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Holomorphic motion and invariant metrics

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1 Introduction

The study of holomorphic motions initiated by Mañé, Sad and Sullivan [16] has attracted much attention since then (cf. [2], [10], [22], [23]). The precise definition is as follows:

Definition. Let E be a subset of $\mathbf{C}$. Let Δ_r denote the open disc $|z| < r$. A holomorphic motion of E is a map

$$f : \Delta_r \times E \to \mathbf{C}$$

with the following properties: 1) $f(0, z) = z$ for all $z \in E$; 2) for every fixed $\lambda \in \Delta_r$, the map $f(\lambda, \cdot) : E \to \mathbf{C}$ is an injection; 3) for every fixed $z \in E$, the map $f(\cdot, z) : \Delta_r \to \mathbf{C}$ is holomorphic.

In other words, a holomorphic motion is a holomorphic family of injections. The original motivation of studying it arises from complex dynamics. From the viewpoint of several complex variables, the study of the graphs

$$\Gamma(f) := \{(\lambda, f(\lambda, z)) \in \mathbf{C}^2 : \lambda \in \Delta_r, z \in E\}, \quad E : \text{ domains}$$

are more natural. A particular interesting case is when $E = \Delta_1$, since $\Gamma(f)$ often serves as the universal covering of a holomorphic family of compact Riemann surfaces with finite punctures, according to the celebrated simultaneous uniformization of Bers. Generally, $\Gamma(f)$ is not biholomorphically equivalent to the unit polydisc (cf. [13]).

In this note, we will show

Theorem 1. *Let $f : \Delta_1 \times \Delta_1 \to \mathbf{C}$ be a holomorphic motion. Then for any $0 < r < 1$, $\Gamma(f|_{\Delta_r \times \Delta_1})$ is a bounded domain of holomorphy which enjoys the following function properties*

(i) *The Carathéodory, Bergman, Kobayashi and Kähler-Einstein metrics are equivalent;*

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(ii) $K \geq C\delta^{-2}|\log\delta|^{-2}$, *where* K *denotes the Bergman kernel and* δ *the Euclidean boundary distance*;

(iii) *All invariant pseudo-distances dominate* $|\log\delta|$.

As applications of Theorem 1 we present the following

Theorem 2. *Let* $\pi : M \to \Delta_1$ *be a holomorphic family of open hyperbolic Riemann surfaces. Then for every* $0 < r < 1$, $\pi^{-1}(\Delta_r)$ *is a complete Kobayashi hyperbolic Stein manifold.*

Theorem 3. *The* L^2 $\bar{\partial}-$*cohomology group of type* (p,q) *with respect to the Bergman metric on* $\Gamma(f|_{\Delta_r\times\Delta_1})$ *is vanishing for* $p+q\neq 2$ *and non-vanishing for* $p+q=2$.

Given a bounded domain in $\mathbf{C}^n$, it is generally very difficult to determine whether the L^2-cohomology group with respect to the Bergman metric is vanishing or not. Besides the trivial polydisc case, only a few results are known, for instance, bounded strongly pseudoconvex domains [8] and bounded symmetric domains [12].

The Kobe-Poincaré uniformization shows that the universal covering of a Reimann surface different from $\mathbf{P}^1$ is either Δ_1 or $\mathbf{C}$. However, one can not expect such a perfect phenomenon still holds for high dimensional complex manifolds. In fact, their universal coverings are completely mysterious expect some special cases (eg. balls, symmetric domains). Based on the Bers theory, Griffiths [11] showed that every point in a projective manifold admits a Zariski neighborhood U such that the universal covering $\tilde{U}$ is topologically a cell and is biholomorphically equivalent to a bounded domain of holomorphy. Griffiths' uniformization was used by Nadel-Tsuji [18] to compactify certain complete Kähler manifolds of finite volume.

Theorem 4. *Let* U *be a Zariski open set in the sense of Griffiths. Then the universal covering* $\tilde{U}$ *is a bounded domain of holomorphy which enjoys the following properties*:

(i) *The Carathéodory, Bergman, Kobayashi and Kähler-Einstein metrics are equivalent*;

(ii) *The Bergman metric of* $\tilde{U}$ *has bounded geometry and it descends to a complete Kähler metric on* U *which has finite volume.*

The geometry interests of (ii) lie in that Cheeger-Gromov [3] extended the L^2 index theory of Atiyah [1] to those non-compact manifolds of finite volume whose universal coverings have bounded sectional curvature and positive injectivity radius.

An interesting consequence of Theorem 4 is the following

Theorem 5. *Let* M *be a projective manifold. Given a point* $p\in M$, *there exists a Zariski open neighborhood* U *of* p *which is complete Kobayashi hyperbolic and hyperbolically embedded into* M.

Hyperbolically embedded complex spaces have important applications in the theory of moduli space of holomorphic maps (cf. [19]).

2 Elementary properties of quasiconformal maps

A homeomorphism f of a domain E in $\mathbf{C}$ is called L–quasiconformal if it is differentiable almost everywhere and

$$|\partial f/\partial \bar{z}| \leq \frac{L-1}{L+1}|\partial f/\partial z| \quad \text{a.e. on } E$$

where $L \geq 1$ is a constant. It is clear that a 1–quasiconformal map is conformal. The smallest L is called the dilatation of f. A L–quasiconformal map f on E is Hölder continuous, that is, for any compact subset F in E,

$$|f(z_2) - f(z_1)| \leq C|z_2 - z_1|^{1/L}$$

for all $z_1, z_2 \in F$. Note that f^{-1} is also L–quasiconformal. Thus

$$|f(z_2) - f(z_1)| \geq C^{-1}|z_2 - z_1|^L.$$

A Beltrami coefficient μ in a planar domain E is an element of the open unit ball in the complex Banach space $L^\infty(E)$. A μ–conformal map f of E is a solution of the Beltrami equation:

$$\partial f/\partial \bar{z} = \mu \partial f/\partial z \tag{1}$$

where the derivatives are taken in the sense of distribution. An important remark is that f depends holomorphically on parameters if μ does. If f_1, f_2 are two μ–conformal maps of E, then $f_2 \circ f_1^{-1}$ is conformal. A basic result is: for any measurable, compactly supported function μ of the plane with $\|\mu\|_\infty < 1$, there exists a solution f of the Beltrami equation (1) of the plane with the property that

$$\frac{f(z)}{z} \to 1 \quad (z \to \infty).$$

The start point is

Proposition 1. (cf. [15], pp. 17) *Let f be a L–quasiconformal map of the plane fixing 0 and ∞. Then for every $r > 0$*

$$\frac{\max_\theta |f(re^{i\theta})|}{\min_\theta |f(re^{i\theta})|} \leq e^{\pi L}.$$

We infer from the above proposition that if f is a L–quasiconformal map of $\mathbf{C}$ fixing ∞, then for any $|z_2 - z_0| = |z_1 - z_0|$:

$$|f(z_2) - f(z_0)| \leq e^{\pi L}|f(z_1) - f(z_0)|. \tag{2}$$

Claim. *Let f be a $L-$quasiconformal map of the plane and E a bounded domain. Then there is a constant C depends only on E such that for any disc $\Delta_r(z_0) \subset\subset E$*

$$\max_{z\in\partial\Delta_r(z_0)} |f(z)-f(z_0)| \leq Ce^{\pi L} \min_{z\in\partial\Delta_r(z_0)} |f(z)-f(z_0)| \tag{3}$$

Proof. Fix a disc Δ' containing the closure of E. Let μ be the Beltrami coefficient of f. Set $\tilde{\mu} = \chi_{\Delta'} \cdot \mu$ where $\chi_{\Delta'}$ is the characteristic function of Δ'. Then there exists a $\tilde{\mu}-$conformal map $\tilde{f}$ of the plane such that $\tilde{f}(z)/z \to 1$ as $z \to \infty$. As $f \circ \tilde{f}^{-1}$ is a conformal map of $\tilde{f}(\Delta')$ and Cauchy's estimate implies that the norm of its derivative is uniformly controllable on $\tilde{f}(E)$, the assertion follows immediately from (2).

The central theorem in the theory of holomorphic motion is the following

Proposition 2. *Let $f : \Delta_1 \times E \to \mathbf{C}$ be a holomorphic motion. Then*

a) (cf. [2]) *every $f(\lambda, \cdot)$ is the restriction to E of a quasiconformal self-map of $\mathbf{C}$, of dilatation not exceeding*

$$L = \frac{1+|\lambda|}{1-|\lambda|}.$$

b) (cf. [22]) *f extends to a holomorphic motion $\tilde{f} : \Delta_1 \times \mathbf{C} \to \mathbf{C}$.*

3 Proof of Theorem 1

Suppose that $f : \Delta_1 \times \Delta_1 \to \mathbf{C}$ is a holomorphic motion. We first note that for every boundary point $(\lambda^*, f(\lambda^*, z^*))$ the holomorphic function

$$(w - f(\lambda, z^*))^{-1}$$

gives a holomorphic function on $\Gamma(f|_{\Delta_r\times\Delta_1})$ which can not be extended through this boundary point. Consequently, $\Gamma(f|_{\Delta_r\times\Delta_1})$ is a domain of holomorphy. By Proposition 2, for every fixed $\lambda \in \Delta_1$, $f(\lambda, \cdot)$ extends to a $\frac{1+|\lambda|}{1-|\lambda|}-$quasiconformal map of the plane. Therefore, $\Gamma(f|_{\Delta_r\times\Delta_1})$ is bounded.

For arbitrary fixed $z^* \in \Delta_1$, take $|z_*| > 1$ so that $\arg z_* = \arg z^*$ such that the middle point $\frac{z^*+z_*}{2} \in \partial\Delta_1$. Let $0 < r < 1$ be given. Fix a number $r < r' < 1$. As f depends holomorphically on λ, it follows from (3) and a compactness argument that there is a constant C_0 depending only on r, r' such that

$$\begin{aligned}
\max_{|z-z^*|=1-|z^*|} |f(\lambda,z)-f(\lambda,z^*)| &\leq C_0 \min_{|z-z^*|=1-|z^*|} |f(\lambda,z)-f(\lambda,z^*)| \\
\max_{|z-z_*|=1-|z^*|} |f(\lambda,z)-f(\lambda,z_*)| &\leq C_0 \min_{|z-z_*|=1-|z^*|} |f(\lambda,z)-f(\lambda,z_*)| \\
\max_{|z-\frac{z^*+z_*}{2}|=1-|z^*|} |f(\lambda,z)-f(\lambda,\frac{z^*+z_*}{2})| &\leq C_0 \min_{|z-\frac{z^*+z_*}{2}|=1-|z^*|} |f(\lambda,z)-f(\lambda,\frac{z^*+z_*}{2})|
\end{aligned}$$

hold for all $\lambda \in \Delta_{r'}$. It follows that

$$\bigcup_{\lambda\in\Delta_r}\left\{(\lambda,w)\in \mathbf{C}^2 : |w-f(\lambda,z^*)| < C_0^{-1}|f(\lambda,\frac{z^*+z_*}{2})-f(\lambda,z^*)|\right\}\subset\Gamma(f) \tag{4}$$

$$C_0^{-1}\leq\frac{|f(\lambda,z^*)-f(\lambda,\frac{z^*+z_*}{2})|}{|f(\lambda,z_*)-f(\lambda,\frac{z^*+z_*}{2})|}\leq C_0 \tag{5}$$

and

$$h(\lambda,w):=\frac{f(\lambda,\frac{z^*+z_*}{2})-f(\lambda,z_*)}{C_0[w-f(\lambda,z_*)]}$$

defines a holomorphic map $\Gamma(f|_{\Delta_r\times\Delta_1})\to\Delta_1$. For an arbitrary fixed point $(\lambda^*,z^*)\in\Delta_r\times\Delta_1$, we define a holomorphic embedding as follows

$$\begin{aligned}\Phi^*: \quad & \{\Gamma(f|_{\Delta_r\times\Delta_1}),(\lambda^*,f(\lambda^*,z^*))\}\to\{\Delta_1^2,0\}\\ & (\lambda,w)\to(\lambda',w')=\left(\frac{\lambda^*-\lambda_*}{2(\lambda-\lambda_*)}-\frac{1}{2},\frac{h(\lambda,w)-h(\lambda,f(\lambda,z^*))}{2}\right),\end{aligned}$$

where $|\lambda_*|>r$, $\arg\lambda_*=\arg\lambda^*$ and $\frac{\lambda_*+\lambda^*}{2}\in\partial\Delta_r$. Let Ω^* denote the image Φ^*. As

$$\begin{aligned}\lambda'(\lambda)-\lambda(\lambda^*) &= \frac{\lambda^*-\lambda}{2(\lambda-\lambda_*)}\\ h(\lambda,w)-h(\lambda,f(\lambda,z^*)) &= C_0^{-1}\frac{f(\lambda,\frac{z^*+z_*}{2})-f(\lambda,z_*)}{f(\lambda,z_*)-f(\lambda,z^*)}\cdot\frac{w-f(\lambda,z^*)}{w-f(\lambda,z_*)},\end{aligned}$$

there exists a constant $0<a<1$ (independent of (λ^*,z^*)) such that

$$\Delta_1^2\supset\Omega^*\supset\Delta_a^2$$

by (4), (5) and the following primary fact

$$\left\{z\in\mathbf{C}:\frac{|z|^2}{|z-1|^2}<\epsilon\right\}\subset\Delta_1,\qquad \text{for } \epsilon\ll 1.$$

The rest steps will be proceeded in a similar way as in [4]. The equivalence of the Carathéodory metric c, Bergman metric b and Kobayashi metric k at $(\lambda^*,f(\lambda^*,z^*))$ follows immediately from the biholomorphic invariance and the following well-known properties:

$$\begin{gathered}c_{\Delta_1^2}\leq c_{\Omega^*}\leq c_{\Delta_a^2}\\ k_{\Delta_1^2}\leq k_{\Omega^*}\leq k_{\Delta_a^2}\end{gathered}$$

and

$$\frac{K_{\Delta_1^2}}{K_{\Delta_a^2}}b_{\Delta_1^2}\leq b_{\Omega^*}\leq\frac{K_{\Delta_a^2}}{K_{\Delta_1^2}}b_{\Delta_a^2}.$$

To see the equivalence of the Kähler-Einstein metric with other canonical metrics, we need to do more. Let $\mathcal{H}_2^{2,0}$ denote the space of L^2 holomorphic $(2,0)$−forms on $\Gamma(f|_{\Delta_r\times\Delta_1})$. For any $s \in \mathcal{H}_2^{2,0}$ with unit L^2 norm, Cauchy's estimates imply

$$\left|\frac{\partial^{\alpha+\beta}s^*}{\partial\lambda'^{\alpha}\partial w'^{\beta}}(0)\right| \leq C_{a,\alpha,\beta}$$

where we write $s = s^* d\lambda' \wedge dw'$ on Δ_a^2. By the well-known extreme property of the Bergman metric, we conclude that $\Gamma(f|_{\Delta_r\times\Delta_1})$ has bounded geometry in the sense of Cheng-Yau [5] with respect to the Bergman metric. By the Schwarz lemma of Yau [24], dV_{KE} is always dominated by dV_B, where dV_{KE} and dV_B denote the volume forms of the Kähler-Einstein and Bergman metrics respectively. Since the Kähler-Einstein metric always dominates the Carathéodory metric (cf. [5]), the proof of (i) is complete.

Let $\tilde{K}$ denote the Bergman kernel form of $\Gamma(f|_{\Delta_r\times\Delta_1})$ and K the Bergman kernel function. Then

$$\begin{aligned}\tilde{K} &= \tilde{K}^* d\lambda' \wedge dw' \wedge d\overline{\lambda'} \wedge d\overline{w'} \\ &= K d\lambda \wedge dw \wedge d\bar{\lambda} \wedge d\bar{w}\end{aligned}$$

which implies

$$\begin{aligned}K &= \tilde{K}^* \left|\det\begin{pmatrix} \partial\lambda/\partial\lambda' & \partial w/\partial\lambda' \\ \partial\lambda/\partial w' & \partial w/\partial w'\end{pmatrix}\right|^2 \\ &= \tilde{K}^* \left|\frac{\lambda_* - \lambda^*}{2(\lambda-\lambda_*)^2}\cdot\frac{f(\lambda,\frac{z_*+z^*}{2}) - f(\lambda,z_*)}{C_0[w-f(\lambda,z_*)]^2}\right|^{-2}. \qquad (6)\end{aligned}$$

Since the ratio of $\tilde{K}$ and dV_{KE} is pinched between two positive uniform constants, (ii) follows from

$$\frac{dV_{KE}}{d\lambda\wedge d\bar{\lambda}\wedge dw\wedge d\bar{w}} \geq C\delta^{-2}|\log\delta|^{-2}$$

(compare [17]). Next consider the holomorphic function

$$\frac{f(\lambda,\frac{z_*+z^*}{2}) - f(\lambda,z_*)}{w - f(\lambda,z_*)}$$

on Δ_a^2. Since it is bounded between two uniform positive constants, the classical Schwarz lemma implies

$$\partial\log\left|\frac{f(\lambda,\frac{z_*+z^*}{2}) - f(\lambda,z_*)}{w - f(\lambda,z_*)}\right|$$

has bounded length at $o \in \Delta_a^2$ w.r.t. the Euclidean metric, hence by (6)

$$\sup|\partial_w \log K|_b < \infty \qquad (7)$$

and (iii) follows immediately from (i), (ii), since the lower estimate of the Bergman distance along horizontal direction is trivial.

Remarks. a) It is not known whether $\partial_\lambda \log K$ is bounded with respect to the Bergman metric.

b) The conclusion of Theorem 1 still holds for a holomorphic motion of planar domains bounded by finite Jordan curves.

c) Every bounded Carathéodory complete domain in $\mathbf{C}^n$ is hyperconvex, i.e., there exists a bounded continuous plurisubharmonic exhaustion function.

d) A bounded domain $\Omega \in \mathbf{C}^n$ is called $B-$regular if every boundary point is a peak point for plurisubharmonic functions. The most important examples of $B-$regular domains are strongly pseudoconvex domains. It is easy to show that $\Gamma(f|_{\Delta_r \times \Delta_1})$ is not boholomorphically equivalent to a $B-$regular domain Ω. In fact, if such a biholomorphic map F exists one can choose a sequence of points $z_j \in \Delta_1$ such that $|z_j| \to 1$ and the sequence of embedded analytic disks $F(\cdot, f(\cdot, z_j))|_{\Delta_1}$ accommodate to the boundary of Ω, violating the maximal principle for psh functions since every boundary point of Ω is a peak point. Contradictory.

4 Proof of Theorem 2

A Riemann surface S is call hyperbolic if there is a Fuchsian group Γ acting freely on Δ_1 such that $S = \Delta_1/\Gamma$. A holomorphic family of Riemann surfaces over Δ_1 with fiber model S consists of a complex manifold M and a holomorphic split submersion π mapping M onto Δ_1 such that there is a map $f : \Delta_1 \times S \to \pi^{-1}(\Delta_1)$ satisfying the following properties: 1) $f(0, z) = z$ for all $z \in E$; 2) for every fixed $\lambda \in \Delta_r$, the map $f(\lambda, \cdot) : E \to \mathbf{C}$ is quasiconformal; 3) for every fixed $z \in E$, the map $f(\cdot, z) : \Delta_r \to \mathbf{C}$ is holomorphic. Generalizing the Bers theory, Earle-Fowler [9] showed that the universal covering of M is a holomorphic motion of the unit disc, thus by Theorem 1 that $\pi^{-1}(\Delta_r)$ is complete Kobayashi hyperbolic for any $r < 1$. The proof of Steinness is essentially due to Ohsawa [20], which is included here for the sake of completeness. Let S be exhausted by an increasing continuous family of open Riemann surfaces $\{S_t\}_{t>0}$. By a theorem of Docquier-Grauert [6], it suffices to show that $\Gamma(f|_{\Delta_r \times S_t})$ is Stein for every $t > 0$. Fix $t' > t$ and $r < r' < 1$. Since every $\pi^{-1}(\lambda)$ is a Stein submanifold of $\pi^{-1}(\Delta_1)$, it has a Stein neighborhood V_λ by Siu's theorem [21] on which admits a strictly plurisubharmonic function ψ_λ. A compactness argument shows that there exists a finite covering $W_1, \cdots, W_l$ of $\overline{\Delta_{r'}}$ together with a partition of unitary χ_α, $1 \le \alpha \le l$ such that

$$C|\lambda|^2 + \sum_\alpha \chi_\alpha \psi_{\lambda_\alpha}$$

is strictly psh on $\Gamma(f|_{\Delta_{r'} \times S_{t'}})$ provided t sufficiently large. On the other hand, $\Gamma(f|_{\Delta_r \times S_t})$ is locally pseudoconvex, thus is Stein by a theorem of Ellencwajg [7].

5 Proof of Theorem 3

We shall first prepare notations on the L^2 $\bar\partial$–cohomology. Let (X,ω) be a complete Kähler manifold of dimension n and $C_0^{p,q}(X)$ the set of compactly supported C^∞ (p,q)–forms on X. We set

$$(u,v)=\int_X u\wedge \overline{*v},\quad \text{for } u,v\in C_0^{p,q}(X)$$

where $*$ denotes the Hodge's star operator. Let $L_{(2)}^{p,q}(X)$ denote the completion of $C_0^{p,q}(X)$ with respect to the L^2 norm $\|\cdot\|=\sqrt{(\cdot,\cdot)}$. By d we denote the exterior derivative, and by $\bar\partial$ the $(0,1)$–component of d. Their maximal closed extensions will be denoted by the same symbol. The L^2 cohomology groups of X are defined by

$$H_{(2)}^{p,q}(X):=\frac{\operatorname{Ker}\bar\partial\cap L_{(2)}^{p,q}(X)}{\operatorname{Im}\bar\partial\cap L_{(2)}^{p,q}(X)}.$$

From now on, we restrict our attention to the case of holomorphic motions. To show that the middle cohomology is non-vanishing, we only need to consider the type $(1,1)$ since other cases are trivial. By the argument in section 2, we find for each point $(\lambda^*,f(\lambda^*,z^*))\in\Gamma(f|_{\Delta_r\times\Delta_1})$, an embedded polydisc Δ_a^2 on which the Bergman and the Euclidean metrics are equivalent. It is important to mention that the first coordinate λ' in the embedded polydiscs remains unchanged when their centers belong to a fixed fiber. Now fix a non-vanishing L^2 holomorphic 1-form s on the fiber at $\lambda=0$ (eg. dw). We try to extend it to a L^2 holomorphic 1-form on $\Gamma(f|_{\Delta_r\times\Delta_1})$ if $H_{(2)}^{1,1}=\{0\}$. This is proceeded as follows: take a locally finite covering $\{\Delta_{a,\alpha}^2\}$ of $\Gamma(f|_{\Delta_r\times\Delta_1})$ among these embedded polydiscs and let χ_α be a partition of unity subordinate to $\{\Delta_{a,\alpha}^2\}$, we set

$$v=\lambda'^{-1}\sum_\alpha \pi_\alpha^*(s)\bar\partial\chi_\alpha$$

where $\pi_\alpha:\Delta_{a,\alpha}^2\to\Delta_{a,\alpha}^2$ is the projection $(\lambda',w')\to\lambda'$. Since on every Δ_{a,α_0}^2, $\pi_\alpha^*(s)=s=\pi_{\alpha_0}^*(s)$ at $\lambda'=0$ and

$$\sum_\alpha \pi_\alpha^*(s)\chi_\alpha=\pi_{\alpha_0}^*+\sum_\alpha(\pi_\alpha^*(s)-\pi_{\alpha_0}^*(s))\chi_\alpha,$$

Cauchy's estimate implies

$$v\in\operatorname{Ker}\bar\partial\cap L_{(2)}^{1,1}.$$

Suppose $H_{(2)}^{1,1}$ vanishes, then there exists a L^2 $(1,0)$–form u such that $\bar\partial u=v$, hence

$$S:=\sum_\alpha \chi_\alpha\pi_\alpha^*(s)-\lambda' u$$

gives a L^2 holomorphic 1-form on $\Gamma(f|_{\Delta_r\times\Delta_1})$ which extends s.

Write

$$S=g_1(\lambda,w)d\lambda+g_2(\lambda,w)dw$$

for some holomorphic functions g_1, g_2 on $\Gamma(f|_{\Delta_r\times\Delta_1})$. Clearly, $g_2 \not\equiv 0$. Now choose a polydisc

$$\Delta_r \times \Delta_\tau \subset \Gamma(f|_{\Delta_r\times\Delta_1}).$$

It follows easily from Hörmander's L^2 theory that the Bergman metric of $\Gamma(f|_{\Delta_r\times\Delta_1})$ is equivalent to the Bergman metric of $\Delta_r \times \Delta_\tau$ on $\Delta_r \times \Delta_{\tau/2}$, whilst the latter is equal to

$$b_{\Delta_r}(\lambda) + b_{\Delta_\tau}(w).$$

As $g_2 dw$ is L^2 on $\Gamma(f|_{\Delta_r\times\Delta_1})$, we see that

$$\int_{\Delta_r\times\Delta_{\tau/2}} (r-|\lambda|)^{-2}|g_2(\lambda,w)|^2 < \infty$$

which implies $g_2 = 0$ on $\Delta_r \times \Delta_{\tau/2}$ hence on $\Gamma(f|_{\Delta_r\times\Delta_1})$, a contradiction.

Clearly, there is no non-vanishing holomorphic function on $\Gamma(f|_{\Delta_r\times\Delta_1})$ which is square-integrable with respect to the Bergman metric. Let

$$S = g_1(\lambda,w)d\lambda + g_2(\lambda,w)dw$$

be any L^2 holomorphic 1-form. Clearly $g_2 \equiv 0$ as the above argument shows. We must show $g_1 \equiv 0$. Note that

$$\begin{aligned} \partial\bar\partial \log K \wedge g_1(\lambda,w)d\lambda &= -\bar\partial\left[\partial \log K \wedge g_1(\lambda,w)d\lambda\right] \\ &= -d\left[\partial_w \log K \wedge g_1(\lambda,w)d\lambda\right] \end{aligned}$$

whilst the term $[\cdots]$ in the second equality is L^2 by (7). As the left side is a L^2 harmonic form, it must be vanishing by the Gaffney trick (cf. 1.1.C' in [12]), consequently $g_1 \equiv 0$. The remaining cases follow from the Serre duality.

6 Proof of Theorem 4

Let us first recall some basic facts about the Teichmüller space. Let Γ be a Fuchsian group acting freely on Δ and $M(\Gamma)$ the open unit ball in the complex Banach space of Beltrami differentials for Γ, i.e., all L^∞ functions μ on Δ_1 satisfying $\|\mu\| < 1$ and

$$(\mu\circ\gamma)\overline{\gamma'}/\gamma' = \mu \quad \text{for all } \gamma \in \Gamma. \tag{8}$$

For each $\mu \in M(\Gamma)$, there exists a unique quasiconformal map w^μ of the plane onto itself that satisfies the Beltrami equation $w_{\bar z} = \mu w_z$ in Δ_1, is conformal on $\mathbf{C} - \overline{\Delta}_1$ and satisfies

$$w^\mu(z) = z + O(|z|^{-1}), \quad \text{as } z \to \infty. \tag{9}$$

We say that μ, $\nu \in M(\Gamma)$ are equivalent if $w^\mu(z) = w^\nu(z)$ when $|z| = 1$. The Teichmüller space $T(\Gamma)$ is the set of equivalent classes in $M(\Gamma)$. Let $\Phi(\mu)$ denote the equivalence class of $\mu \in M(\Gamma)$. A fundamental fact in the Teichmüller theory is that

$T(\Gamma)$ has a unique complex structure so that the map $\Phi : M(\Gamma) \to T(\Gamma)$ is a holomorphic submersion. We define the Bers fiber space $F(\Gamma)$ by

$$F(\Gamma) = \{(\Phi(\mu), z) \in T(\Gamma) \times \mathbf{C} : \mu \in M(\Gamma), \quad z \in w^{\mu}(\Delta_1)\}.$$

Clearly, it is a complex manifold.

In [11], Griffiths constructed the Zariski neighborhood U of a given point in a projective manifold V by induction on the dimension n of V, precisely, U can be realized as a holomorphic family of Riemann surfaces C_s with genus g and m punctures over a quasi-projective manifold S such that

a) $3g - 3 + m > 0$;

b) the universal covering $\tilde{S}$ of S is biholomorphically equivalent to a bounded domain of holomorphy in $\mathbf{C}^{n-1}$.

The case $n = 1$ is just the uniformization of Riemann surfaces. The step from $n-1$ to n uses the Bers simultaneous unformization as follows. Note that $\pi_S : U \to S$ lifts to a holomorphic family of algebraic curves, say $U_{\tilde{S}}$, over $\tilde{S}$ with fibers $C_{\tilde{s}} = \pi_{\tilde{S}}^{-1}(\tilde{s})$. Fixing $\tilde{s}_0 \in \tilde{S}$, one chooses a Fuchsian group Γ such that

$$C_{\tilde{s}_0} = \Delta_1/\Gamma.$$

For every $\tilde{s} \in \tilde{S}$, there exists a quasiconformal map $f_{\tilde{s}} : C_{\tilde{s}_0} \to C_{\tilde{s}}$ which depends holomorphically on $\tilde{s}$. Every $f_{\tilde{s}}$ lifts to a quasiconformal map $w^{\tilde{s}} : \Delta_1 \to \Delta_1$ fixing $1, -1, i$ with complex dilatation

$$\mu^{\tilde{s}}(z) = w^{\tilde{s}}_{\bar{z}}(z)/w^{\tilde{s}}_{z}(z)$$

satisfying (7), and $w^{\tilde{s}}$ can be extended to a quasiconformal map of the plane to itself so that it is conformal outside the unit disc such that (8) holds, furthermore, it depends holomorphically on $\tilde{s}$. Thus there is a holomorphic map $\Psi : \tilde{S} \to T(\Gamma)$ where

$$\tilde{s} \to \Phi(\mu^{\tilde{s}})$$

such that the pull-back of $F(\Gamma)$ by Ψ, say $\tilde{U}_{\tilde{S}}$, is a bounded domain of holomorphy in $\mathbf{C}^n$. Since $\tilde{S}$ is simply-connected, the fundamental groups of $C_{\tilde{s}_0}$ and the fibration $U_{\tilde{S}}$ are isometric, the universal covering $\tilde{U}$ of U is biholomorphically equivalent to $\tilde{U}_{\tilde{S}}$.

To get our assertion, we shall assume the following more

For every point $\tilde{s}_0 \in \tilde{S}$, there is a holomorphic embedding Θ of $\tilde{S}$ into Δ_1^{n-1} such that $\Theta(\tilde{s}_0) = 0$ and the image of $\tilde{S}$ contains a polydisc Δ_a^{n-1} with $a < 1$ a uniform constant.

Again the case $n = 1$ is trivial. Observe that $\tilde{U}$ is biholomorphic to a holomorphic family of Jordan domains over $\Theta(\tilde{S})$ such that its restriction to the polydisc Δ_a^{n-1}

defines a holomorphic motion of the unit disc such that the dilatations of $w^{\bar{s}}$ are bounded by a uniform constant on Δ_a^{n-1}. Since all $w^{\bar{s}}$ are conformal outside Δ_1 (i.e., the dilatations are 1) and satisfy (8), it follows form Proposition 1 and the arguments in section 3 that for every point $q \in \tilde{U}$, there is a holomorphic embedding $\tilde{\Theta}$ of $\tilde{U}$ into Δ_1^n such that $\tilde{\Theta}(q) = 0$ and $\tilde{\Theta}(\tilde{U})$ contains a polydisc $\Delta_{a'}^n$ for some uniform constant $a' < 1$, completing the induction step from $n-1$ to n. A similar argument as in section 3 implies (i) and that the Bergman metric has bounded geometry. As the Bergman metric on $\tilde{U}$ descends to a complete Kähler metric on U which is equivalent to the Kobayashi metric, hence has finite volume since U is quasi-projective.

7 Proof of Theorem 5

Let $X \subset Y$ be two complex manifolds. Let $\overline{X}$ denote the closure of X in Y. X is called to be hyperbolically embedded in Y if for any two points $x, y \in \overline{X}$, there exists an open neighborhood U of x, V of y such that

$$d_X(U \cap X, V \cap Y) > 0$$

where d_X denotes the Kobayashi distance on X.

Suppose now M is a projective manifold. By [11], one can choose and ample divisor D such that

a) D has only simple normal crossings;
b) $K_M + D$ is ample where K_M denotes the canonical line bundle of M;
c) The universal covering of $M - D$ is biholomorphically equivalent to a bounded domain of holomorphy as in Theorem 4.

Let $D = \sum_{j=1}^{l} D_j$ be the decomposition into irreducible components and σ_j a holomorphic section of $[D_j]$ defining D_j. It follows from b) that there exists a volume form Ω on M such that

$$-\mathrm{Ric}\,\Omega - \sum_{j=1}^{l} \partial\bar{\partial} \log \|\sigma_j\|^2$$

is positive definite on M. Set

$$\Psi = \Omega / \prod_{j=1}^{l} \|\sigma_j\|^2 (\log \|\sigma_j\|)^2.$$

After taking a suitable constant multiple of $\|\cdot\|$, we may assume $-\mathrm{Ric}\,\Psi$ dominates some fixed Kähler metric ω on M. R. Kobayashi [14] showed that there exists a complete Kähler-Einstein metric on $M - D$ which is equivalent to $-\mathrm{Ric}\,\Psi$. By c) and Theorem 4, $M - D$ is complete hyperbolic and the Kobayashi metric on $M - D$ is equivalent to the Kähler-Einstein metric, hence dominates ω, which implies that $M - D$ can be hyperbolically embedded into M.

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