数理解析研究所講究録
1480 巻 2006 年 166-178



# Form factors, correlation functions and vertex operators in the eight-vertex model at reflectionless points*

鈴鹿医療科学大学　桑野泰宏 (Yas-Hiro Quano) †

*Department of Clinical Engineering, Suzuka University of Medical Science*
*Kishioka-cho 1001-1, Suzuka 510-0293, Japan*

**Abstract**

The eight-vertex model at the reflectionless points is considered on the basis of Smirnov's axiomatic approach. Integral formulae for form factors of the eight-vertex model can be obtained in terms of those of the eight-vertex SOS model, by using vertex-face transformation. The resulting formulae have very simple forms at the reflectionless points, and suggest us the free field representation of type II vertex operators in the eight-vertex model.

## 1 Introduction

In this paper we wish to construct form factors in the eight-vertex model at the reflectionless points. Form factors are originally defined as matrix elements of local operators. Through the study of form factors in the sine-Gordon model, Smirnov [1] found three axioms as sufficient conditions for the local commutativity of local fields in the model. Thus, following Smirnov, any objects that satisfy Smirnov's three axioms are referred to as 'form factors'.

For fixed a local operator $\mathcal{O}$, let

$$F_m^{(i)}(\mathcal{O};\zeta_1,\cdots,\zeta_{2m}) = \sum_{\mu_1\cdots\mu_{2m}} v_{\mu_1}^*\otimes\cdots\otimes v_{\mu_{2m}}^* F_m^{(i)}(\zeta_1,\cdots,\zeta_{2m})_{\mu_1\cdots\mu_{2m}}. \tag{1.1}$$

Then Smirov's axioms are as follows [1]:

1. $S$-matrix symmetry:

$$F_m^{(i)}(\cdots,\zeta_{j+1},\zeta_j,\cdots)P_{j\,j+1} = F_m^{(i)}(\cdots,\zeta_j,\zeta_{j+1},\cdots)S_{j\,j+1}(u_j-u_{j+1}), \tag{1.2}$$

*Based on a talk given in the conference 'Solvable Lattice Models 2004 – Recent Progress in Solvable Lattice Models –', RIMS, Kyoto University, 23 July 2004.

†email: quanoy@suzuka-u.ac.jp

where $\zeta_j = x^{-u_j}$, and $P$ is the permutation operator $(x \otimes y)P = y \otimes x$.

2. cyclicity:

$$F_m^{(i)}(\zeta', x^{-2}\zeta_{2m}) = F_m^{(1-i)}(\zeta_{2m}, \zeta')P_{1\,2} \cdots P_{2m-1\,2m}. \tag{1.3}$$

where $\zeta' = (\zeta_1, \cdots, \zeta_{2m-1})$.

3. annihilation pole condition

$$\operatorname*{Res}_{\zeta_{2m}=\varepsilon x^{-1}\zeta_{2m-1}} F_m^{(i)}(\zeta)\frac{d\zeta_{2m}}{\zeta_{2m}} \tag{1.4}$$

$$= \varepsilon^i \left( F_{m-1}^{(i)}(\zeta'') \otimes u_\varepsilon^* - F_{m-1}^{(1-i)}(\zeta'') \otimes u_\varepsilon^* S_{2m-1\,1}(u_{2m-1} - u_1) \cdots S_{2m-1\,2m-2}(u_{2m-1} - u_{2m-2}) \right),$$

Here, $\zeta = (\zeta_1, \cdots, \zeta_{2m})$, and $\zeta'' = (\zeta_1, \cdots, \zeta_{2m-2})$; and $u_\varepsilon^* = v_+^* \otimes v_-^* + \varepsilon v_-^* \otimes v_+^*$.

The first two axioms imply the $q$-KZ equation [2] of level 0:

$$\begin{aligned} F_m^{(i)}(\zeta_1, \cdots, x^2\zeta_j, \cdots, \zeta_{2m}) &= F_m^{(1-i)}(\zeta)S_{j\,j+1}(u_j - u_{j+1}) \cdots S_{j\,2m}(u_j - u_{2m}) \\ &\times S_{j\,1}(u_j - u_1 - 2) \cdots S_{j\,j-1}(u_j - u_{j-1} - 2). \end{aligned} \tag{1.5}$$

Lashkevich and Pugai [3, 4] used the vertex-face correspondence [5] in order to construct the correlation functions of the eight-vertex/XYZ model in terms of those of the eight-vertex SOS model [6]. The author constructed another simplified expression for the eight-vertex/XYZ correlation function, by solving Bootstrap equations [7]. Shiraishi [8] constructed the formulae of the correlation functions of the XYZ model without using the vertex-face correspondence.

Concerning form factors in the eight-vertex model, Lashkevich [9] found a bosonization recipe to construct integral representations of the form factors in the eight-vertex model. In principle, all form factors corresponding to all local fields can be constructed, but they take very complicated forms. We wish to construct simpler expressions in terms of the eight-vertex SOS model form factors [10]. This paper is the first trial for that purpose.

## 2 Basic definitions

Let us consider the $Z$-invariant eight-vertex model [11] on a planar rectangular lattice. The state variables are associated with four edges around each vertex. Here the local state on an edge takes two possible values $(+)$ and $(-)$, respectively. The product of four states on the four edges around each vertex should be $+$ sign. This is called the generalized ice condition.

Each straight line on the lattice carries a rapidity, or a spectral parameter. Let $V = \mathbb{C}v_+ \otimes \mathbb{C}v_-$, and let $V_u$ be a copy of $V$ with a rapidity $u$. Then the $R$-matrix $R^{V_{u_1},V_{u_2}}$ can be regarded as an endomorphism on $V_{u_1} \otimes V_{u_2}$. It is due to the Lorentz invariance that $R^{V_{u_1},V_{u_2}}$

depends only upon the difference of the rapidities $u_1 - u_2$. In what follows we thus denote $R^{V_{u_1},V_{u_2}}$ by $R(u_1 - u_2)$.

The convention of the matrix elements of $R(u) \in \mathrm{End}(V \otimes V)$ are as follows:

$$R(u)v_{\varepsilon_1} \otimes v_{\varepsilon_2} = \sum_{\varepsilon_1',\varepsilon_2'=\pm} v_{\varepsilon_1'} \otimes v_{\varepsilon_2'} R(u)_{\varepsilon_1\varepsilon_2}^{\varepsilon_1'\varepsilon_2'}. \tag{2.1}$$

For fixed $x = e^{-\epsilon}$ $(\epsilon > 0)$ and $r > 1$, the explicit expression of the entries of $R(u)$ is given as follows:

$$R(u) = \frac{1}{\tilde{\kappa}(u)}\tilde{R}(u) = \frac{1}{\bar{\kappa}(u)}\overline{R}(u) = \frac{1}{\bar{\kappa}(u)}\begin{bmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{bmatrix}, \tag{2.2}$$

where

$$\begin{aligned} \tilde{\kappa}(u) &= \frac{[1]}{[1-u]}\bar{\kappa}(u) = \zeta^{\frac{r-1}{r}}\frac{\rho(z)}{\rho(z^{-1})}, \qquad (z = \zeta^2 = x^{-2u}) \\ \rho(z) &= \frac{(x^4z;x^4,x^{2r})_\infty (x^{2r}z;x^4,x^{2r})_\infty}{(x^2z;x^4,x^{2r})_\infty (x^{2r+2}z;x^4,x^{2r})_\infty}, \qquad (a;p_1,\cdots,p_n)_\infty = \prod_{k_i\geqslant 0}(1-ap_1^{k_1}\cdots p_n^{k_n}), \\ [u] &= x^{\frac{u^2}{r}-u}\Theta_{x^{2r}}(x^{2u}), \quad \Theta_p(z) = (z;p)_\infty(pz^{-1};p)_\infty(p;p)_\infty = \sum_{n\in\mathbf{Z}} p^{n(n-1)/2}(-z)^n, \end{aligned} \tag{2.3}$$

$$\begin{aligned} a(u) &= \frac{\theta_2(\frac{u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_1(\frac{1-u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}{\theta_2(0;\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_1(\frac{1}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}, \quad & b(u) &= \frac{\theta_1(\frac{u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_2(\frac{1-u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}{\theta_2(0;\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_1(\frac{1}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}, \\ c(u) &= \frac{\theta_2(\frac{u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_2(\frac{1-u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}{\theta_2(0;\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_2(\frac{1}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}, \quad & d(u) &= -\frac{\theta_1(\frac{u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_1(\frac{1-u}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}{\theta_2(0;\frac{\pi\sqrt{-1}}{2\epsilon r})\theta_2(\frac{1}{2r};\frac{\pi\sqrt{-1}}{2\epsilon r})}, \end{aligned} \tag{2.4}$$

$$\theta_{1,2}(u;\tau) = \sqrt{\mp 1}q^{\frac{1}{4}}\zeta^{-1}\Theta_{q^2}(\pm z), \qquad \theta_{3,4}(u;\tau) = \Theta_{q^2}(\mp qz). \qquad (q = e^{\sqrt{-1}\pi\tau},\ z = \zeta^2 = e^{2\sqrt{-1}\pi u})$$

The most important property of the $R$-matrix is the Yang-Baxter equation [12]:

$$R_{12}(\zeta_1/\zeta_2)R_{13}(\zeta_1/\zeta_3)R_{23}(\zeta_2/\zeta_3) = R_{23}(\zeta_2/\zeta_3)R_{13}(\zeta_1/\zeta_3)R_{12}(\zeta_1/\zeta_2), \tag{2.5}$$

where the subscript of the $R$-matrix denotes the spaces on which $R$ nontrivially acts.

From (2.3) $\tilde{\kappa}(0) = 1 = \bar{\kappa}(0)$. It is easy to see $\bar{\kappa}(1-u) = \bar{\kappa}(u)$, which implies $\bar{\kappa}(1) = 1$. From these and (2.4) we have

$$R(0) = P = \begin{bmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{bmatrix}, \quad R(1) = \begin{bmatrix} & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & \end{bmatrix}. \tag{2.6}$$

The unitarity relation

$$R_{12}(u)R_{21}(-u) = 1, \tag{2.7}$$

and the crossing symmetries

$$R_{21}^{t_1}(1-u) = \sigma_1^x R_{12}(u)\sigma_1^x \tag{2.8}$$

are also important.

For fixed $\epsilon > 0$ and $r > 1$, the region $0 < u < 1$ is called the principal regime. This regime is one of antiferroelectric regions because of $c > a+b+|d|$. In the low temperature limit $\epsilon \to +\infty$ ($c \gg a+b+|d|$), only $c$-type configuration is permitted at each vertex. Thus, there are two ground states in the principal regime.

Let us consider the half infinite pure tensor vector $\cdots \otimes v_{\varepsilon_3} \otimes v_{\varepsilon_2} \otimes v_{\varepsilon_1}$ along a half infinite row. The ground state corresponds to the sequence $\varepsilon_j = (-1)^{j+i}$ ($i = 0, 1$). Fix the ground state labeled by $i$. Then at finite temperature $\epsilon > 0$, any state configurations differ from that of $i$-th ground states by altering a finite number of spins. Otherwise, the system has infinitely high energy. Thus, the space of states $\mathcal{H}_i$ is the subspace of '$\cdots \otimes V \otimes V \otimes V$' spanned by

$$\cdots \otimes v_{\varepsilon_3} \otimes v_{\varepsilon_2} \otimes v_{\varepsilon_1}, \qquad \varepsilon_j = (-1)^{j+i} \ (j \gg 1).$$

Let us remind the definitions of the eight-vertex SOS model and the intertwining vectors. The eight-vertex SOS model is a face model [5] which is defined on the square lattice with a site variable $k_j \in \mathbb{Z}$ attached to each site $j$. We call $k_j$ a local state or a height and impose the condition that heights of adjoining sites differ by one. Local Boltzmann weight of this model is given for a state configuration $\begin{array}{cc} c & d \\ b & a \end{array}$ round a face. Here the four states $a, b, c$ and $d$ are ordered clockwise from the SE corner. The weights are assumed to be the functions of the spectral parameter $u$ and the nonzero Boltzmann weights are given as follows:

$$\begin{aligned}
W\left[\begin{array}{cc|c} k\pm2 & k\pm1 & \\ & & u \\ k\pm1 & k & \end{array}\right] &= \frac{1}{\bar{\kappa}(u)} = \frac{1}{\bar{\kappa}(u)}\frac{[1-u]}{[1]}, \\
W\left[\begin{array}{cc|c} k & k\pm1 & \\ & & u \\ k\pm1 & k & \end{array}\right] &= \frac{1}{\bar{\kappa}(u)}\frac{[1][k\pm u]}{[1-u][k]} = \frac{1}{\bar{\kappa}(u)}\frac{[k\pm u]}{[k]}, \\
W\left[\begin{array}{cc|c} k & k\mp1 & \\ & & u \\ k\pm1 & k & \end{array}\right] &= \frac{1}{\bar{\kappa}(u)}\frac{[u][k\pm1]}{[1-u][k]} = \frac{1}{\bar{\kappa}(u)}\frac{[u][k\pm1]}{[1][k]}.
\end{aligned} \tag{2.9}$$

In regime III ($0 < u < 1$) the ground state of the eight-vertex SOS model (2.9) can be labeled by an integer $l$, whose local states are $l$ or $l+1$. In what follows, we fix one of the ground state (labeled by, say, $l$), and consider any configurations which differ from that of the $l$-th ground state by changing a finite number of local states. Let us call a path $p = (k_1, k_2, k_3, \cdots)$ an admissible path, if $|k_{j+1} - k_j| = 1$ $(j = 1, 2, 3, \cdots)$ holds. Let $\mathcal{H}_{l,k}^{(i)}$ $(i = 0, 1)$ be the space of admissible paths satisfying the initial condition $k_1 = k$ and the following boundary condition

$$k_j = \begin{cases} l & \text{if } j \equiv 1 - i \pmod 2 \\ l+1 & \text{if } j \equiv i \pmod 2 \end{cases} \qquad (j \gg 1).$$

Note that $i \equiv k - l \pmod 2$.

The intertwining vectors

$$t_k^{k\pm1}(u)^\varepsilon = \frac{(\sqrt{-1})^{k-l+1/2}\varepsilon^{k-l}}{\sqrt{2}} f(u)\theta_{\bar\varepsilon}\left(\tfrac{k\mp u}{2r}; \tfrac{\pi\sqrt{-1}}{2\varepsilon r}\right), \qquad (\overline{+}, \overline{-}) = (3, 4), \tag{2.10}$$

map the eight-vertex SOS model in regime III onto the eight-vertex model in principal regime. Here, the normalization factor $f(u)$ satisfies the relation

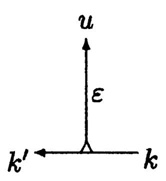
$$[u]f(u)f(u-1) = \frac{\pi}{\epsilon r} e^{\frac{\epsilon r}{2}}. \tag{2.11}$$

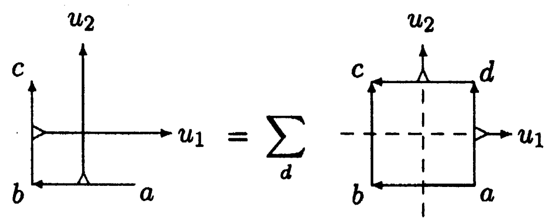
The explicit expression of $f(u)$ is as follows:

$$f(u) = \frac{x^{-\frac{u^2}{2r}+\frac{r-1}{2r}u+\frac{1}{4}}}{C\sqrt{(x^{2r};x^{2r})_\infty}} \frac{(x^{4+2u};x^4,x^{2r})_\infty (x^{2r+2-2u};x^4,x^{2r})_\infty}{(x^{2+2u};x^4,x^{2r})_\infty (x^{2r-2u};x^4,x^{2r})_\infty} \tag{2.12}$$

Then we have the so-called vertex-face correspondence:

$$R(u_1 - u_2) t_b^c(u_1) \otimes t_a^b(u_2) = \sum_d W\left[\begin{array}{cc|c} c & d & \\ & & u_1 - u_2 \\ b & a & \end{array}\right] t_a^d(u_1) \otimes t_d^c(u_2). \tag{2.13}$$

$t_k^{k'}(u)^\varepsilon =$ 


Let us introduce the following dual intertwining vectors:

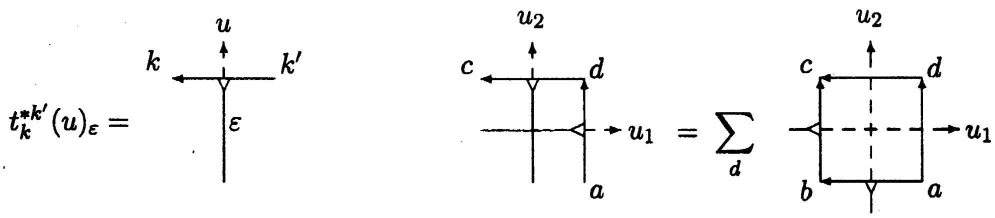
$$\begin{aligned} t_k^{*k'}(u;\epsilon, r) &= t_k^{*k'}(u) = \sum_{\varepsilon=\pm} t_k^{*k'}(u)_\varepsilon v_\varepsilon^*, \\ t_k^{*k'}(u)_\varepsilon &= \frac{1}{[k]} t_k^{k'}(u+1)^{-\varepsilon}. \end{aligned} \tag{2.14}$$

From the following inversion relations

$$\sum_{\varepsilon=\pm} t_k^{*k'}(u)_\varepsilon t_{k''}^{k}(u)^\varepsilon = \delta_{k''}^{k'}, \qquad \sum_{k'=k\pm1} t_{k'}^{k}(u)^\varepsilon t_k^{*k'}(u)_{\varepsilon'} = \delta_{\varepsilon'}^{\varepsilon}, \tag{2.15}$$

the dual vertex-face correspondence holds:

$$t_d^{*a}(u_1) \otimes t_c^{*d}(u_2) R(u_1 - u_2) = \sum_b W\left[\begin{array}{cc|c} c & d & \\ & & u_1 - u_2 \\ b & a & \end{array}\right] t_c^{*b}(u_1) \otimes t_b^{*a}(u_2). \tag{2.16}$$

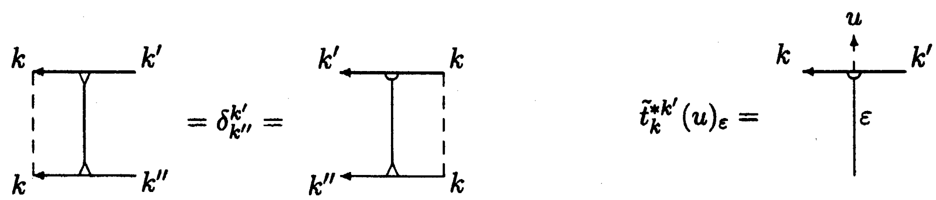



We also introduce another dual intertwining vector

$$\begin{aligned} \tilde{t}_k^{*k'}(u;\epsilon,r) &= \tilde{t}_k^{*k'}(u) = \sum_{\varepsilon=\pm} \tilde{t}_k^{*k'}(u)_\varepsilon v_\varepsilon^*, \\ \tilde{t}_k^{*k'}(u)_\varepsilon &= \frac{1}{[k']} t_{k'}^{k}(u-1)^{-\varepsilon}, \end{aligned} \tag{2.17}$$

that satisfies the following inversion relations:

$$\sum_{\varepsilon=\pm} \tilde{t}_{k'}^{*k}(u)_\varepsilon t_k^{k''}(u)^\varepsilon = \delta_{k'}^{k''}, \qquad \sum_{k'=k\pm1} t_k^{k'}(u)^\varepsilon \tilde{t}_{k'}^{*k}(u)_{\varepsilon'} = \delta_{\varepsilon'}^{\varepsilon}. \tag{2.18}$$




For fixed $r > 1$, let

$$S(u) = -R(u;\epsilon, r-1), \qquad W'\left[\begin{array}{cc|c} c & d & \\ & & u \\ b & a & \end{array}\right] = -W\left[\begin{array}{cc|c} c & d & \\ & & u \\ b & a & \end{array}\right]\Bigg|_{r\mapsto r-1}, \tag{2.19}$$

and

$$t'^{*k}_{k'}(u) := t_{k'}^{*k}(u;\epsilon, r-1). \tag{2.20}$$

Then we have

$$t'^{*a}_d(u_1) \otimes t'^{*d}_c(u_2) S(u_1 - u_2) = \sum_d W'\left[\begin{array}{cc|c} c & d & \\ & & u_1 - u_2 \\ b & a & \end{array}\right] t'^{*b}_c(u_1) \otimes t'^{*a}_b(u_2). \tag{2.21}$$

Note that the normalization factor in (2.19) is given by

$$\bar{\kappa}^*(\zeta) = -\bar{\kappa}(\zeta; x, x^{2(r-1)}) = \zeta^{-\frac{r}{r-1}} \frac{g^*(z)}{g^*(z^{-1})}, \qquad g^*(z) = g^*(x^4 z^{-1}). \tag{2.22}$$

The explicit expression of $g^*(z)$ is given as follows:

$$g^*(z) = \frac{\{z\}'_\infty \{x^4 z^{-1}\}'_\infty \{x^{2r+2} z\}'_\infty \{x^{2r+6} z^{-1}\}'_\infty}{\{x^2 z\}'_\infty \{x^6 z^{-1}\}'_\infty \{x^{2r} z\}'_\infty \{x^{2r+4} z^{-1}\}'_\infty}, \qquad \{z\}'_\infty = (z; x^4, x^4, x^{2(r-1)}). \tag{2.23}$$

The eight-vertex model is on the 'reflectionless point' if $r = 1 + 1/N$ $(N = 1, 2, 3, \cdots)$ and therefore the $S$-matrix becomes (anti-)diagonal. When $r = 2$ $(N = 1)$ the XYZ model is equivalent to the double Ising model [12], as is well known.

## 3 Form factors in the eight-vertex SOS model

In this section we construct integral formulae for form factors in the eight-vertex SOS model. The first two axioms for form factors in the eight-vertex SOS model are as follows:

1. $W'$–symmetry

$$\begin{aligned} & F_m^{(l,k)}(\cdots, \zeta_{j+1}, \zeta_j, \cdots)_{\cdots l_{j-1} l_j l_{j+1} \cdots} \\ = & \sum_{l'_j} W' \left[ \begin{array}{cc|c} l_{j+1} & l'_j & \\ & & u_j - u_{j+1} \\ l_j & l_{j-1} & \end{array} \right] F_m^{(l,k)}(\cdots, \zeta_j, \zeta_{j+1}, \cdots)_{\cdots l_{j-1} l'_j l_{j+1} \cdots} \end{aligned} \tag{3.1}$$

2. Cyclicity

$$F_m^{(l,k)}(\zeta', x^{-2}\zeta_{2m})_{l \cdots l' l} = F_m^{(l',k)}(\zeta_{2m}, \zeta')_{l' l \cdots l'}. \tag{3.2}$$

Here, we only consider the case $l_0 = l_{2m}$ for $2m$-pt SOS form factors. These two imply the $q$-KZ equation of level 0:

$$\begin{aligned} & F_m^{(l_0,k)}(\zeta_1, \cdots, x^2\zeta_j, \cdots, \zeta_{2m})_{l_0 l_1 \cdots l_{2m-1} l_0} = \sum_{l'_1 \cdots l'_{j-1} l'_{j+1} \cdots l'_{2m}} W' \left[ \begin{array}{cc|c} l_j & l'_{j-1} & \\ & & u_j - u_{j-1} - 2 \\ l_{j-1} & l_{j-2} & \end{array} \right] \\ & \times \prod_{k=1}^{j-2} W' \left[ \begin{array}{cc|c} l'_{k+1} & l'_k & \\ & & u_j - u_k - 2 \\ l_k & l_{k-1} & \end{array} \right] W' \left[ \begin{array}{cc|c} l'_1 & l'_{2m} & \\ & & u_j - u_{2m} \\ l_0 & l_{2m-1} & \end{array} \right] \\ & \times \prod_{k=j+1}^{2m} W' \left[ \begin{array}{cc|c} l'_{k+1} & l'_k & \\ & & u_j - u_k \\ l_k & l_{k-1} & \end{array} \right] F_m^{(l'_1,k)}(\zeta_1, \cdots, \zeta_j, \cdots, \zeta_{2n})_{l'_1 \cdots l'_{j-1} l_j l'_{j+1} \cdots l'_{2m} l'_1}. \end{aligned} \tag{3.3}$$

Set

$$F_m^{(l,k)}(\zeta)_{l l_1 \cdots l_{2m-1} l} = c_m \prod_{1 \leqslant j < k \leqslant 2m} \zeta_j^{-\frac{r}{r-1}} g^*(z_j/z_k) \overline{F}_m^{(l,k)}(\zeta)_{l l_1 \cdots l_{2m-1} l}. \tag{3.4}$$

Here $c_m$ is a constant, and the function $g^*(z)$ is a scalar function defined by (2.23).

Let

$$A_\pm := \{a | l_a = l_{a-1} \pm 1,\ 1 \leqslant a \leqslant 2m\}. \tag{3.5}$$

Then the number of the elements of $A$ is equal to $m$ because $l_0 = l_{2m}$. Let us introduce the following meromorphic function

$$Q'_m(w|\zeta)_{ll_1\cdots l_{2m-1}l} = \prod_{\substack{a,b\in A_- \\ a<b}} [v_a - v_b + 1]' \prod_{a\in A_-} \frac{[u_a - v_a - \frac{1}{2} + l_a]'}{[u_a - v_a - \frac{3}{2}]'} \left( \prod_{j=a+1}^{2m} \frac{[u_j - v_a - \frac{1}{2}]'}{[u_j - v_a - \frac{3}{2}]'} \right), \quad (3.6)$$

$$[u]' = x^{\frac{u^2}{r-1} - u} \Theta_{x^{2(r-1)}}(x^{2u}),$$

where $w_a = x^{-2v_a}$ and $z_j = \zeta_j^2 = x^{-2u_j}$. Here we use slightly different $Q'_m(w|\zeta)_{ll_1\cdots l_{2m-1}l}$ from the one we used in [10].

The integral part $\overline{F}_m^{(l,k)}$ in (3.4) is given as follows:

$$\overline{F}_m^{(l,k)}(\zeta)_{ll_1\cdots l_{2m-1}l} = \prod_{a\in A_-} \oint_{C'_a} \frac{dw_a}{2\pi\sqrt{-1}w_a} \Psi'^{(i)}_m(w|\zeta) Q'_m(w|\zeta)_{ll_1\cdots l_{2m-1}l}. \quad (3.7)$$

Here, $i \equiv k - l \pmod 2$, and the kernel has the form

$$\Psi'^{(i)}_m(w|\zeta) = \vartheta^{(i)}_m(w|\zeta) \prod_{a\in A_-} \prod_{j=1}^{2n} x^{-\frac{(v_a-u_j)^2}{2(r-1)}} \psi'\left(\frac{w_a}{z_j}\right) \prod_{1\leqslant j<k\leqslant 2n} x^{-\frac{(u_j-u_k)^2}{4(r-1)}}, \quad (3.8)$$

where

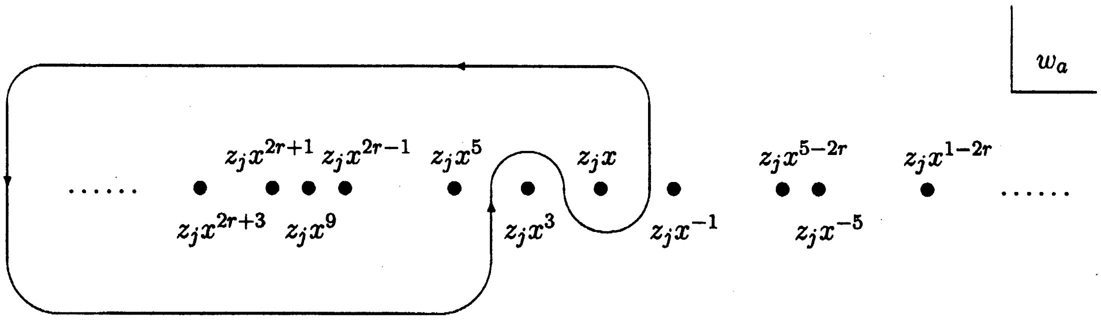
$$\psi'(z) = \frac{(x^{2r+1}z; x^4, x^{2(r-1)})_\infty (x^{2r+1}z^{-1}; x^4, x^{2(r-1)})_\infty}{(xz; x^4, x^{2(r-1)})_\infty (xz^{-1}; x^4, x^{2(r-1)})_\infty}, \quad (3.9)$$

$$\begin{aligned} \vartheta^{(i)}_m(w|\zeta) &= \left( (-1)^m \prod_{a\in A_-} w_a^{-1} \prod_{j=1}^{2m} \zeta_j \right)^i \Theta_{x^8}\left( -x^{2+4i} \prod_{a\in A_-} w_a^{-2} \prod_{j=1}^{2m} z_j \right) \\ &\times \prod_{j=1}^{2m} \zeta_j^{-n(1-\frac{1}{r})-\frac{1}{2r}} \prod_{a\in A_-} x^{-mv_a} \prod_{\substack{a,b\in A_- \\ a<b}} w_a^{-1}\Theta_{x^2}(w_a/w_b). \end{aligned} \quad (3.10)$$

The integrand may have poles at

$$w_a = \begin{cases} x^{\pm(1+4n_1+2(r-1)n_2)} z_j & (1 \leqslant j \leqslant 2m, n_1, n_2 \in \mathbb{Z}_{\geqslant 0}), \\ x^{3+2(r-1)n_3} z_j & (a \leqslant j \leqslant 2m, n_3 \in \mathbb{Z}). \end{cases} \quad (3.11)$$

We choose the integration contour $C'_a$ with respect to $w_a$ ($a \in A_-$) to be along a simple closed curve oriented counter-clockwise that encircles the points $x^{1+4n_1+2(r-1)n_2} z_j$ ($1 \leqslant j \leqslant 2m, n_1, n_2 \in \mathbb{Z}_{\geqslant 0}$) and $x^{3+2(r-1)n_3} z_j$ ($a \leqslant j \leqslant 2m, n_3 \in \mathbb{Z}_{>0}$), but not $x^{-1-4n_1-2(r-1)n_2} z_j$ ($1 \leqslant j \leqslant 2m, n_1, n_2 \in \mathbb{Z}_{\geqslant 0}$) nor $x^{3-2(r-1)n_3} z_j$ ($a \leqslant j \leqslant 2m, n_3 \in \mathbb{Z}_{\geqslant 0}$). Thus, the contour $C'_a$ actually depends on the variables $z_j$, and therefore strictly, it should be written $C'_a(z)$. The LHS of (3.2) represents the analytic continuation with respect to $\zeta_{2m}$.




$(1 \leqslant j \leqslant 2n)$

Then $F_m^{(l,k)}(\zeta)_{ll_1\cdots l_{2m-1}l}$ satisfies level 0 $q$-KZ equations [10], and therefore it can be identified a form factor in the eight-vertex SOS model.

# 4 Form factors in the eight-vertex model

Let us introduce $F_m^{(i)}(\zeta)$, the form factors in the eight-vertex model through the vertex-face transformation as follows:

$$F_m^{(l_0,k)}(\zeta)_{l_0l_1\cdots l_{2m-1}l_{2m}} = \sum_{\mu_1,\cdots,\mu_{2m}} F_m^{(i)}(\zeta)_{\mu_1\cdots\mu_{2m}} t'^{l_1}_{l_0}(u_1-u_0)^{\mu_1}\cdots t'^{l_{2m}}_{l_{2m-1}}(u_{2m}-u_0)^{\mu_{2m}}. \tag{4.1}$$

Here $i \equiv k - l_0 \pmod 2$, and

$$t'^{k}_{k'}(u) := t^{k}_{k'}(u;\epsilon, r-1). \tag{4.2}$$

Let us remind (2.20) and let us introduce

$$\tilde{t}'^{*k}_{k'}(u) := \tilde{t}^{*k}_{k'}(u;\epsilon, r-1). \tag{4.3}$$

Then the following inversion relations hold:

$$\sum_{\varepsilon=\pm} t'^{*k'}_{k}(u)_\varepsilon t'^{k}_{k''}(u)^\varepsilon = \delta^{k'}_{k''}, \qquad \sum_{k'=k\pm1} t'^{k}_{k'}(u)^\varepsilon t'^{*k'}_{k}(u)_{\varepsilon'} = \delta^{\varepsilon}_{\varepsilon'}, \tag{4.4}$$

$$\sum_{\varepsilon=\pm} \tilde{t}'^{*k}_{k'}(u)_\varepsilon t'^{k''}_{k}(u)^\varepsilon = \delta^{k''}_{k'}, \qquad \sum_{k'=k\pm1} t'^{k'}_{k}(u)^\varepsilon \tilde{t}'^{*k}_{k'}(u)_{\varepsilon'} = \delta^{\varepsilon}_{\varepsilon'}. \tag{4.5}$$

It follows from (4.4) and (4.5) that the relation (4.1) is equivalent to

$$\begin{aligned} F_m^{(i)}(\zeta) &= \sum_{l_0,\cdots,l_{2m-1}} F_m^{(l_0,k)}(\zeta)_{l_0l_1\cdots l_{2m-1}l_{2m}} t'^{*l_0}_{l_1}(u_1-u_0)\otimes\cdots\otimes t'^{*l_{2m-1}}_{l_{2m}}(u_{2m}-u_0) \\ &= \sum_{l_1,\cdots,l_{2m}} F_m^{(l_0,k)}(\zeta)_{l_0l_1\cdots l_{2m-1}l_{2m}} \tilde{t}'^{*l_0}_{l_1}(u_1-u_0)\otimes\cdots\otimes \tilde{t}'^{*l_{2m-1}}_{l_{2m}}(u_{2m}-u_0). \end{aligned} \tag{4.6}$$

Thus, the $S$-matrix symmetry (1.2) for $F_m^{(i)}(\zeta)$ follows from the $W'$-symmetry (3.1) for $F_m^{(l_0,k)}(\zeta)$.

It is evident from (4.4) and (4.5) that one of the sufficient conditions of (1.3), the cyclicity for $F_m^{(i)}(\zeta)$, is as follows:

$$\begin{aligned}&\sum_{\substack{l_{2m}=\\ l_{2m-1}\pm 1}} \tilde{t}'^{*l_{2m-1}}_{l_{2m}}(u_{2m}-u_0+2)_\mu F_m^{(l_0,k)}(\zeta', x^{-2}\zeta_{2m})_{l_0\cdots l_{2m-1}l_{2m}}\\ &= \sum_{l'=l_0\pm 1} t'^{*l'}_{l_0}(u_{2m}-u_0)_\mu F_m^{(l',k)}(\zeta_{2m},\zeta')_{l'l_0\cdots l_{2m-1}}.\end{aligned} \tag{4.7}$$

The strategy is as follows. We have an expression for only the case $l_{2m}=l_0$. Thus, first let $l_{2m-1}=l_0\pm 1$ and solve (4.7). Then we will obtain formulae for $l_{2m}=l_0\pm 2$. Next let $l_{2m-1}=l_0\pm 3$ and solve (4.7). Then we will obtain formulae for $l_{2m}=l_0\pm 4$. Repeating this procedure, we will obtain the general formulae for $l_{2m}\equiv l_0$ (mod 2).

For generic $r$, not (4.7) but (4.8) does holds:

$$\begin{aligned}&\sum_{\substack{l_{2m}=\\ l_{2m-1}\pm 1}} \tilde{t}'^{*l_{2m-1}}_{l_{2m}}(u_{2m}-u_0)_\mu F_m^{(l_0,k)}(\zeta', x^{-2}\zeta_{2m})_{l_0l_1\cdots l_{2m-1}l_{2m}}\\ &= \sum_{l'=l_0\pm 1} t'^{*l'}_{l_0}(u_{2m}-u_0)_\mu F_m^{(l',k)}(\zeta_{2m},\zeta')_{l'l_0l_1\cdots l_{2m-1}}.\end{aligned} \tag{4.8}$$

Here, for $l_{2m}=l+2s\geqslant l$, let

$$A'_-=A_-\sqcup\{-1,\cdots,-s\},$$

and $l_{-i}=l+2(i-1)$ for $1\leqslant i\leqslant s$. Then the meromorphic function $Q'_m(w|\zeta)_{ll_1\cdots l_{2m-1}l+2s}$ is defined as follows [13]:

$$\begin{aligned}Q'_m(w|\zeta)_{ll_1\cdots l_{2m-1}l+2s} &= \prod_{a\in A_-}\frac{[u_a-v_a-\frac{1}{2}+l_a]'}{[u_a-v_a-\frac{3}{2}]'}\left(\prod_{j=a+1}^{2m}\frac{[u_j-v_a-\frac{1}{2}]'}{[u_j-v_a-\frac{3}{2}]'}\right)\\ \times \prod_{\substack{a,b\in A'_-\\ a<b}}[v_a-v_b+1]' &\prod_{a'=-1}^{-s}\frac{[u_0-v_{a'}-\frac{1}{2}+l_{a'}]'}{[u_0-v_{a'}-\frac{3}{2}]'}\left(\prod_{j=1}^{2m}\frac{[u_j-v_{a'}-\frac{1}{2}]'}{[u_j-v_{a'}-\frac{3}{2}]'}\right).\end{aligned} \tag{4.9}$$

The meromorphic function $Q'_m(w|\zeta)_{ll_1\cdots l_{2m-1}l-2s}$ for $l_{2m}=l-2s\leqslant l$ can be defined similarly, see [13]. In order to derive (4.8) we use the relation (2.11) with $r$ replaced by $r-1$, and the addition theorems

$$\theta_i\left(\tfrac{l+2s-1-(u-u_0)}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right)\frac{[u-v-\frac{1}{2}+l+2(s-1)]'}{[u-v-\frac{3}{2}]'}-\theta_i\left(\tfrac{l+2s-1+(u-u_0)}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right)$$
$$\times\ \frac{[u_0-v-\frac{1}{2}+l+2(s-1)]'}{[u_0-v-\frac{3}{2}]'}=\frac{[u_0-u]'[l+2s-1]'}{[u_0-v-\frac{3}{2}]'[u-v-\frac{3}{2}]'}\theta_i\left(\tfrac{l+u+u_0-2v+2s-4}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right),$$

$$\theta_i\left(\tfrac{l-(u-u_0+1)}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right)\frac{[u-v-\frac{1}{2}+l]'}{[u-v-\frac{3}{2}]'}-\theta_i\left(\tfrac{l+(u-u_0+1)}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right)$$
$$\times\ \frac{[u_0-v-\frac{3}{2}+l]'}{[u_0-v-\frac{3}{2}]'}\frac{[u-v-\frac{1}{2}]'}{[u-v-\frac{3}{2}]'}=\frac{[u_0-u-1]'[l]'}{[u_0-v-\frac{3}{2}]'[u-v-\frac{3}{2}]'}\theta_i\left(\tfrac{l+u+u_0-2v-2}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right),$$

where $i=3,4$.

When $r=r_N=1+1/N$ $(N=1,2,3,\cdots)$, the eight-vertex model is called reflectionless. At $r=r_N$,

$$\tilde{t}_l^{\prime *l'}(u)=\tilde{t}_l^{\prime *l'}(u+2)$$

holds. Thus, (4.8) implies (4.7) at $r=r_N$. Furthermore, the sum with respect to $l_j$ can be carried out when $r=r_N$, by rewriting $F_m^{(l,k)}$ as $2m$-fold integral form:

$$\overline{F}_m^{(i)}(\zeta)_{\mu_1\cdots\mu_{2m}} = \prod_{a=1}^{2m}\oint_{C'}\frac{dw_a}{2\pi\sqrt{-1}w_a}\Psi_m^{(i)}(w|\zeta)Q_m^{(i)}(w|\zeta)_{\mu_1\cdots\mu_{2m}}. \tag{4.10}$$

Here,

$$\begin{aligned}Q_m^{(i)}(w|\zeta)_{\mu_1\cdots\mu_{2m}} &= \prod_{a=1}^{2m}\frac{1}{[u_0-v_a-\frac{3}{2}]'[u_a-v_a-\frac{3}{2}]'}\left(\prod_{j=a+1}^{2m}\frac{[u_j-v_a-\frac{1}{2}]'}{[u_j-v_a-\frac{3}{2}]'}\right)\\ &\times\prod_{j=1}^{2m}\theta_{\overline{\mu_j}}\left(\tfrac{i+u+u_0-2v}{2(r-1)};\tfrac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right)\prod_{\substack{a<b\\a,b=1}}^{2m}[v_a-v_b+1]',\end{aligned}\tag{4.11}$$

and $\overline{+}=3$, $\overline{-}=4$.

The resulting formulae suggest us that the free field representation of the type II vertex operators[1] are as follows:

$$\Psi_\mu^{*(1-i,i)}(\zeta) = \oint_{C'}\frac{dw}{2\pi\sqrt{-1}w}\psi^*(\zeta)B(w)\frac{\theta_{\bar{\mu}}\left(\frac{i+u+u_0-2v}{2(r-1)};\frac{\pi\sqrt{-1}}{2\epsilon(r-1)}\right)}{[u_0-v-\frac{3}{2}]'[u-v-\frac{3}{2}]'}. \tag{4.12}$$

where

$$\psi^*(\zeta)=\zeta^{\frac{r}{2(r-1)}}:\exp\left(\sqrt{\frac{r}{2(r-1)}}(\sqrt{-1}Q+P\log z)+\sum_{m\neq0}\frac{\alpha_m}{m}z^{-m}\right),$$

$$B(w)=w^{\frac{r}{(r-1)}}:\exp\left(-\sqrt{\frac{2r}{(r-1)}}(\sqrt{-1}Q+P\log w)-\sum_{m\neq0}\frac{\alpha_m}{m}\frac{[2m]_x}{[m]_x}z^{-m}\right).$$

[1]Concerning the terminology type I and II, see e.g., [14].

Here we use the bosonic oscillators with the following commutation relations:

$$\begin{aligned}[]
[\alpha_m, \alpha_n] &= m\frac{[m]_x[rm]_x}{[2m]_x[(r-1)m]_x}\delta_{m+n,0}, \quad [m]_x := \frac{x^m - x^{-m}}{x - x^{-1}}, \\
[Q, P] &= \sqrt{-1}.
\end{aligned} \tag{4.13}$$

## 5 Summary and discussion

In this paper, we tried to construct the form factors in the eight-vertex model as solutions to level 0 $q$-KZ equation, or Smirnov's axioms. The $q$-KZ equation was reduced to (4.7). Up to now, eq. (4.7) has been solved only at reflectionless points $r = 1 + 1/N$ ($N = 1, 2, 3, \cdots$). On these points, we further succeeded to construct the free field representation of the type II vertex operators.

Let us list a few open problems.

1) Obtain the type I vertex operators at reflectionless points, which should commute the type II ones with some scalars, and which themselves should satisfy appropriate commutation relations.
2) Solve (4.7) for generic $r > 1$.
3) Find the link with Shiraishi's work, in which the type I and II vertex operators can be constructed from the representations of the deformed $W(D_{N+1})$ or $W(B_l^{(m)} \otimes B_m)$ algebra.

Shiraishi's bosonization is phenomenological in the sense that the relation between the eight-vertex model and the deformed $W$ algebra is unclear, at least up to now. In a joint work with M. Lashkevich, we study to show that the form factors at reflectionless points can be obtained without integrals on the basis of vertex-face transformation method. Throughout this study, we wish to give theoretical account of Shiraishi's scheme.

## Acknowledgements

The author would like to thank M. Jimbo, H. Konno, M. Lashkevich, A. Nakayashiki, Ya. Pugai and J. Shiraishi for useful discussion. This work was supported in part by a Grant-in-Aid for Scientific Research from JSPS, Japan Society for the Promotion of Science (No. 15540218).

## References

[1] Smirnov, F. A. *Form factors in completely integrable models of quantum field theory*; Advanced Series in Mathematical Physics Vol **14**; World Scientific: Singapore, 1992.

[2] Frenkel I. B. ; Reshetikhin, N. Y. Quantum affine algebras and holonomic difference equations. *Commun. Math. Phys.* 1992, **146**, 1–60.

[3] Lashkevich, M. ; Pugai, Ya. Free field construction for correlation functions of the eight vertex model. *Nucl. Phys.* 1998, **B516**, 623–651.

[4] Lashkevich, M. ; Pugai, Ya. Nearest neighbor two-point correlation function of the $Z$-invariant eight vertex model. *JETP Lett.* 1998, **68**, 257–262.

[5] Baxter, R. J. Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors. *Ann. Phys.* (NY) 1973, **76**, 1–24; II. Equivalence to a generalized ice-type lattice model. *ibid.*, 25–47; III. Eigenvectors of the transfer matrix and Hamiltonian. *ibid.*, 48–71.

[6] Lukyanov, S. ; Pugai, Ya. Multi-point local height probabilities in the integrable RSOS model. *Nucl. Phys.* 1996, **B473**[FS], 631–658.

[7] Quano, Y-H. Bootstrap equations and correlation functions for the Heisenberg XYZ antiferromagnet. *J. Phys. A: Math. Gen.* 2002, **35**, 9549–9572.

[8] Shiraishi, J. (2003) Free field constructions for the elliptic algebra $\mathcal{A}_{q,p}(\widehat{\mathfrak{sl}_2})$ and Baxter's eight-vertex model. math.QA/0302097.

[9] Lashkevich, M. Free field construction for the eight-vertex model: representation for form factors. *Nucl. Phys.* 2002, **B621**, 587–621.

[10] Quano, Y-H. Quantum Knizhnik–Zamolodchikov equations of level 0 and form factors in SOS model. *Prog. Theo. Phys.* 2004, **111**, 361–370.

[11] Baxter, R. J. Solvable eight-vertex model on an arbitrary planar lattice. *Phil. Trans. Roy. Soc.* (London) 1978, **289A**, 315–346.

[12] Baxter, R. J. *Exactly Solved Models in Statistical Mechanics*; Academic Press: London, 1982.

[13] Quano, Y-H. Difference equations for correlation functions and form factors of the eight-vertex/XYZ model, to appear in *Progress in Ferromagnetism Research*, Nova Science Publ: NY.

[14] Jimbo, M. ; Miwa, T. *Algebraic analysis of solvable lattice models*; CBMS Regional Conferences Series in Mathematics Vol **85**; AMS: Providence, RI, 1994.