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On classification of horizontal loops in the standard Engel space

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1 Introduction

Submanifolds in manifolds with some geometric structures are interesting objects for geometry. For example, Legendrian submanifolds in contact manifolds and Lagrangian submanifolds in symplectic manifolds. In this paper, we study submanifolds in Engel manifolds. We observe properties of loops, namely embedded circles, which are always tangent to the standard Engel structure in the standard Engel 4-space. We obtain the classification of such loops up to horizontal homotopy.

Engel structures are interesting object for differential topology. An *Engel structure* is a distribution of rank 2 on a 4-dimensional manifold which is maximally non-integrable (see Section 2 for precise definition). A distribution is a subbundle of the underlying manifold. Engel structures have an important property like contact structures. All Engel structures are locally equivalent. Therefore, global study is important for Engel structures ([M], [Aj1],..., etc). Recently, sufficient condition for the existence of an Engel structure is obtained by Vogel [V]: There exists an

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Engel structure on a 4-dimensional manifold if and only if the manifold is parallelizable. Then, Engel manifolds must be going to be studied as a object for global differential topology. An embedded circle in the standard Engel space which is everywhere tangent to the Engel structure is studied in this note. We call such a circle a *horizontal loop*. Horizontal loops might be an interesting issue to Engel topology like Legendrian knots to contact topology. A contact structure on a 3-dimensional manifold is a distribution of rank 2 which is completely non-integrable. A Legendrian knot is an embedded circle into a 3-dimensional contact manifold which is everywhere tangent to the contact structure. Legendrian knots are important for contact topology. They take important roles in constructions and classifications of contact manifolds. Therefore, horizontal loops should be good tools for constructions and classifications of Engel structures. Engel structures and contact structures on 3-dimensional manifolds are so closely related that mutual contributions between Engel topology and 3-dimensional contact topology are expected. By reducing horizontal loops in the standard Engel space to Legendrian knots, we obtain the classification of horizontal loops.

2 Engel structures and horizontal curves

2.1 Basic definitions

An Engel structure is a maximally non-integrable distribution of rank two on a 4-dimensional manifold. Generally, it is defined as follows. Let M be a 4-dimensional manifold, and D a distribution, or a subbundle of the tangent bundle TM, of rank 2. We can regard D as a locally free sheaf of vector fields on M. Let $[X, Y]$ denote a sheaf of vector fields generated by all Lie brackets $[X, Y]$ of vector fields X, Y which are cross-sections of D. Set $D^2 := D + [D, D]$, and $D^3 := D^2 + [D^2, D^2]$. Then, an *Engel structure* on M is defined as a distribution $D \subset TM$ of rank 2 which satisfies the following conditions:

$$\operatorname{rank} D_p^2 = 3, \qquad \operatorname{rank} D_p^3 = 4 \tag{2.1}$$

at any point $p \in M$.

A certain Engel manifold is constructed from a 3-dimensional contact manifold. A contact structure is a completely non-integrable distribution of corank one on an odd-dimensional manifold. Let E be a contact structure on a 3-dimensional manifold N. By taking fibrewise porjectivization of the contact structure E, we obtain a new 4-dimensional manifold $\mathbb{P}E = \cup_{x \in N} \mathbb{P}(E_x)$. On the 4-dimensional manifold $\mathbb{P}E$, an Engel structure $D(E)$ is defined as $D(E)_q := (d\pi)^{-1} l$, where $\pi \colon \mathbb{P}E \to M$ is a canonical projection, $q = (p, l) \in \mathbb{P}E$ is a point, and $l \in T_p M$ is a line (see [M]). Such a procedure is called a *Cartan prolongation* (see [BCG3], [M], [Aj1]).

An Engel structure has a characteristic direction. Let D be an Engel structure on a 4-dimensional manifold M. From this Engel structure D, a line field is defined as follows: $L(D) := \{X \in D \mid [X, D^2] \subset D^2\}$. The line field $L(D)$ is called the *Engel line field*. It is known that a contact structure is induced from an even-contact structure D^2 on an embedded manifold $N \subset M$ which is transverse to the Engel line field $L(D)$. The contact structure is obtained as $D^2 \cap TN$. Such a procedure is called a *deprolongation* (see [M], [BCG3]).

In this paper, we work just in the standard Engel space $(\mathbb{R}^4, E)$, that is, an ordinary 4-dimensional space $\mathbb{R}^4$ endowed with the standard Engel structure. The *standard Engel structure* on $\mathbb{R}^4$ is defined as a kernel of the following pair ω_1, ω_2 of 1-forms:

$$\omega_1 = dy - z dx, \qquad \omega_2 = dz - w dx, \tag{2.2}$$

where $(x, y, z, w) \in \mathbb{R}^4$ are coordinates. Let E denotes the standard Engel structure on $\mathbb{R}^4$:

$$E := \{\omega_1 = 0,\ \omega_2 = 0\} = \operatorname{Span}\left\{ \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}, \frac{\partial}{\partial w} \right\}.$$

We call the 4-dimensional space $(\mathbb{R}^4, E)$ endowed with the standard Engel structure the *standard Engel space.* It is clear that the standard Engel structure E actually satisfies the condition (2.1) of the definition. In this case, the Engel line fields is $L(E) = \mathrm{Span}\{\partial/\partial w\}$. With respect to the standard Engel structure on $\mathbb{R}^4$, the induced contact structure on $\mathbb{R}^3 \subset \mathbb{R}^4$, the (x, y, z)-space, is $C = \{\omega_1 = dy - zdx = 0\}$. It is also called the *standard contact structure* on $\mathbb{R}^3$.

A *horizontal curve* $\Gamma \subset M$ in a Engel manifold (M, D) is a curve which is tangent to the Engel structure D everywhere: $T_p\Gamma \subset D_p$ at any $p \in M$. Horizontal loops in the standard Engel space $(\mathbb{R}^4, E)$ are dealt with in this paper. A *horizontal loop* in $(\mathbb{R}^4, E)$ is an embedding $\gamma\colon S^1 \to (\mathbb{R}^4, E)$ of an oriented circle S^1 into $(\mathbb{R}^4, E)$ which satisfies the condition $\gamma_*(T_pS^1) \subset (E)_{\gamma(p)}$. In other words, γ satisfies the conditions $\gamma^*\omega_1 = 0$ and $\gamma^*\omega_2 = 0$, where ω_1, ω_2 are 1-forms defined as equations (2.2). For horizontal loops, we introduce an equivalence relation. Two horizontal loops γ_0, $\gamma_1\colon S^1 \to (\mathbb{R}^4, E)$ are said to be *horizontally homotopic* if there exists a smooth mapping $H\colon S^1 \times I \to (\mathbb{R}^4, E)$ which satisfies that, setting $H_t(s) := H(s,t)$, $H_0 = \gamma_0$ and $H_1 := \gamma_1$, and that $H_t\colon S^1 \to (\mathbb{R}^4, E)$ is a horizontal loop for any $t \in I = [0, 1]$.

Similarly to horizontal curves in Engel manifolds, a curve in a contact 3-manifold which is everywhere tangent to the contact structure is called a *Legendrian curve.* Studying Legendrian knots is one of important issues in contact topology (see [B], [El], [EF], [Et] for example).

2.2 Horizontal projections and rotation number

The horizontal projection is defined as follows. Let (x, y, z, w) be coordinates of the standard Engel space $(\mathbb{R}^4, E)$. Set $P_3 := \{y = 0\} \cong \mathbb{R}^3$, and $P_2 := \{y = 0,\ z = 0\} \cong \mathbb{R}^2$. Let $p_1\colon \mathbb{R}^4 \to P_3$, $(x, y, z, w) \mapsto (x, z, w)$, and $p_2\colon P_3 \to P_2$, $(x, z, w) \mapsto (x, w)$ be the canonical projections. We call them the first and the second *horizontal projections.*

We define an invariant, the rotation number, of a horizontal loop in the standard Engel space. It is similar to the rotation number for Legendrian knots (see [B]). Let $\gamma\colon S^1 \to (\mathbb{R}^4, E)$ be a horizontal loop. The projected curve $p_2 \circ p_1 \circ \gamma\colon S^1 \to P_2 \cong \mathbb{R}^2$ is an immersed plane curve because the standard Engel structure is transverse to y and z-axes. Then, we can calculate the degree of the immersed oriented curve $p_2 \circ p_1 \circ \gamma\colon S^1 \to P_2 \cong \mathbb{R}^2$ with respect to x and w-axes, in other words, with respect to a trivialization of the Engel structure E. We call it the *rotation number* of γ. Let $r(\gamma)$ denote it. This definition is independent of the choice of the trivialization by a similar reason to the case of Legendrian knots (see [B]). It is because $H_1(\gamma)$ is null in $H_2(\mathbb{R}^4)$. Then, the rotation number is invariant under diffeomorphisms preserving Engel structure, and horizontal homotopies of horizontal loops.

We can lift a certain immersed closed plane curve to a horizontal loop in the standard Engel space. We call this procedure a *horizontal lift*. Let $g\colon S^1 \to P_2 \cong \mathbb{R}^2$, $g(s) = (g_1(s), g_4(s))$, be an immersed closed curve in the (x, w)-plane. Suppose that the algebraic area bounded by the immersed curve is zero:

$$\int_{g(S^1)} g_4(s)dg_1(s) = 0. \tag{2.3}$$

The condition guarantees that the lifted curve is closed. We remark that horizontal projections of horizontal loops in $(\mathbb{R}^4, E)$ and horizontal projections of Legendrian closed curves satisfy this condition. Then, the given immersed plane curve $g(s) = (g_1(s), g_4(s))$ satisfying the condition (2.3) above is lifted to a Legendrian immersed circle with singular points in the (x, z, w)-space $P_3 \cong \mathbb{R}^3$ with the standard contact structure $C' = \{\omega_2 = dz - wdx = 0\}$. Further, if the obtained Legendrian circle satisfies a similar condition to equation (2.3), then it can be lifted to a horizontal loop in the standard Engel space.

Example 2.1. If an immersed curve as the top of Figure 1 is given, we can lift the curve from the top to the bottom of Figure 1.

We use this horizontal lift in an argument in Section 6.

3 Classification result

The following is one of main results in this paper.

Theorem A. *Let $\gamma_0, \gamma_1 \colon S^1 \to (\mathbb{R}^4, E)$ be horizontal loops in the standard Engel space. These loops γ_0 and γ_1 are horizontally homotopic if and only if their rotation numbers coincide*: $r(\gamma_0) = r(\gamma_1)$

According to this Theorem, we can classify horizontal loops as the following Remark B in Section 4. We claim that there actually is a horizontal loop with rotation number k for any integer $k \in \mathbb{Z}$.

4 Vertical projections

The vertical projection is defined as follows. Let (x, y, z, w) be coordinates of the standard Engel space $(\mathbb{R}^4, E)$ as above. Set $Q_3 := \{w = 0\} \cong \mathbb{R}^3$, and $Q_2 := \{w = 0,\ z = 0\} \cong \mathbb{R}^2$. Let $\pi_1 \colon \mathbb{R}^4 \to Q_3$, $(x, y, z, w) \mapsto (x, y, z)$, and $\pi_2 \colon Q_3 \to Q_2$, $(x, y, z) \mapsto (x, y)$ be the canonical projections. We call them the first and the second *vertical projections*.

We should remark that the first projection is along the direction of an Engel line field, and the second one is along the Legendrian fiber with respect to the induced standard contact structure on $Q_3 \cong \mathbb{R}^3$ (see Subsection 2.1). Let $\gamma \colon S^1 \to (\mathbb{R}^4, E)$ be a horizontal lop in the standard Engel space, and $C = \{\omega_1 = dy - z dx = 0\}$ an induced standard contact structure on $Q_3 \cong \mathbb{R}^3$. Then, $\pi_1 \circ \gamma \colon S^1 \to (\mathbb{R}^3, E)$ is a Legendrian immersion with singularities. At a point where γ is tangent to w-direction, the projection $\pi_1 \circ \gamma$ has a singular point. Such Legendrian curves with singularities are studied by Ishikawa [I] and Zhitomirskiĭ [Zh]. We call the image of further projection $\pi_2 \circ \pi_1 \circ \gamma \colon S^1 \to Q_2 \cong \mathbb{R}^2$ a *front* of γ. It is a wave front of a Legendrian knot $\pi_1 \circ \gamma$. The mapping $\pi_2 \circ \pi_1 \circ \gamma$ is an immersion with singular points. The singular points are cusps of type $(2, 5)$, that is, points where $\pi_2 \circ \pi_1 \circ \gamma$ is locally diffeomorphic to a curve $t \mapsto (t^2, t^5)$ at $t = 0$.

Contrary to the vertical projection, we can reconstruct a horizontal curve in the standard Engel space from a certain plane curve. We call this procedure a *vertical lift.* It is based on the prolongation procedure explained in Subsection 2.1. Let $f\colon S^1 \to \mathbb{R}^2 \cong Q_2$, $f(s) = (f_1(s), f_2(s))$, be an immersion with cusps of type $(2,5)$. Assume that the plane curve f is *nonvertical,* in other words, it satisfies $df_1(s)/ds \neq 0$ at regular points and $d^2 f_1(s)/ds^2 \neq 0$ at cusp points. We remark that the vertical projection of a horizontal loop is actually nonvertical. Then, we can lift the given plane curve $f(s) = (f_1(s), f_2(s))$ to a Legendrian curve $\bar{f}\colon S^1 \to \mathbb{R}^3 \cong Q_3$, $\bar{f}(s) = (f_1(s), f_2(s), f_3(s))$, in $\mathbb{R}^3$ with the standard contact structure $C = \{\omega_1 = dy - zdx = 0\}$ whose vertical projection is the given $f(t)$. Its z-coordinate $f_3(s)$ is obtained as a slope of the curve $f(s)$. Precisely, it is written down as follows:

$$f_3(s) = \begin{cases} \dfrac{df_2}{dt}(s) \Big/ \dfrac{df_1}{dt}(s) & \text{at regular points,} \\ \dfrac{d^2 f_2}{dt^2}(s) \Big/ \dfrac{d^2 f_1}{dt^2}(s) & \text{at cusp points.} \end{cases} \tag{4.1}$$

It is clear, from the definition of the standard contact structure, that the obtained curve $\bar{f}(s)$ is Legendrian. Furthermore, we can lift the obtained Legendrian curve $\bar{f}(s) = (f_1(s), f_2(s), f_3(s))$ in $(\mathbb{R}^3 \cong Q_3, C)$ to a horizontal loop $\tilde{f}(s) = (f_1(s), f_2(s), f_3(s), f_4(s))$ in the standard Engel space $(\mathbb{R}^4, E)$ whose vertical projection is the given plane curve $f(s)$. Its w-coordinate is obtained similarly to above. It is obtained as a slope of a plane curve induced from $\bar{f}(s)$ by the projection to the (x,z)-plane. Note that the projected plane curve $\mathrm{pr} \circ \bar{f}\colon S^1 \to \mathbb{R}^2$ is a nonvertical immersion with cusps of type $(2,3)$, namely locally equivalent to a curve $t \mapsto (t^2, t^3)$. It is because of the assumption that $f(s)$ is an immersion cusps of type $(2,5)$ (see Example 4.1 bellow). Then, the w-coordinate of

$\tilde{f}(s)$ is obtained as follows:

$$f_4(s) = \begin{cases} \dfrac{df_3}{dt}(s) \Big/ \dfrac{df_1}{dt}(s) & \text{at regular points,} \\ \dfrac{d^2 f_3}{dt^2}(s) \Big/ \dfrac{d^2 f_1}{dt^2}(s) & \text{at cusp points.} \end{cases} \tag{4.2}$$

It is clear that the obtained curve $\tilde{f}(s)$ is horizontal with respect to the standard Engel structure $E = \{dy - zdx = 0,\ dz - wdx = 0\}$.

Example 4.1. We observe these lifts at a cusp point. Suppose that a non-vertical cusp of type $(2,5)$ on the (x,y)-plane $\mathbb{R}^2 \cong Q_2$ is parameterized as $g(s) = \left(t^2, t^5\right)$. Then, it is lifted to a space curve $\bar{g}(s) := \left(t^2, t^5, 5t^3/2\right)$ in $\mathbb{R}^3 \cong Q_3$ which is Legendrian for the standard contact structure $C = \{dy - zdx = 0\}$. The projection $\mathrm{pr} \circ \bar{g}(s) = \left(t^2, 5t^3/2\right)$ to the (x,z)-plane has a cusp of type $(2,3)$ at $t = 0$, where $\mathrm{pr}\colon (x,y,z) \mapsto (x,z)$ is a canonical projection. Then, $\bar{g}(s)$ is lift to a horizontal smooth curve $\tilde{g}(s) := \left(t^2, t^5, 5t^3/2, 15t/4\right)$ in the standard Engel space $\left(\mathbb{R}^4, E\right)$.

Example 4.2. As a global circle, an immersed circle on (x,y)-plane with cusps of type $(2,5)$ is lifted as in Figure 1.

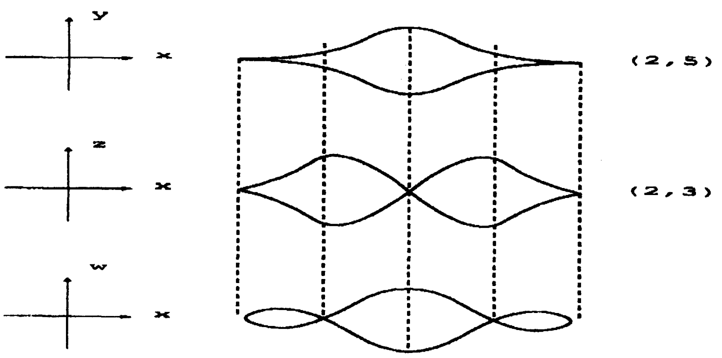


Figure 1: projections and lifts.

The rotation number of this curve is 1. We can count by using the picture in Figure 1 on the (x,w)-plane.

By using vertical lifts, examples of horizontal loops which have all integer as their rotation numbers.

Remark B. For any integer $n \in \mathbb{Z}$, there is a horizontal loops γ whose rotation number is the given integer n: $r(\gamma) = n$. In fact, we have the following catalogue of fronts of horizontal loops (see Figure 2). By lifting an F_k-type front vertically, we obtain a horizontal loop $\bar{F}_k$ with rotation number $r\left(\bar{F}_k\right) = k$. Thus, we conclude that horizontal loops in

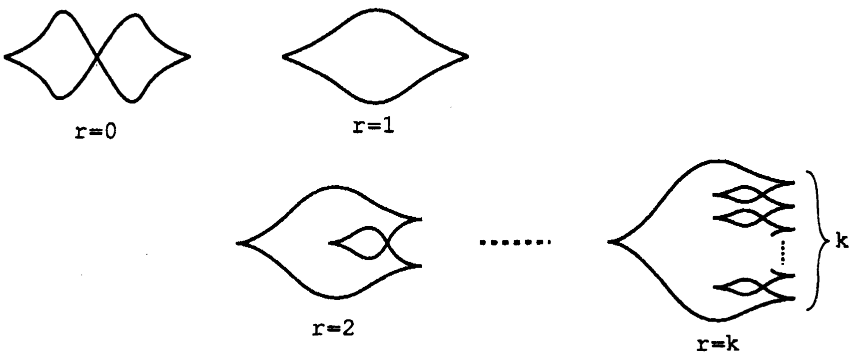


Figure 2: Catalog of fronts

the standard Engel space are classified by the rotation number, and that for any integer, there corresponds a class.

We can calculate rotation numbers of horizontal loops obtained from fronts of type F_k (see Figure 2). According to Example 4.2, a cusp of a front corresponds to an immersed curve with one twist with the reversed orientation. Thus, in order to calculate the rotation number of a horizontal curve from a front of type F_k, we have only to count the degree of an immersed curve in Figure 3. As a result, we obtain that the rotation

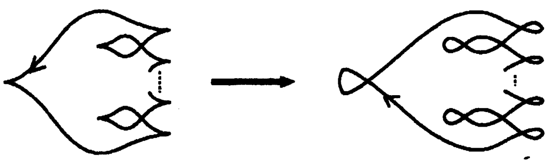


Figure 3: How to count rotation numbers

number of a horizontal loop from a front of type F_k is k.

5 Engel Reidemeister moves

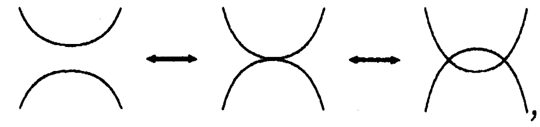
In order to prove Theorem A, we need the following result. We discuss Reidemeister moves for Engel horizontal loops. It is not only a useful tool, but an interesting result as itself.

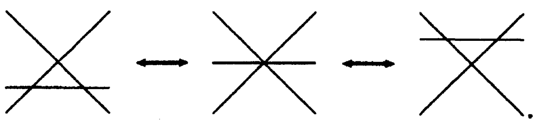
Theorem C (Engel Reidemeister moves). *Let $\gamma_0, \gamma_1 \colon S^1 \to (\mathbb{R}^4, E)$ be horizontal loops in the standard Engel space. These loops are horizontally homotopic if and only if the front $\pi_2 \circ \pi_1 \circ \gamma_0 (S^1)$ is moved to the other front $\pi_2 \circ \pi_1 \circ \gamma_1 (S^1)$ via a finite sequence of the following moves:*

0 *an isotopy of $\mathbb{R}^2$ preserving the curve non-vertical,*

I 

II-(1) 

II-(2) 

III 

Note that moves I and II-(2) never occur in Legendrian Reidemeister moves.

6 Observations

We can observe the results of this note from a view point of homotopy principle (h-principle for short). One of the most famous examples of h-principle is a Whitney's theorem on immersions of a circle into a plane ([W1], [Am]): regular homotopy classes of immersed circles are classified by their rotation numbers. The rotation number for horizontal curves is also the main tool for the classification in Theorem A in this note. And, the rotation number for horizontal curves is defined by taking a projection to a plane. Then, by taking such a projection, we obtain a version of a Whitney's theorem with some conditions for some area. In other words, we find a new relation which satisfies h-principle. On the other hand, in order to show h-principle there is a strong machinery, Gromov's method (see [G], [Am]). A relation, homotopy among Legendrian immersions, similar to the relation in this note is studied by that method in [G]. The results in this note should also be proved by that method.

The result of this note can be generalized to higher dimensional cases. The notion of Engel structure is a special case of the notion of Goursat structure. The Goursat structure on a 3-manifold is a contact structure. However, the Darboux-type theorem, or local triviality, does not hold in higher dimensional cases. We extend Theorem A to the higher dimensional cases.

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