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Weak Solutions to the Navier-Stokes-Poisson Equations

Dedicated to Professor Tai-Ping Liu on his 60th birthday

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We study the global existence of weak solutions to the Navier-Stokes-Poisson equation

$$\begin{aligned} &\rho_t + \nabla\cdot(\rho u) = 0 \\ &(\rho u)_t + \nabla\cdot(\rho u\otimes u) + \rho\nabla\Phi + a\nabla\rho^\gamma = \mu\Delta u + (\lambda+\mu)\nabla(\nabla\cdot u) \\ &\Delta\Phi = 4\pi g\left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\right) \qquad \text{in } \Omega\times(0,T) \end{aligned} \tag{1}$$

with the initial-boundary condition

$$\begin{aligned} &u=0, \quad \frac{\partial\Phi}{\partial\nu}=0 \qquad \text{on } \partial\Omega\times(0,T) \\ &\rho|_{t=0}=\rho_0(x), \quad (\rho u)|_{t=0}=q_0(x) \qquad \text{in } \Omega, \end{aligned} \tag{2}$$

where $\Omega\subset\mathbf{R}^3$ is a bounded domain with $C^{2,\theta}$ boundary $\partial\Omega$ $(0<\theta<1)$, ν the outer normal vector, $\rho=\rho(x,t)$ the density,

$$u=u(x,t)=(u^1(x,t),u^2(x,t),u^3(x,t))$$

the velocity, $\Phi=\Phi(x,t)$ the Newtonian gravitational potential, $\gamma>1$ the adiabatic constant, $\mu>0$ and λ the viscosity constants satisfying $\lambda+\frac{2}{3}\mu\geq 0$, $a=e^S$ the constant determined by the entropy S, and $g>0$ the gravitational constant. Physically, this system describes the motion of compressible viscous isentropic gas flow under the self-gravitational force. Such a fluid may be formulated as the Euler-Poisson equation, and several mathematical studies have been done for this Euler-Poisson equation [16, 17, 2, 10], where the viscosity is neglected, the equation is considered in the whole space $\mathbf{R}^3$, and the solution admits to have the compact support. Then, the contact angle between fluid and vaccume is not zero in the equilibrium state, and establishing the existence of the solution in an appropriate function space including the equilibrium state is not easy because of this, even locally in time with spherically symmetry in space.

In a series of papers [20, 18, 19], T. Makino and his co-workers studied such a problem of vaccume for spherically symmetric Navier-Stokes-Poisson equation (of a specified pair of the viscosity constants) with solid core. Here, we study the Navier-Stokes-Poisson equation on the fixed domain Ω without radial symmetry or solid core, and show the existence of the weak solution in a resonable function space including the equilibrium state, emphasizing that the vaccume region $\left\{x \in \overline{\Omega} \mid \rho(x,t) = 0\right\}$ can exist inside this domain Ω although the equilibrium state is everywhere positive in this problem.

Similarly to the above mentioned equations, equation (1) is provided with the properties of the conservation of total mass $M = \int_\Omega \rho$ and the decrease of total energy E;

$$
\begin{aligned}
E &= \int_\Omega \left(\frac{\rho}{2} |u|^2 + \frac{P}{\gamma - 1} \right) + \frac{g}{2} \int\int_{\Omega\times\Omega} G(x,y)\rho(x)\rho(y)dxdy \\
&= \frac{a}{\gamma-1} \|\rho\|_\gamma^\gamma + \frac{1}{2} \|\sqrt{\rho}u\|_2^2 - \frac{1}{8\pi g} \|\nabla\Phi\|_2^2,
\end{aligned}
$$

and here, $P = a\rho^\gamma$ and $G = G(x,y)$ denote the pressure and the Green's function of the Poisson part, respectively, so that $\Phi(x) = g\int_\Omega G(x,y)\rho(y)dy$ if and only if

$$
\Delta\Phi = 4\pi g \left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho \right) \quad \text{in } \Omega, \qquad \frac{\partial\Phi}{\partial\nu} = 0 \quad \text{in } \partial\Omega, \qquad \int_\Omega \Phi = 0. \tag{3}
$$

In this Poisson equation, $\rho \in L^\gamma(\Omega)$ implies $\Phi_{x_i} \in L^2(\Omega)$ $(i = 1,2,3)$ for $\gamma \geq \frac{6}{5}$. More precisely,

$$
\|\nabla\Phi\|_2 \leq gK \|\rho\|_{\frac{6}{5}} \tag{4}
$$

by $L^{\frac{6}{5}}$ elliptic estimate and Sobolev's inequality, where K is a constant determined by Ω. In accordance with the energy E given above, therefore, $\gamma > \frac{6}{5}$

and $\gamma = \frac{6}{5}$ are the subcritical and critical exponents of the equilibrium, respectively. In more detail, the equilibrium state is realized by $u = 0$, and hence it holds that (3) and

$$\Phi + \frac{a\gamma}{\gamma - 1}\rho^{\gamma-1} = \text{constant} \qquad \text{in } \Omega.$$

Then, this problem has a variational solution in the case of $\gamma > \frac{6}{5}$, while it does not admit a solution if $1 < \gamma < \frac{6}{5}$ and $\Omega \subset \mathbf{R}^3$ is star-shaped [21].

Our result on the non-equilibrium state, on the other hand, is regarded as the generalization of Feireisl, Nevotný, and Petzeltová [12] concerning the Navier-Stokes equation without the Poisson term ; more precisely,

Theorem 1 *Let $T > 0$ and $\gamma > \frac{3}{2}$. Then, given $\rho_0 \in L^\gamma(\Omega)$ and $|q_0^i|^2/\rho_0 \in L^1(\Omega)$ with $\rho_0 = \rho_0(x) \geq 0$ and $q_0^i(x) = 0$ for x of $\rho_0(x) = 0$, we have a finite energy weak solution ρ, u, Φ to (1) satisfying the following.*

1. *$\rho = \rho(x,t) \geq 0$, $\rho \in L^\infty(0,T; L^\gamma(\Omega))$, $u^i \in L^2(0,T; H_0^1(\Omega))$.*
2. *$E = E(t) \in L^1_{loc}(0,T)$.*
3. *$\frac{dE}{dt} + \mu \|\nabla u\|_2^2 + (\lambda + \mu)\|\nabla \cdot u\|_2^2 \leq 0$ in $\mathcal{D}'(0,T)$.*
4. *The first two equations of (1) hold in $\mathcal{D}'(\Omega \times (0,T))$.*
5. *$\Phi(\cdot,t) = g\int_\Omega G(\cdot,y)\rho(y,t)dy$ for a.e. $t \in (0,T)$.*
6. *The first equation of (1) holds in $\mathcal{D}'(\mathbf{R}^3 \times (0,T))$ if the zero extension is taken outside Ω to ρ, u.*
7. *The first equation of (1) is satisfied in the sense of the renormalized solution, i.e.,*

$$\frac{d}{dt}b(\rho) + \nabla \cdot (b(\rho)u) + (b'(\rho)\rho - b(\rho))\nabla \cdot u = 0 \tag{5}$$

in $\mathcal{D}'(\Omega \times (0,T))$ for any $b \in C^1(\mathbf{R})$ such that $b'(z) = 0$ if $|z|$ is large.

Similarly to the Navier-Stokes equation without the Poisson term, any finite energy weak solution satisfies

$$\begin{aligned} &\rho \in C([0,T]; L^{\gamma}_{weak}(\Omega)) \\ &\rho u^i \in C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)) \end{aligned} \tag{6}$$

by the first two equations of (1) and consequently, the initial conditions make sense in the above weak solution.

In fact, we have $\rho u^i \in L^2(0,T; L^p(\Omega))$ for $\frac{1}{p} = \frac{5}{6} - \frac{1}{\gamma}$ by Sobolev's inequality, and therefore,

$$\left| \frac{d}{dt} \int_\Omega \rho\phi \right| = \left| \int_\Omega \rho u \cdot \nabla\phi \right| \leq C \left\| \nabla\phi \right\|_{p'}$$

for $\phi \in C_0^\infty(\Omega)$, where $\frac{1}{p'} + \frac{1}{p} = 1$. This implies the continuity of $t \in [0,T] \mapsto \int_\Omega \rho\phi$ for $\phi \in L^{\gamma'}(\Omega)$ by $\rho \in L^\infty(0,T; L^\gamma(\Omega))$, where $\frac{1}{\gamma'} + \frac{1}{\gamma} = 1$, and hence $\rho \in C([0,T]; L^\gamma_{weak}(\Omega))$. The second relation of (6) is shown similarly, because the energy inequality guarantees $\sqrt{\rho}u \in L^\infty(0,T; L^2(\Omega))$ by (4) and $\gamma \geq \frac{6}{5}$, and therefore, $\rho u \in L^\infty(0,T; L^{\frac{2\gamma}{\gamma+1}}(\Omega))$ follows from $\rho \in L^\infty(0,T; L^\gamma(\Omega))$.

B. Ducomet, E. Feireisl, H. Petzeltová, and I. Straskrabá [7] prove the above Theorem 1 in the case that the solution Φ of the Poisson equation $\Delta\Phi = 4\pi g\rho$ is expressed by the corresponding Green function provided ρ is extended to be zero outside Ω.

We shall follow the scheme [12, 7] to construct our weak solution. In more detail, taking large $\beta > 0$, we construct an approximate solution $u = u_{\delta,\varepsilon}(x,t)$ to

$$\begin{aligned} &\rho_t + \nabla \cdot (\rho u) = \varepsilon\Delta\rho \\ &(\rho u^i)_t + \nabla \cdot (\rho u^i u) + a(\rho^\gamma)_{x_i} + \rho\Phi_{x_i} + \delta(\rho^\beta)_{x_i} + \varepsilon\nabla u^i \cdot \nabla\rho \\ &= \mu\Delta u^i + (\lambda+\mu)(\nabla\cdot u)_{x_i} \\ &\Delta\Phi = 4\pi g\left(\rho - \frac{1}{|\Omega|}\int_\Omega \rho\right) \qquad \text{in } \Omega \times (0,T) \end{aligned} \tag{7}$$

with (2) by the Faedo-Galerkin method. Then, we obtain the actual solution, by vanishing the artificial viscosity ε and then the artificial pressure δ. The condition $\gamma > \frac{3}{2}$ is necessary to take the limit of the Faedo-Galerkin approximation and construct the solution to (7), using the div-curl lemma. This restriction is the same as that of [12], because the total mass conservation $\|\rho(\cdot,t)\|_1 = M$ guarantees

$$\|\rho\|_{\frac{6}{5}} \le M^{1-\theta} \|\rho\|_\gamma^\theta$$

for $\frac{1-\theta}{1} + \frac{\theta}{\gamma} = \frac{5}{6}$, and therefore, if $\gamma > \frac{4}{3}$, then $2\theta < \gamma$ and the contribution of the Poisson term $-\frac{1}{8\pi g} \|\nabla\Phi\|_2^2$ of the total energy E is absorbed into that of the pressure term $\int_\Omega P$ by (4). We note that this condition $\gamma > \frac{4}{3}$ is actually weaker than $\gamma > \frac{3}{2}$.

Also, we use the following lemma on L^1 convergence used in taking limits of the approximate solution.

Lemma 0.1 *If $\Phi : \mathbf{R} \to (-\infty, +\infty]$ is a proper, lower semi-continuous, convex function, $D \subset \mathbf{R}^m$ is a domain with bounded measure, and*

$$\sup_k \|v_k\|_p < +\infty \tag{8}$$

$$v_k \to v \qquad \textit{weakly in } L^1(D) \tag{9}$$

$$\Phi(v_k) \to \overline{\Phi(v)} \qquad \textit{weakly in } L^1(D) \tag{10}$$

$$\int_D \Phi(v) = \int_D \overline{\Phi(v)} \tag{11}$$

with $p > 1$, then it holds that

$$v_k \to v \qquad \textit{strongly in } L^1(D). \tag{12}$$

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