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PLAYERS' INFORMATION IN TWO-PLAYER GAMES OF "SCORE SHOWDOWN" **

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ABSTRACT. There are some games widely played in the routine world of gambles, roulette, quiz show and sports excersizes. The object of the games is to get the highest score among the players, from one or two chances of sampling. The two-player games of "Keep-or-Exchange" and "Risky Exchange", where three types of information are provided to the players, are investigated. The results are compared and some open problems in this area are mentioned.

1 Two-player Game of "Score Showdown".

Consider the two players I and II (sometimes they are denoted by 1 and 2, respectively). Let X_j (Y_j) be the random variable observed by I (II) at the j-th observation, $j = 1, 2$. We assume that X_1, X_2, Y_1, Y_2 are *i.i.d.* each with uniform distribution in $[0, 1]$.

The game is played as follows. I [II] observes $X_1 = x$ [$Y_1 = y$] and chooses one of either A (*i.e.*, accepts the observed value) or R (*i.e.*, rejects his observed value and samples a new random variable).

The *score* for player I is defined by

$$S_1(X_1, X_2) = \begin{cases} X_1, \\ \varphi(X_1, X_2), \end{cases} \quad \text{if } X_1 \text{ is } \begin{cases} \text{accepted,} \\ \text{rejected and } X_2 \text{ is sampled,} \end{cases}$$

and the score $S_2(Y_1, Y_2)$ for II, is defined similarly, with X_is replaced by Y_is.

We call the game of "Keep-or-Exchange", "Risky Exchange" and "Showcase Showdown", when

$$\varphi(X_1, X_2) = X_2, X_2 I(X_2 > X_1), \text{ and } (X_1 + X_2) I(X_1 + X_2 \le 1),$$

respectively. Here $I(e)$ is the indicator of the event e. For simplicity, we denote these games GKE, GRE and GSS, respectively. The name of GSS comes from Ref.[1].

After each player chooses his (or her) R or A, showdown is made, the scores are compared, and the player with the higher score than the opponent becomes the *winner*. Each player aims to maximize the probability of his (or her) winning.

We consider the three information types, under which the players decide their choices of either R or A.

I^{10-01} means that I observes $X_1 = x$, II observes $Y_1 = y$, and each player doesn't inform his observed value to his opponent.

I^{11-11} means that I observes $X_1 = x$, II observes $Y_1 = y$, and each player informs his observed value to his opponent.

I^{10-11} means that I observes $X_1 = x$, II observes $Y_1 = y$, and I informs his $X_1 = x$ to II, but II

doesn't inform his $Y_1 = y$ to I.
The GKEs (GREs) under these three information types are solved in Sections 2, 3 and 4 (Sections 5,6 and 7). The most important difference between GKE and GRE is that "draw" occurs with positive probability in the latter, but it doesn't occur in the former.

In Section 9 we discuss the games GKE and GRE under information I^{10-11}, in which the "first-mover" I adopts some randomization in his strategy in order to restore his disadvantage. The results in Theorem 1~6, 3B and 6B are compared in Sections 8 and 9. Some open probrems in this area are mentioned in the final Section 10.

2. Game of "Keep-or-Exchange" under I^{10-01} (略)
3. Game of "Keep-or-Exchange" under I^{11-11} (略)
4. Game of "Keep-or-Exchange" under I^{10-11} (略)
5. Game of "Risky Exchange" under I^{10-01} (略)

6 Game of "Risky Exchange" under I^{11-11}

Define state (x, y) as the same as in Section 3. Let $p_{AR}(x, y)$ $[q_{AR}(x, y)]$ denote the winning probability for I [II] when the players' choices are A by I and R by II in state (x, y). Other three probabilities $p_{RA}(x, y)$ $[q_{RA}(x, y)]$ *etc* are defined similarly. Also let $h_{RA}(x, y)$ *etc* denote the probability of draw, similarly. Hereafter, we shall sometimes omit the state description, for simplicity. We evidently have, for $\forall(x, y)$,

$$p_{AA} + q_{AA} + h_{AA} = 1 \tag{6.1}$$

and other three equations, and

$$h_{AA} = h_{AR} = h_{RA} = 0, \qquad h_{RR} = P(X_2 < x, Y_2 < y) = xy. \tag{6.2}$$

Furthermore we find that

$$p_{AA} = I(x, y), \tag{6.3}$$

$$p_{AR} = P[\{y < Y_2 < x\} \cup \{Y_2 < y\}] = (x - y)I(x > y) + y, \tag{6.4}$$

$$p_{RA} = P\{X_2 > x, X_2 > y\} = 1 - x \vee y, \tag{6.5}$$

$$\begin{aligned} p_{RR} &= P[(X_2 > x) \cap \{(X_2 > Y_2 > y) \cup (Y_2 < y)\}] \\ &= \frac{1}{2}(1 + y^2) - xy - \frac{1}{2}(x - y)^2 I(x > y). \end{aligned} \tag{6.6}$$

Therefore, from Eqs (6.1)~(6.6), players in state (x, y) face the bimatrix game with the payoff bimatrix

(6.7) $M(x, y) =$

	R		A	
R	p_{RR},	$1 - xy - p_{RR}$	$1 - x \vee y$,	$x \vee y$
A	$(x - y)I(x > y) + y$,	$\bar{y} - (x - y)I(x > y)$	$I(x > y)$,	$I(x < y)$

$$= \begin{cases} \begin{array}{|cc|cc|} \hline \frac{1}{2}(1+y^2)-xy, & \frac{1}{2}(1-y^2) & \bar{y}, & y \\ \hline y, & \bar{y} & 0, & 1 \\ \hline \end{array} (\equiv \mathbf{M}(x,y \mid x \le y), \text{say}), & \text{if } x \le y\ ; \\ \\ \begin{array}{|cc|cc|} \hline \frac{1}{2}(1-x^2), & \frac{1}{2}(1+x^2)-xy & \bar{x}, & x \\ \hline x, & \bar{x} & 1, & 0 \\ \hline \end{array} (\equiv \mathbf{M}(x,y \mid x > y), \text{say}), & \text{if } x > y. \end{cases}$$

Theorem 5 *Solution to* GRE *under* I^{11-11}, *in state* (x,y), *is as follows ;*

Case	Eq. strategy-pair	Eq, val.$\mathbf{M}(x,y)$
$y > x \vee (\sqrt{2}-1)$	saddle pt. R-A	$\bar{y},\ \ y$
$x < y < \sqrt{2}-1$	R-R	$\frac{1}{2}(1+y^2)-xy,\ \frac{1}{2}(1-y^2)$
$y < x < \sqrt{2}-1$	R-R	$\frac{1}{2}(1-x^2),\ \frac{1}{2}(1+x^2)-xy$
$x > y \vee (\sqrt{2}-1)$	A-R	$x,\ \ \bar{x}$

The winning probabilities for the players and the probability of draw are

$$P(\text{draw}) = \frac{1}{4}(17-12\sqrt{2}) \approx 0.00736,$$

$$P(W_1) = P(W_2) = \frac{1}{2}\{1-P(\text{draw})\} = \frac{1}{8}(12\sqrt{2}-13) \approx 0.49632.$$

(See Figure 1e.)

Proof. For the bimatrix $\mathbf{M}(x,y \mid x \le y)$ we note that

$$\begin{array}{ll} \frac{1}{2}(1-y^2) > (<)\ y, & \text{if } y < (>)\ \sqrt{2}-1 \\ \frac{1}{2}(1+y^2)-xy > (<)\ y, & \text{if } y < (>)\ 1+x-\sqrt{2x+x^2} \equiv k(x), \text{say}. \end{array}$$

And $k(x)$ is convex, decreasing with $k(0)=1, k(\sqrt{2}-1)=\sqrt{2}-1$, and $k(1)=2-\sqrt{3}\approx 0.268$.

So, for the bimatrix $\mathbf{M}(x,y \mid x > y)$, we evidently have

$$\begin{array}{ll} \frac{1}{2}(1-x^2) > (<)\ x, & \text{if } x < (>)\ \sqrt{2}-1 \\ \frac{1}{2}(1+x^2)-xy > (<)\ x, & \text{if } x < (>)\ 1+y-\sqrt{2y+y^2} = k(y). \end{array}$$

Therefore, by combining the above facts about $\mathbf{M}(x,y)$, we get the table in the theorem.

The probabilities we want to find are

$$\begin{aligned} P(\text{draw}) &= \int_0^{\sqrt{2}-1}\int_0^{\sqrt{2}-1} h_{RR}\,dxdy = \int_0^{\sqrt{2}-1}\int_0^{\sqrt{2}-1} xy\,dxdy = \frac{1}{4}(\sqrt{2}-1)^4 = \frac{1}{4}(17-12\sqrt{2}), \\ P(W_1) &= \iint\limits_{0<x<y<\sqrt{2}-1} \left\{\frac{1}{2}(1+y^2)-xy\right\} dxdy + \iint\limits_{0<y<x<\sqrt{2}-1} \frac{1}{2}(1-x^2)\,dxdy + \int_{\sqrt{2}-1}^1 x^2dx + \int_{\sqrt{2}-1}^1 y\bar{y}dy \\ &= \frac{1}{8}(4\sqrt{2}-5)+\sqrt{2}-1 = \frac{1}{8}(12\sqrt{2}-13) \end{aligned}$$

and

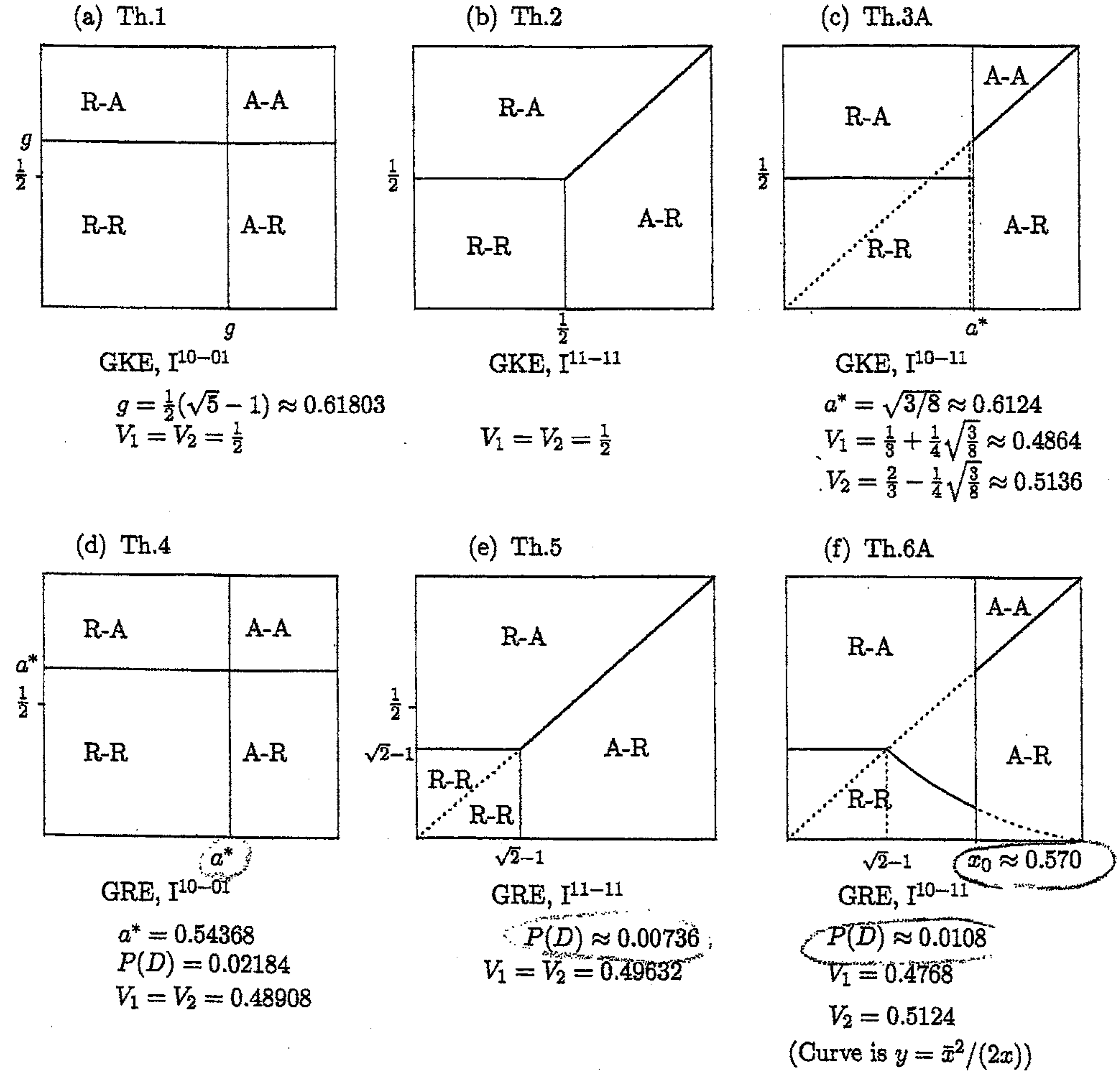
$$P(W_2) = 1 - P(W_1) - P(\text{draw}) = 1 - \frac{1}{8}(12\sqrt{2} - 13) - \frac{1}{4}(17 - 12\sqrt{2}) = \frac{1}{8}(12\sqrt{2} - 13).$$

The result $P(W_1) = P(W_2)$ is consistent with our common sence. □

7. Game of "Risky Exchange" under I^{10-11} (略)

8. Comparison between Theorems 1 ~ 6.

Figure 1. Optimal choice-pairs in GKE and GRE.



We abserve that (1) The decision threshould in GRE in each information type is smaller than that in GKE. (2) There doesn't exsit the optimal A-A pair under I^{11-11} ; and (3) In each of GKE and GRE under I^{10-11}, the border of the optimal A and R regions for II is more complicate than theose under I^{10-01} and I^{11-11}. And, we find that $P(\text{draw}) > 0$ and $P(W_1) < P(W_2)$.

9 More about Games under Information I^{10-11}.

Under information I^{10-11}, player I has an advantage over player I. It would seem natural that I would randomize his decision threshold in order to improve his disadvantage due to the leakage

of his "hand" to his opponent. The situation is like in poker. See, for example, Ref.[2 ; Section 6]. Standing at this viewpoint, the next

Assumption B. *Player I, in state $X_1 = x$, chooses R if $x < \bar{a}$, chooses A if $x > a$ and employs the mixed strategy $\left(R, A \,;\, \frac{a-x}{a-\bar{a}} \cdot \frac{x-\bar{a}}{a-\bar{a}}\right)$, if $\bar{a} < x < a$, for some $a \in [\frac{1}{2}, 1]$ which he must determine beforehand,*

instead of **Assumption A** (stated in Section 4), is worth studying.

The best choice of a is not yet derived. Next two theorems show that the two extremes $a = \frac{1}{2}$ and $a = 1$ belong to the worst choices for I.

Theorem 3B. *Solution to* GKE *under information* I^{10-11} *and Assumption* B.

(i). *Case* $a = \frac{1}{2}$.

The optimal strategy for II in state $(y \mid x)$ is ;

Choose A (R), *if* $x < \frac{1}{2}$ *and* $y > (<)\ \frac{1}{2}$,

Choose A (R), *if* $x > \frac{1}{2}$ *and* $y > (<)\ x$,

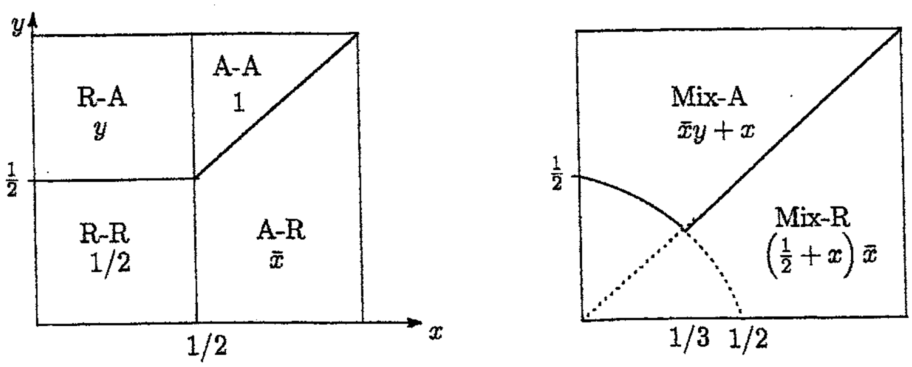
The winning probabilities are

$$P(W_1) = 1 - P(W_2) = \frac{23}{48} \approx 0.4792.$$

(ii). *Case* $a = 1$.

Players' optimal strategy-pairs and winning probabilities for II are as shown in Figure 2. *We obtain*

(9.1) $$P(W_2) = 1 - P(W_1) = \frac{1}{6} + \frac{16}{81} + \left(-\frac{13}{1296} + \frac{1}{2}\log\frac{3}{2}\right) \approx 0.5569.$$



Case $a = 1/2$ Case $a = 1$

Figure 2. Optimal strategy-pair

Theorem 6B (略)

(Mix. means I's mixed strategy $(R, A\ ;\ \bar{x}, x)$. The curve is $\xi(x) \doteqdot \frac{3}{2} + x - \bar{x}^{-1}$.
II's winning prob. are mentioned therewith.)

10 Final Remark.

Three-player games under various information are of interest. GKE under $\mathrm{I}^{100-010-001}$ and $\mathrm{I}^{100-110-111}$ are solved in Ref.[4 ; Theorem 3] and Ref.[4 ; Theorem 2], respectively. GRE under $\mathrm{I}^{100-010-001}$ is solved in Ref.[6 ; Theorem 1] (The meaning of the information types in three-player games will be understood by referring to those in two-player games mentioned in Section 1). Several games, for example, GKE and GRE both under $\mathrm{I}^{111-111-111}$ remain to be solved.

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