, Lü, and O'Regan [1]. In [1], they considered the problem

$$(|u'|^{p-2}u')' + \lambda F(x,u) = 0, \quad 0 < x < 1, \quad u(0) = u(1) = 0$$

and obtained explicit intervals of values of λ such that the problem has at least one or two positive solutions. Later, Sánchez [22] considered problem (1.1)–(1.2) in the case where a is nonnegative and measurable in $(0,1)$, and derived the existence and nonexistence results of positive solutions. Their approaches in [1], [22] are based on the fixed point theorem in cones. In this paper, we investigate the existence of sign-changing solutions of (1.1)–(1.2) by an approach based on the shooting method together with the qualitative theory for half-linear differential equations. As a consequence, we characterize the value of λ such that the problem has solutions with prescribed numbers of zeros.

Let λ_k be the k-th eigenvalue of

$$\left\{\begin{array}{l} (|\varphi'|^{p-2}\varphi')' + \lambda a(x)|\varphi|^{p-2}\varphi = 0, \quad 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0, \end{array}\right. \tag{1.4}$$

and let φ_k be an eigenfunction corresponding to λ_k. It is known that

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \lambda_{k+1} < \cdots, \quad \lim_{k\to\infty} \lambda_k = \infty,$$

and that φ_k has exactly $k-1$ zeros in $(0,1)$. (See, e.g., [2], [3], [11].) For convenience, we put $\lambda_0 = 0$.

For each $k \in \mathbf{N}$, we denote by S_k^+ (respectively S_k^-) the set of all solutions u for (1.1)–(1.2) which has exactly $k-1$ zeros in $(0,1)$ and satisfies $u'(0) > 0$ (respectively $u'(0) < 0$).

First we consider the nonexistence of solutions in the class S_k^+ or S_k^- for each $k \in \mathbf{N}$. Throughout of this paper, we agree that $1/0 = \infty$ and $1/\infty = 0$.

Theorem 1. *Let $k \in \mathbf{N}$. Assume either $\lambda \in (0, \lambda_k/f^*)$ or $\lambda \in (\lambda_k/f_*, \infty)$. Then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$.*

By the property $\lambda_k < \lambda_{k+1}$ for $k = 1, 2, \ldots$, Theorem 1 implies that if $\lambda \in (\lambda_{k-1}/f_*, \infty)$, then $S_j^+ = \emptyset$ and $S_j^- = \emptyset$ for each $j = 1, 2, \ldots, k-1$, and that if $\lambda \in (0, \lambda_k/f^*)$, then $S_j^+ = \emptyset$ and $S_j^- = \emptyset$ for each $j = k, k+1, \ldots$. We can show that the number of zeros of nontrivial solutions of (1.1)–(1.2) is finite. Hence we obtain the following corollary.

Corollary 1. *Assume that there exists an integer $k \in \mathbf{N}$ such that $\lambda_{k-1}/f_* < \lambda_k/f^*$. If $\lambda \in (\lambda_{k-1}/f_*, \lambda_k/f^*)$, then problem (1.1)–(1.2) has no nontrivial solution.*

Next we consider the existence of solutions belonging the class S_k^+ or S_k^- in the case $f_0 \neq f_\infty$.

Theorem 2. *Assume that* $f_0 \neq f_\infty$. *If* $\lambda \in (\lambda_k/f_\infty, \lambda_k/f_0)$ *or* $\lambda \in (\lambda_k/f_0, \lambda_k/f_\infty)$ *for some* $k \in \mathbf{N}$, *then* $S_k^+ \neq \emptyset$ *and* $S_k^- \neq \emptyset$.

Remark. Let us consider, for instance, the case where

$$f_* = f_0 < f_\infty = f^*. \tag{1.5}$$

In this case, by Theorem 1, we find that $S_k^+ = \emptyset$ and $S_k^- = \emptyset$ if either $\lambda \in (0, \lambda_k/f_\infty)$ or $\lambda \in (\lambda_k/f_0, \infty)$. Hence, in this case, λ_k/f_∞ and λ_k/f_0 are critical value of existence of solutions in S_k^+ and S_k^-. For example, if $f(s)/|s|^{p-2}s$ is nondecreasing, then (1.5) holds.

Corollary 2. *Assume that either the following* (i) *or* (ii) *holds:*

$$\text{(i) } f_0 = 0,\ f_\infty = \infty, \quad \text{(ii) } f_0 = \infty,\ f_\infty = 0.$$

Then, for every $\lambda > 0$ *and* $k \in \mathbf{N}$, $S_k^- \neq \emptyset$ *and* $S_k^- \neq \emptyset$. *In particular, problem* (1.1)–(1.2) *has infinity many solutions for each* $\lambda > 0$.

Remark. (i) In the autonomous case $a(x) \equiv$ const, related results have been obtained by using a time-mapping method. See, e.g., [17] and [23].

(ii) Let us consider the case $p = 2$. In this case, Theorem 2 has been obtained in [14], [16] provided $f_0, f_\infty \in (0, \infty)$, and the results in Corollary 2 was obtained in [15] with the additional condition. In [14], [15], [16], bifurcation techniques were used. For the case $\lambda = 1$, a related result has been obtained in [21] by the shooting method. However, in [15], [21], it is required that $f(s)$ is increasing and $f(s)/s$ is nondecreasing on $(0, s_0]$ for some $s_0 > 0$ in the case $f_0 = \infty$. Thus Theorem 2 and Corollary 2 are even new when $p = 2$.

(iii) Under the condition (i) and (ii) in Corollary 2, Wang [24] has showed the existence of at least one positive solution of (1.1) subject to nonlinear boundary conditions by using a fixed point theorem in cones.

(vi) Recently, Huy and Thanh [9] considered the problem

$$(|u'|^{p-2}u')' + \lambda f(x, u, u') = 0, \quad 0 < x < 1 \quad \text{and} \quad u(0) = u(1) = 0,$$

and obtained intervals of values of λ such that the problem has at least one solution. See also Milakis [18].

Finally, let us consider the existence of solutions in the case $f_0 = f_\infty$. In the case $f_0 = f_\infty \in (0, \infty)$, we require that either $f_0 = f_\infty = f_*$ or $f_0 = f_\infty = f^*$, and that

$$\frac{f(s)}{|s|^{p-2}s} \not\equiv f_0 \quad \text{on } (0, \infty) \quad \text{and} \quad \frac{f(s)}{|s|^{p-2}s} \not\equiv f_0 \quad \text{on } (-\infty, 0). \tag{1.6}$$

It is clear that we have $f_* < f^*$, if (1.6) holds.

Theorem 3. *Let $k \in \mathbf{N}$.* (i) *Assume that $f_0 = f_\infty = f^* \in (0, \infty)$ and* (1.6) *holds. Then there exists $\delta_k \in (\lambda_k/f^*, \lambda_k/f_*)$ such that, if $\lambda \in (\lambda_k/f^*, \delta_k)$, then problem* (1.1)–(1.2) *has at least four solutions u_k^+, v_k^+, u_k^-, and v_k^- such that u_k^+, $v_k^+ \in S_k^+$ and u_k^-, $v_k^- \in S_k^-$.*

(ii) *Assume that $f_0 = f_\infty = f_* \in (0, \infty)$ and* (1.6) *holds. Then there exists $\delta_k \in (\lambda_k/f^*, \lambda_k/f_*)$ such that, if $\lambda \in (\delta_k, \lambda_k/f_*)$, then problem* (1.1)–(1.2) *has at least four solutions u_k^+, v_k^+, u_k^-, and v_k^- such that u_k^+, $v_k^+ \in S_k^+$ and u_k^-, $v_k^- \in S_k^-$.*

Remark. In the case (i), if $\lambda \in (0, \lambda_k/f^*)$ then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$ by Theorem 1. It also follows from Theorem 1 that, in the case (ii), if $\lambda \in (\lambda_k/f_*, \infty)$ then $S_k^+ = \emptyset$ and $S_k^- = \emptyset$.

Theorem 4. (i) *Assume that $f_0 = f_\infty = \infty$. Then there exists a sequence $\{\delta_k\}$ satisfying $0 < \delta_1 < \delta_2 < \cdots$ with $\lim_{k\to\infty} \delta_k = \infty$ such that if $\lambda \in (0, \delta_k)$ then problem* (1.1)–(1.2) *has solutions $\{u_j^+, v_j^+, u_j^-, v_j^-\}_{j=k}^\infty$ with u_j^+, $v_j^+ \in S_j^+$ and u_j^-, $v_j^- \in S_j^-$ for each $j = k, k+1, \ldots$.*

(ii) *Assume that $f_0 = f_\infty = 0$. Then there exists a sequence $\{\delta_k\}$ satisfying $0 < \delta_1 < \delta_2 < \cdots$ such that if $\lambda > \delta_k$ then the problem* (1.1)–(1.2) *has solutions $\{u_j^+, v_j^+, u_j^-, v_j^-\}_{j=1}^k$ with u_j^+, $v_j^+ \in S_j^+$ and u_j^-, $v_j^- \in S_j^-$ for each $j = 1, 2, \ldots, k$.*

Remark. In the cases $f_0 = f_\infty = \infty$ and $f_0 = f_\infty = 0$, the existence of at least two positive solutions has been obtained by [1], [22]. In the case $f_0, f_\infty \notin \{0, \infty\}$, we refer to [8].

(ii) Kong and Wang [12] considered the case where $a(x)$ is allowed to have singularity at $x = 0$ or 1, and showed the existence of at least two positive solutions in the cases $f_0 = f_\infty = 0$ and $f_0 = f_\infty = \infty$ with some additional conditions. In the case where f in (1.1) depends on u and u', we refer to [25].

By a change of variable (see, e.g., [20]), it can be shown that the existence of solution of problem (1.1)–(1.2) is equivalent to the existence of radially symmetric solutions of the following Dirichlet problem for quasilinear elliptic equations in annular domains:

$$\Delta_p u + a(|x|)f(u) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where $\Omega = \{x \in \mathbf{R}^N : R_1 < |x| < R_2\}$ with $0 < R_1 < R_2$ and $N \geq 2$. Concerning the existence of positive solutions for problem (1.7), we refer to [5], [6], [7].

In the proofs of Theorems 1–4, our argument is based on the shooting method together with the qualitative theory for half-linear differential equations. First we consider the solution $u(x;\mu)$ of (1.1) satisfying the initial condition

$$u(0) = 0 \quad \text{and} \quad u'(0) = \mu,$$

where $\mu \in \mathbf{R}$ is a parameter, and then we investigate the behavior of the solution $u(x;\mu)$ as $\mu \to 0$ and $\mu \to \infty$ by making use of the properties of solutions for half-linear differential equations of the form

$$(|v'|^{p-2}v')' + c(x)|v|^{p-2}v = 0, \tag{1.8}$$

where c is continuous. In particular, we will employ the generalized Prüfer transformation, the Sturumian theorems, and Picone type identity for (1.8) in our arguments.

During the recent years it was shown that there is striking similarity in the qualitative behavior of the solutions of (1.8) and the second order linear differential equation $v'' + c(x)v = 0$ which is the special case of (1.8) when $p = 2$. The Sturm comparison theorem for the half-linear differential equation (1.8) is formulated as following:

Lemma 1.1. *Consider a pair of half-linear differential equations*

$$(|u'|^{p-2}u')' + c(x)|u|^{p-2}u = 0, \quad x_1 \le x \le x_2, \tag{1.9}$$

and

$$(|U'|^{p-2}U')' + C(x)|U|^{p-2}U = 0, \quad x_1 \le x \le x_2, \tag{1.10}$$

where c, $C \in C[x_1, x_2]$. Suppose that $C(x) \ge c(x)$ for $x \in (x_1, x_2)$, and that a nontrivial solution u of (1.9) satisfies $u(x_1) = u(x_2) = 0$. Then every nontrivial solution U of (1.10) has a zero in (x_1, x_2) or it is a multiple of the solution u. The last possibility is excluded if $C(x) \not\equiv c(x)$ for $x \in (x_1, x_2)$.

For the proof, we refer to [3, Theorem 1.2.4]. (See also [2], [4] and [13].) We will give some variants of Lemma 1.1 in Section 3 below. The following Picone type identity for the equations (1.9) and (1.10) is introduced by Jaroš and Kusano [10], and it can be shown by a direct computation and Young's inequality. See also [2] and [3].

Lemma 1.2. *Define Φ and P, respectively, by*

$$\Phi(u) = |u|^{p-2}u \quad \textit{and} \quad P(u,v) = \frac{|u|^p}{p} - uv + \frac{|v|^q}{q} \ge 0,$$

where $q = p/(p-1)$. Let u and U be solutions of (1.9) and (1.10), respectively. Then

$$\begin{aligned}&\left[\frac{u}{\Phi(U)}\left(\Phi(u')\Phi(U) - \Phi(u)\Phi(U')\right)\right]' \\ &= [C(x) - c(x)]|u|^p + pP\left(u', \Phi\left(\frac{uU'}{U}\right)\right) \quad \textit{for } x \in [x_1, x_2].\end{aligned}$$

In particular, we have

$$\left[\frac{u}{\Phi(U)}\Big(\Phi(u')\Phi(U) - \Phi(u)\Phi(U')\Big)\right]' \ge [C(x) - c(x)]|u|^p$$

for $x \in [x_1, x_2]$.

In proving the existence of solutions with prescribed numbers of zeros, the generalized Prüfer transformation plays a fundamental role. This transformation involves the generalized sine function and the generalized cosine function.

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