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Jack symmetric functions and representations of the Virasoro algebra

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1 Introduction

In this talk, we consider the relationships between the Jack symmetric functions and representations of the Virasoro algebra based on my collaboration with J. Shiraishi, D. Arnaudon, L. Frappat and E. Ragoucy [1]. In this section, let us summarize some of basic properties of the Fock representations of the Virasoro algebra and the Jack symmetric functions.

We denote the Virasoro algebra by operators L_n's ($n \in \mathbf{Z}$) and central charge c with commutation relations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{n(n^2-1)}{12}\, c\, \delta_{n+m,0}, & (1)\\ [L_n, c] &= 0. & (2) \end{aligned}$$

Then, we have a representation on the Fock space called the Feigin-Fuchs representation as follows [2].

In terms of the bosonic operators a_n ($n \in \mathbf{Z}$) with usual commutation relations $[a_n, a_m] = n\delta_{n+m,0}$, the Fock space $\mathcal{F}_A$ is defined by

$$\mathcal{F}_A = \mathbf{C}[a_{-1}, a_{-2}, a_{-3}, \cdots]|A\rangle, \tag{3}$$

where $|A\rangle$ is a vacuum vector defined as

$$a_0|A\rangle = A|A\rangle, \qquad a_n|A\rangle = 0 \ \ (n \in \mathbf{Z}_{>0}). \tag{4}$$

We also define dual of the Fock $\mathcal{F}_A^*$ space by

$$\mathcal{F}_A^* = \langle A|\mathbf{C}[a_1, a_2, a_3, \cdots], \tag{5}$$

where we have used the dual of the vacuum vector $\langle A|$.

If we parametrize the central charge c as

$$c = 1 - \frac{6(\beta-1)^2}{\beta}, \tag{6}$$

then we have a representation given by

$$L_n = \frac{1}{2}\sum_{k\in\mathbf{Z}} : a_{n-k}a_k : - A_{1,1}(n+1)a_n, \tag{7}$$

with the usual normal ordered product $:\cdot:$, and notation

$$A_{r,s} = \frac{1}{\sqrt{2}}\left(r\sqrt{\beta} - \frac{s}{\sqrt{\beta}}\right). \tag{8}$$

Note that in this notation, we have a relation $L_0|A_{r+1,s+1}\rangle = h_{r,s}|A_{r+1,s+1}\rangle$, where

$$h_{r,s} = \frac{(r\beta - s)^2 - (\beta-1)^2}{4\beta} = \frac{1}{2}A_{r+1,s+1}A_{r-1,s-1}. \tag{9}$$

In 1995, Mimachi and Yamada [3] found that inside the Fock space, another kind of integrable system — the Calogero-Sutherland model — emerges as singular vectors of the Virasoro algebra. The Calogero-Sutherland models describe the motion of particles with inverse square repulsive potential $U = 1/r^2$ [4]. Especially we consider the quantum N-body model on the circle of length L whose Hamiltonian is given by

$$H_{CS} = -\sum_{i=1}^{N}\frac{1}{2}\frac{\partial^2}{\partial q_i^2} + \beta(\beta-1)\sum_{i<j}\frac{\left(\frac{\pi}{L}\right)^2}{\sin^2\frac{\pi}{L}(q_i - q_j)}, \tag{10}$$

where $0 \le q_i \le L$ are the coordinates of i-th particle, and β in the coupling constant is eventually identified with β which appeared in the central charge of the Virasoro algebra (6). Then it is known that all the excited states of the Schrödinger equation can be written as a product of ground state and the symmetric polynomials of $x_i = \exp\left(\frac{2\pi\sqrt{-1}}{L}q_i\right)$ called Jack symmetric polynomials ([5], see also [6]). If degree of symmetric polynomial is lesser than the number of variables N, then we can represent these polynomials uniquely in terms of the power sums $p_n = \sum_{i=1}^{N} x_i^n$ instead of using coordinates x_i. After expressing symmetric polynomials by power sums, we can take the number of variables to be infinite inside the power sums. Resulting symmetric polynomials with infinitely many variables are often called "symmetric functions".

These Jack symmetric functions form orthogonal basis of the space of symmetric functions Λ. Inner product on Λ is defined by using product of power sums $p_\lambda =$

$\prod_{i=1}^{l(\lambda)} p_{\lambda_i}$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \cdots) := (\cdots, 3^{m_3}, 2^{m_2}, 1^{m_1})$ is a partition and $l(\lambda)$ is a length of partition. Then we define

$$\langle p_\lambda, p_\mu \rangle := \delta_{\lambda,\mu} \beta^{-l(\lambda)} \prod_i i^{m_i} m_i!. \tag{11}$$

There are different choices for normalization of Jack symmetric function, however the normalization called "integral form" denoted by J_λ is suitable for our purpose. The norm of Jack symmetric functions was calculated by Stanley, and it takes following form in our normalization;

$$\langle J_\lambda, J_\mu \rangle = \delta_{\lambda,\mu} \frac{1}{\beta^{2|\lambda|}} \prod_{s\in\lambda} \Big(a(s) + (l(s)+1)\beta\Big)\Big((a(s)+1) + l(s)\beta\Big), \tag{12}$$

where $a(s)$ and $l(s)$ are the arm length and the leg length of s, respectively. It is known that we can identify the Jack symmetric functions as elements of the Fock space through the following two identifications,

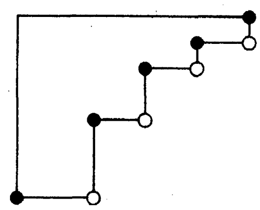
$$p_n \longleftrightarrow \sqrt{\frac{2}{\beta}} a_{-n}|A\rangle, \quad p_n \longleftrightarrow \langle A| a_n \frac{1}{\sqrt{2\beta}}. \tag{13}$$

These identifications work because it preserve the inner product.

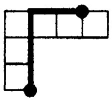
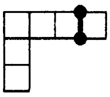
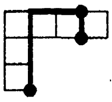
Mimachi and Yamada showed that singular vector of degree rs in the Fock space $\mathcal{F}_{r+1,s+1}$ is proprtional to the Jack symmetric function with rectangular partition $J_{(s^r)}$. Recently [1] extend this result further and fully used Jack symmtric functions as a basis of the Fock space. As a result, some combinatorial properties of the Virasoro algebra emerge, and [1] conjectured the action of Virasoro generators on this Jack basis (see [7] for action on Schur symmetric functions). In rest of this talk, I will give some examples of the formula to complement that paper.

2 Actions of L_n $(n > 0)$

We first consider the action of type $\langle A_{r+1,s+1}|J_\lambda L_n := \langle J_\lambda|L_n$ $(n > 0)$. To describe the results, it is convenient to use some terminologies. For a given diagram, we define *outer corners* (white circles) and *inner corners* (black circles) as follows:

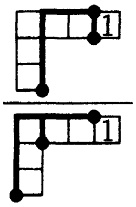


We put "hooks" on diagram, and assign a term $(m+n\beta)$ to each hook with horizontal length m and vertical length n. If there are multiple hooks on a diagram, then we take product of all terms. Some examples are given below:

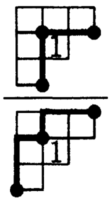
 $= (2+3\beta)$,  $= (0+1\beta)$,  $= \beta\cdot(2+3\beta)$.

Our first example is the action of L_1 on J.

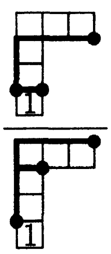
$\langle J$$|L_1 = \sqrt{2\beta}\Big(\langle J$$|\dfrac{\beta\cdot(2+3\beta)}{(2+\beta)(3+3\beta)}A_{r-1,s-4}$



$+\langle J$$|\dfrac{2\cdot 2\beta}{(2+\beta)(1+2\beta)}A_{r-2,s-2}$



$+\langle J$$|\dfrac{(3+2\beta)\cdot 1}{(3+3\beta)(1+2\beta)}A_{r-4,s-1}\Big)$.



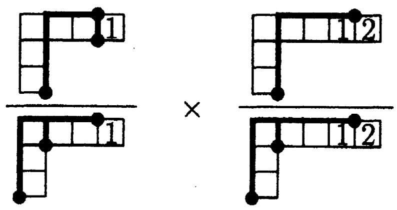
First of all, we notice from above example that the action of L_1 has the effect of adding one more box to each possible places on the partition , and coordinates of added boxes appear in the term like $A_{r-*,s-*'}$, i.e. if added box is on i-th row and j-th column then the term $A_{r-i,s-j}$ appears. The rest part of the above equation, rational functions of β, is explained by the diagrams inserted there. Two diagrams beside the horizontal line stand for the numerator and the denominator respectively.

In the numerator, we join upper left corner of box 1, the added box, and all the outer corners of original diagram $(3,1,1)$. In the denominator, on the other hand, we join the upper left corner of box 1 and all the inner corners of original partition. According to [1], above example should be compared with the following equation:

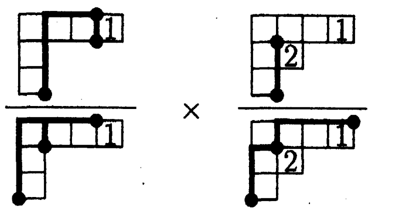
$$p_1 \cdot J_{(3,1,1)} = \left(\frac{\beta \cdot (2+3\beta)}{(2+\beta)(3+3\beta)} J_{(4,1,1)} + \frac{2 \cdot 2\beta}{(2+\beta)(1+2\beta)} J_{(3,2,1)} + \frac{(3+2\beta) \cdot 1}{(3+3\beta)(1+2\beta)} J_{(3,1,1,1)} \right).$$

Let us now go on to example for action of L_2;

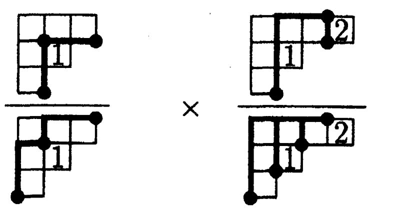
$$\langle J_{(3,1,1)} | L_2 = \sqrt{2\beta} \cdot \beta \left(\langle J_{(5,1,1)} | \frac{\beta(2+3\beta)}{(2+\beta)(3+3\beta)} \cdot \frac{(3+3\beta)}{(3+\beta)(4+3\beta)} A_{r-1,s-5} \right.$$



$$- \langle J_{(4,2,1)} | \frac{\beta(2+3\beta)}{(2+\beta)(3+3\beta)} \cdot \frac{2\beta}{(3+\beta)(1+2\beta)} A_{r-2,s-2}$$



$$+ \langle J_{(4,2,1)} | \frac{2 \cdot 2\beta}{(2+\beta)(1+2\beta)} \cdot \frac{\beta(2+3\beta)}{(1+\beta)(2+2\beta)(3+3\beta)} A_{r-1,s-4}$$



$$\left. + \text{other 7 terms} \right).$$

In this case, we notice that if we act L_2 on given Jack symmetric function, then result is a sum over the Jack symmetric functions with two more boxes added to the original Young diagram. We also notice that there are two terms proportional to $\langle J_{(4,2,1)} |$. If we act L_2 on $\langle J_{(3,1,1)}|$, then primarily we obtain the result as a sum of these two terms, however we cannot find any combinatorial properties as it is. The decomposition given above can be interpreted as two possible ways to obtain $J_{(4,2,1)}$ from $J_{(3,1,1)}$. Diagrams inserted between the equation explain the situation. "1" and "2" assigned on the boxes means the order of addition, and left and right diagrams stand for the first and second factor of rational function of β, respectively. When we add first box to the diagram, then we have a rational function by the same rule as in the case L_1. In the second addition, we obtain a rational function by almost the same rule as in case L_1, however, in the numerator, we do not join upper left corner of box "2" with outer corner created by addition of box "1". The order of addition also affect the term like $A_{r-*,s-*'}$ if second addition is at i-th row and j-th column then we have a term $A_{r-i,s-j}$. Finally, as for over all constants, we have a term $\sqrt{2\beta}\beta^{n-1}$ when we consider the action of L_n, and extra minus signs appear on R.H.S. when row of box 2 is greater than that of 1.

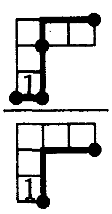
Compare above equation with following formula for multiplication by p_2;

$$
\begin{aligned}
p_2 \cdot J_{(3,1,1)} &= \beta\Bigg(\frac{\beta(2+3\beta)}{(2+\beta)(3+3\beta)}\cdot\frac{(3+3\beta)}{(3+\beta)(4+3\beta)}J_{(5,1,1)} \\
&\quad -\frac{\beta(2+3\beta)}{(2+\beta)(3+3\beta)}\cdot\frac{2\beta}{(3+\beta)(1+2\beta)}J_{(4,2,1)} \\
&\quad +\frac{2\cdot 2\beta}{(2+\beta)(1+2\beta)}\cdot\frac{\beta(2+3\beta)}{(1+\beta)(2+2\beta)(3+3\beta)}J_{(4,2,1)} \\
&\quad +\text{other 7 terms}\Bigg).
\end{aligned}
$$

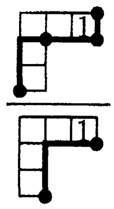
3 Actions of L_n $(n < 0)$

Actions of L_n for $n < 0$ can be understood in a similar manner as in the case $n > 0$ discussed in last section. First we consider the action of L_{-1}.

$$
\langle J_{(3,1,1)}|L_{-1} = \frac{1}{\sqrt{2\beta}}\cdot\frac{1}{\beta}\Bigg(\langle J_{(3,1)}|\frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}A_{r+3-(3+3),\,s+3-(1+1)}
$$



$$+\langle J_{\yng(2,1,1)} | \frac{\beta \cdot 2(3+2\beta)}{(2+2\beta)} A_{r+3-(1+1),\, s+3-(3+3)} \Bigg) .$$



From above example, we notice that when we act L_{-1} on given Jack symmetric function, then we have a sum over the Young diagrams which can be created by removing one box from original one. This situation is explained in the diagrams inserted between above equations (box with number "1" is a box to be removed). These diagrams also explain the rational function of β (like $\frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}$) --- in the numerator, we join the lower right corner of box "1" with all the inner corners of original diagram by hook, and in the denominator, we join the lower right corner of box "1" with all the other outer corners of original diagram.

The term like $A_{r-*,s-*'}$ is a bit more complicated than the case considered in the last section. If the removed box is at r_1-th row and s_1-th column, then we have

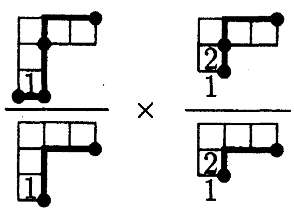
$$A_{r+3\cdot 1-(r_1+r_1),\, s+3\cdot 1-(s_1+s_1)} . \tag{14}$$

Finally, as for the overall constant, we have a term $\frac{1}{\sqrt{2\beta}} \cdot \frac{1}{n\beta^n}$ when we consider the action of L_{-n}. Compare above result with following equation;

$$\frac{\partial}{\partial p_1} J_{\yng(3,1,1)} = \frac{1}{\beta} \left(\frac{(2+3\beta)2\beta \cdot 1}{(2+2\beta)} J_{\yng(3,1)} + \frac{\beta \cdot 2(3+2\beta)}{(2+2\beta)} J_{\yng(2,1,1)} \right) . \tag{15}$$

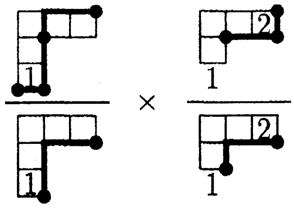
Next, we consider the action of L_{-2};

$$\langle J_{\yng(3,1,1)} | L_{-2} = \frac{1}{\sqrt{2\beta}} \cdot \frac{1}{2\beta^2} \Bigg(-\langle J_{\yng(3)} | \frac{(2+3\beta)2\beta \cdot 1}{(2+2\beta)} \cdot \frac{(2+2\beta)\beta}{(2+\beta)}$$



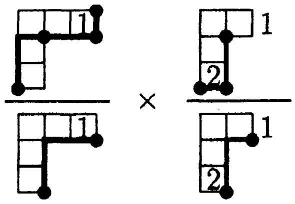
$$\times A_{2r+3(1+2)-(2+3)-(2+2),\, 2s+3(1+2)-(1+1)-(1+1)}$$

$$-\langle J_{\boxplus} \,|\, \frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}\cdot\frac{\beta\cdot 2}{(2+\beta)}$$



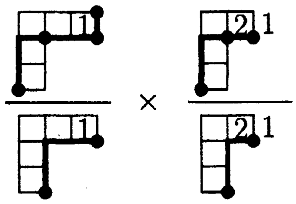
$$\times A_{2r+3(1+2)-(1+3)-(1+1),\,2s+3(1+2)-(3+1)-(3+3)}$$

$$+\langle J_{\boxplus} \,|\, \frac{\beta\cdot 2(3+2\beta)}{(2+2\beta)}\cdot\frac{2\beta\cdot 1}{(1+2\beta)}$$



$$\times A_{2r+3(1+2)-(3+1)-(3+3),\,2s+3(1+2)-(1+3)-(1+1)}$$

$$+\langle J_{\boxminus} \,|\, \frac{\beta\cdot 2(3+2\beta)}{(2+2\beta)}\cdot\frac{1\cdot(2+2\beta)}{(1+2\beta)}$$



$$\times A_{2r+3(1+2)-(1+1)-(1+1),\,2s+3(1+2)-(2+3)-(2+2)}\Big).$$

As in the case L_2, above equation can be obtained by similar graphical procedure used in L_{-1}. Especially, in the numerator, we do not join the lower right corner of box "2" and the inner corner created by removal of box "1". If we remove the box (r_1, s_1) at first and next we remove the box (r_2, s_2), then we have a term

$$A_{2r+3(1+2)-(r_2+r_1)-(r_2+r_2),\,2s+3(1+2)-(s_2+s_1)-(s_2+s_2)}. \tag{16}$$

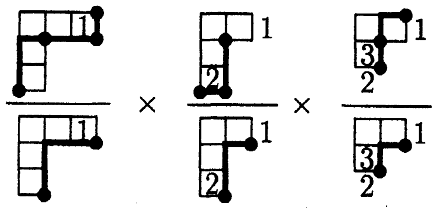
Extra minus sign appears in R.H.S. when the row of box "1" is greater than that of box "2". Compare this result with following equation;

$$\frac{\partial}{\partial p_2}J_{\boxplus} = \frac{1}{2\beta^2}\left(-\frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}\cdot\frac{(2+2\beta)\beta}{(2+\beta)}J_{\square\square\square}\right.$$

$$
\begin{aligned}
&-\frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}\cdot\frac{\beta\cdot 2}{(2+\beta)}J_{(2,1)}\\
&+\frac{\beta\cdot 2(3+2\beta)}{(2+2\beta)}\cdot\frac{2\beta\cdot 1}{(1+2\beta)}J_{(2,1)}\\
&+\frac{\beta\cdot 2(3+2\beta)}{(2+2\beta)}\cdot\frac{1\cdot(2+2\beta)}{(1+2\beta)}J_{(1,1,1)}\Bigg).
\end{aligned}
$$

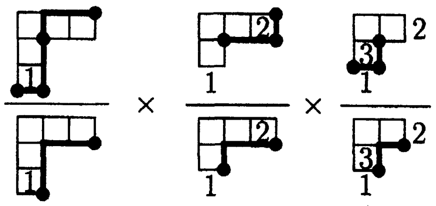
To make these rules clearer, we consider the action of L_{-3} as a more complicated example;

$$
\langle J_{(3,1,1)}|L_{-3} = \frac{1}{\sqrt{2\beta}}\cdot\frac{1}{3\beta^3}\Bigg(-\langle J_{(2)}|\frac{\beta\cdot 2(3+2\beta)}{(2+2\beta)}\cdot\frac{2\beta\cdot 1}{(1+2\beta)}\cdot\frac{(1+2\beta)\beta}{(1+\beta)}
$$



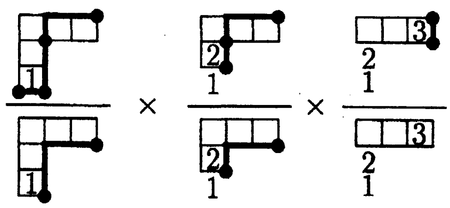
$$
\times A_{3r+3(1+2+3)-(2+1)-(2+3)-(2+2),\,3s+3(1+2+3)-(1+3)-(1+1)-(1+1)}
$$

$$
-\langle J_{(2)}|\frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}\cdot\frac{\beta\cdot 2}{(2+\beta)}\cdot\frac{\beta\cdot 1}{(1+\beta)}
$$



$$
\times A_{3r+3(1+2+3)-(2+3)-(2+1)-(2+2),\,3s+(1+2+3)-(1+1)-(1+3)-(1+1)}
$$

$$
+\langle J_{(2)}|\frac{(2+3\beta)2\beta\cdot 1}{(2+2\beta)}\cdot\frac{(2+2\beta)\beta}{(2+\beta)}\cdot\frac{\beta}{1}
$$



$$
\times A_{3r+(1+2+3)-(1+3)-(1+2)-(1+1),\,3s+(1+2+3)-(3+1)-(3+1)-(3+3)}
$$

$$
+\langle J_{(1,1)}|\frac{\beta\cdot 2(3+2\beta)}{(2+2\beta)}\cdot\frac{1\cdot(2+2\beta)}{(1+2\beta)}\cdot\frac{1}{1}
$$

$$
\times A_{3r+3(1+2+3)-(3+1)-(3+1)-(3+3),\,3s+3(1+2+3)-(1+3)-(1+2)-(1+1)}
$$

$$
\begin{aligned}
&-\langle J_{\boxminus} | \frac{\beta \cdot 2(3+2\beta)}{(2+2\beta)} \cdot \frac{2\beta \cdot 1}{(1+2\beta)} \cdot \frac{\beta \cdot 1}{(1+\beta)} \\
&\times A_{3r+3(1+2+3)-(1+1)-(1+3)-(1+1),3s+3(1+2+3)-(2+3)-(2+1)-(2+2)} \\
&-\langle J_{\boxminus} | \frac{(2+3\beta)2\beta \cdot 1}{(2+2\beta)} \cdot \frac{\beta \cdot 2}{(2+\beta)} \cdot \frac{1 \cdot (2+\beta)}{(1+\beta)} \\
&\times A_{3r+3(1+2+3)-(1+3)-(1+1)-(1+1),3s+3(1+2+3)-(2+1)-(2+3)-(2+2)} \Bigg).
\end{aligned}
$$

As in the case L_{-2}, some of the hooks do not appear in the numerators — when we remove the box "2", we do not join the lower right corner of "2" and inner corner created by removal of box "1", and also we do not join that of box "3" and inner corner created by removal of box "2". If we remove boxes (r_1, s_1), (r_2, s_2), (r_3, s_3) in this order, then we have a term

$$A_{3r+3(1+2+3)-(r_3+r_1)-(r_3+r_2)-(r_3+r_3),3s+3(1+2+3)-(s_3+s_1)-(s_3+s_2)-(s_3+s_3)}. \tag{17}$$

Extra minus sign appear in R.H.S. when the number $\#\{i \in \{1,2\} | r_{i+1} < r_i\}$ is odd.

As for the action of L_{-n}, we remark that if we remove the boxes (r_1, s_1), (r_2, s_2), $\cdots$, (r_n, s_n) in this order, we have a term

$$A_{\sum_{k=1}^{n} [r+3k-(r_n+r_k)], \sum_{k=1}^{n} [s+3k-(s_n+s_k)]}. \tag{18}$$

4 Singular vectors of the Virasoro algebra

As an application of actions of L_n operators on Jack symmetric functions, we consider the singular vectors of the Virasoro algebra on the Fock space. See [8, 2, 7, 3] for earlier works on this subject. To derive the formula $L_n | J_\lambda \rangle$ — rightward action of L_n — we use the relation

$$\Big(\langle J_\lambda | L_n \Big) | J_\mu \rangle = \langle J_\lambda | \Big(L_n | J_\mu \rangle \Big) \tag{19}$$

and orthogonality of Jack symmetric functions. As a result, in $L_n | J_\lambda \rangle$ $(n > 0)$, term $A_{r-i,s-j}$ appears when first removal is on (i, j). We also notice that in $L_n | J_\lambda \rangle$, we have factorized rational function of β whose fuctors have form $(n + m\beta)$ $(n, m \geq 0)$ just as in the case for $\langle J_\lambda | L_n$.

Then, we can reconstruct the Mimachi-Yamada theorem by using Virasoro action on Jack symmetric functions as follows [1]. First of all, from the commutation relations of the Virasoro algebra, we simply have to check the actions of L_1 and L_2

are 0 to obtain the singular vectors. We take the Fock space $\mathcal{F}_{A_{r+1,s+1}}$ and consider its vectors of the form $|J_{(n^m)}\rangle$. Then we have [9]

$$L_1|J_{(n^m)}\rangle = \sqrt{\frac{2}{\beta}}\cdot\frac{1}{\beta}mn\beta A_{r-m,s-n}|J_{(n^{m-1},n-1)}\rangle, \tag{20}$$

$$\begin{aligned} L_2|J_{(n^m)}\rangle &= \sqrt{\frac{2}{\beta}}\cdot\frac{1}{\beta^2}\left(mn\beta\cdot\frac{(n-1)(1+m\beta)}{(1+\beta)}A_{r-m,s-n}|J_{(n^{m-1},n-2)}\rangle\right.\\ &\left.\quad -mn\beta\cdot\frac{(m-1)\beta(n+\beta)}{(1+\beta)}A_{r-m,s-n}|J_{(n^{m-2},(n-1)^2)}\rangle\right). \end{aligned} \tag{21}$$

From the definition of $A_{r,s}$, we have $A_{0,0} = 0$. Thus we can conclude $L_1|J_{(s^r)}\rangle = L_2|J_{(s^r)}\rangle = 0$, i.e. $|J_{(s^r)}\rangle$ is a singular vector of $\mathcal{F}_{A_{r+1,s+1}}$. This singular vector is usually denoted as $|\chi_{r,s}\rangle$.

Next we consider singular vectors in the region $c \leq 1$, that is, the region $\beta > 0$, in more detail [9]. To do this, it is convenient to take parametrization of the central charge as

$$\beta = \frac{p}{q} > 0, \tag{22}$$

where p and q are mutually prime positive integers. To find a vectors which will vanish under the action of L_1, terms like $(n + m\beta)$ is not important, and we have only to consider terms like $A_{r-*,s-*'}$. If there are more than one corners on Young diagram, then action of L_1 will result in sum of more than one terms each proportional to different factors $A_{r-*,s-*'}$ — this is because all such corners have different coordinates. So, we cannot make action of L_1 on these kind of Jack symmetric functions vanish merely adjust integers r, s in $\mathcal{F}_{A_{r+1,s+1}}$, because set of Jack symmetric functions forms linearly independent basis of the space of symmetric functions. Thus all the singular vectors have to be proportional to Jack symmetric functions with rectangular type partitions. Then from equation

$$A_{m-(m+kq),n-(n+kp)} = A_{-kq,-kp} = 0, \qquad k \in \mathbf{Z}_{>0}, \tag{23}$$

we have candedates for singular vectors on $\mathcal{F}_{A_{m+1,n+1}}$ $(n, m \in \mathbf{Z}_{>0})$ as

$$|J_{(n^m)}\rangle,\ |J_{(n+p)^{(m+q)}}\rangle,\ |J_{(n+2p)^{(m+2q)}}\rangle,\ \cdots,\ |J_{(n+kp)^{(m+kq)}}\rangle,\ \cdots. \tag{24}$$

Now we can easily verify that all elements of the above sequence also vanish under the action of L_2.

Finally, we compare the singular vectors on Verma module and Fock space [1]. This again shows the combinatorial aspects of the Virasoro algebra. To fix the comparison uniquely, we first normalize singular vectors in the Verma module as

$$|\chi_{r,s}\rangle = \left(c_1 L_{-n} + c_2 L_{-(n-1)} L_{-1} + \cdots + 1 \times L_{-1}^n\right)|A_{r+1,s+1}\rangle, \tag{25}$$

i.e., we set the coefficient in front of L_{-1}^n to be unity. Then, if we bosonize $|\chi_{r,s}\rangle$, we have

$$|\chi_{r,s}\rangle = \prod_{(i,j)\in(s^r)} (i\beta - j)|J_{(s^r)}\rangle, \tag{26}$$

where the product is taken over all the boxes on partition (s^r). If we further take dual of $|\chi_{r,s}\rangle$ in the Verma module, and consider its bosonization. Then we have

$$\langle\chi_{r,s}| = 0 \tag{27}$$

in general.

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