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Stanley–Reisner rings with large multiplicities

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1. Introduction

Throughout this report, let Δ be a **simplicial complex** on the vertex set $V = [n] := \{1, 2, \ldots, n\}$, that is, $\Delta \subseteq 2^V$ such that

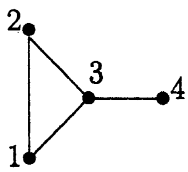
$$(a)\ \{i\} \in \Delta \text{ for all } i \in V, \qquad (b)\ F \in \Delta, G \subseteq F \Longrightarrow G \in \Delta.$$

An element of Δ is called a **face** of Δ. For a face $F \in \Delta$, the **dimension** of F is defined by $\dim F = \#(F) - 1$, where $\#(F)$ denotes the cardinality of F. A face of dimension i is called an i-face. We also define the **dimension** of Δ by $\dim \Delta = \max\{\dim F : F \in \Delta\}$. A simplicial complex Δ is **pure** if all facets (maximal faces) has the same dimension.

Let k be a field of any characteristic. Let $S = k[X_1, \ldots, X_n]$ be a polynomial ring with n variables over k. We regard the ring S as a homogeneous k-algebra with $\deg X_i = 1$. For a simplicial complex Δ, the **Stanley–Reisner ideal** I_Δ is defined by

$$I_\Delta = (X_{i_1} \cdots X_{i_p} : 1 \le i_1 < \cdots < i_p \le n,\ \{i_1, \ldots, i_p\} \notin \Delta)S.$$

The ring $k[\Delta] = S/I_\Delta$ is called the **Stanley–Reisner ring** of Δ. For example,

$\Delta =$ 

$$k[\Delta] = \frac{k[X_1, X_2, X_3, X_4]}{(X_1X_4,\ X_2X_4,\ X_1X_2X_3)}$$

The Hilbert series of $k[\Delta]$ can be written as in the following form:

$$\begin{aligned} F(k[\Delta], \lambda) &= \sum_{i \ge 0} \dim_k k[\Delta]_i\, \lambda^i \\ &= f_{-1} + \frac{f_0 \lambda}{1-\lambda} + \frac{f_1 \lambda^2}{(1-\lambda)^2} + \cdots + \frac{f_{d-1}\lambda^d}{(1-\lambda)^d} \\ &= \frac{h_0 + h_1\lambda + \cdots + h_d \lambda^d}{(1-\lambda)^d}, \end{aligned}$$

where $f_i = f_i(\Delta)$ denotes the number of i-faces of Δ and $f_{-1} = 1$. Hence $\dim k[\Delta] = d$ (the **Krull dimension**) and the **multiplicity** $e(k[\Delta])$ is equal to f_{d-1}, the number of $(d-1)$-faces in Δ. In particular, $e(k[\Delta]) \le \binom{n}{d}$.

Let $A = S/I$ be a homogeneous k-algebra with $\dim A = d$ with the unique homogeneous maximal ideal $\mathfrak{m} = (X_1, \ldots, X_n)S/I$ or a d-dimensional Noetherian local ring with the unique maximal ideal $\mathfrak{m}$. Then the ith local cohomology module $H^i_{\mathfrak{m}}(A)$ with support $V(\mathfrak{m})$ is defined by

$$H^i_{\mathfrak{m}}(A) := \varinjlim_j \operatorname{Ext}^i_A(A/\mathfrak{m}^j, A).$$

Then it is well known that $H^d_{\mathfrak{m}}(A) \ne 0$. We also define the **depth** of A by

$$\operatorname{depth} A = \min\{i \in \mathbb{Z}_{\ge 0} \,:\, H^i_{\mathfrak{m}}(A) \ne 0\}.$$

By the above remark, we always have $\operatorname{depth} A \le \dim A$. If the equality holds, then A is said to be a **Cohen–Macaulay** ring. We say that A satisfies **Serre's condition** (S_2) if $\operatorname{depth} A_P \ge \min\{2, \dim A_P\}$ for all prime ideals P in A. The Cohen–Macaulay property is very important notion in the theory of commutative algebra.

The purpose of this report is to give an answer to the following question with respect to Cohen–Macaulay property of Stanley–Reisner rings:

Question. Let Δ be a $(d-1)$-dimensional simplicial complex on $V = [n]$. If $e(k[\Delta])$ is sufficiently large (that is, $e(k[\Delta])$ is close to $\binom{n}{d}$), then is $k[\Delta]$ Cohen–Macaulay?

Now let us observe the above question in some special cases. First we consider the case where $e(k[\Delta]) = \binom{n}{d}$. Then Δ is certainly Cohen–Macaulay. Indeed, we can characterize such a complex; see below. Recall that the i-**skeleton** of Δ is defined by $\Delta^{(i)} = \{F \in \Delta \,:\, \dim F \le i\}$. It is also well known that $\Delta^{(i)}$ is Cohen–Macaulay if so is Δ.

Proposition 1.1 ([5, Proposition 1.2]). *Let Δ be a $(d-1)$-dimensional simplicial complex on V. Then the following conditions are equivalent:*

(1) $e(k[\Delta]) = \binom{n}{d}$.
(2) $\operatorname{indeg} I_\Delta := \min\{i \in \mathbb{Z} \,:\, (I_\Delta)_i \ne 0\} = d+1$.
(3) $I_\Delta = (X_{i_1} \cdots X_{i_{d+1}} \,:\, 1 \le i_1 < \cdots < i_{d+1} \le n)$. *That is, Δ is the $(d-1)$-skeleton of the standard $(n-1)$-simplex 2^V.*

When this is the case, $k[\Delta]$ is Cohen–Macaulay.

Next we consider the case of $\dim \Delta = 1$. Let Δ be a 1-dimensional simplicial complex on $V = [n]$, and put $e = e(k[\Delta])$. Then Δ can be regarded as a simple graph having n points and e edges. $k[\Delta]$ is also Cohen–Macaulay if and only if $\widetilde{H}_0(\Delta; k) = 0$, that is, Δ is connected. Thus, in this case, the above question says that

"If a graph has sufficiently many edges, then is it connected ?".

Of course, this is true! To be precise, the graph is connected whenever $e \geq \binom{n-1}{2} + 1$. Similarly, any graph without isolated points is connected whenever $e \geq \binom{n-2}{2} + 2$.

The main result in this report is the following theorem, which generalizes the above observations.

Theorem 1.2 (See [3, 6, 7]). *Let Δ be a $(d-1)$-dimensional simplicial complex on V. Suppose that one of the following conditions is satisfied:*

(1) $e(k[\Delta]) \geq \binom{n}{d} - (n-d)$;

(2) $e(k[\Delta]) \geq \binom{n}{d} - 2(n-d) + 1$ *and Δ is pure;*

(3) $e(k[\Delta]) \geq \binom{n}{d} - 3(n-d) + 2$ *and $k[\Delta]$ satisfies Serre's condition (S_2).*

Then $k[\Delta]$ is Cohen–Macaulay.

2. Sketch of the proof of the main result

In this section, we give a sketch of the proof of Theorem 1.2. We first recall some definitions and terminology which we need later. Throughout this section, let Δ be a $(d-1)$-dimensional simplicial complex on $V = [n]$, unless otherwise specified. Let $k[\Delta] = S/I_\Delta$ denote the Stanley–Reisner ring of Δ, where $S = k[X_1, \ldots, X_n]$ is a homogeneous polynomial ring over a field k, and put $c = n - d$.

For a face F of Δ and a subset $W \subseteq V$, let us define several subcomplexes of Δ as follows:

$$\begin{aligned} \Delta_W &= \{G \in \Delta \,:\, G \subseteq W\}, \\ \operatorname{link}_\Delta F &= \{G \in \Delta \,:\, F \cup G \in \Delta,\ F \cap G = \emptyset\}, \\ \operatorname{star}_\Delta F &= \{G \in \Delta \,:\, F \cup G \in \Delta\}. \end{aligned}$$

These complexes are called the **restriction** to W, the **link** of F, and the **star** of F, respectively.

Take a graded minimal free resolution of an arbitrary homogeneous ideal I $(0 \neq I \subseteq (X_1, \ldots, X_n)^2)$ over S :

$$0 \to \bigoplus_{j \in \mathbf{Z}} S(-j)^{\beta_{p,j}(I)} \xrightarrow{\varphi_p} \cdots \xrightarrow{\varphi_1} \bigoplus_{j \in \mathbf{Z}} S(-j)^{\beta_{0,j}(I)} \xrightarrow{\varphi_0} I \to 0,$$

where $p = \operatorname{pd}_S I$. In general, $n - d - 1 \leq p$, and the equality holds if and only if $A := S/I$ is Cohen–Macaulay.

Let $\mu(I)$ denote the minimal number of generators of I, that is, $\mu(I) = \sum \beta_{0,j}(I)$. Moreover,

$$\begin{aligned} \operatorname{indeg} I &= \min\{j \in \mathbb{Z} \,:\, \beta_{0,j}(I) \neq 0\} = \min\{j \in \mathbb{Z} \,:\, \beta_{1,j}(A) \neq 0\}, \\ \operatorname{rt}(I) &= \max\{j \in \mathbb{Z} \,:\, \beta_{0,j}(I) \neq 0\} = \max\{j \in \mathbb{Z} \,:\, \beta_{1,j}(A) \neq 0\}, \\ \operatorname{reg} I &= \max\{j - i \in \mathbb{Z} \,:\, \beta_{i,j}(I) \neq 0\} \end{aligned}$$

are called the **initial degree** of I, the **relation type** of I and the **regularity** of I, respectively. By definition, it is easy to see that $\operatorname{reg} I \geq \operatorname{indeg} I$. If equality holds (and $\operatorname{indeg} I = q$), then I (or A) has (q-)**linear resolution**. For a given integer $r \geq 0$, a homogeneous ideal I **satisfies** $(N_{q,r})$**-condition** if the graded minimal free resolution of I over S can be written as in the following shape:

$$\cdots \to \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{r,j}} \to S(-q-r+1)^{\beta_{r-1}} \to \cdots \to S(-q)^{\beta_0} \longrightarrow I \to 0.$$

Note that I satisfies $(N_{q,r})$ for $r > \operatorname{pd}_S I$ if and only if it has q**-linear resolution**. A homogeneous ideal I satisfies $(N_{*,r})$ if it satisfies $(N_{q,r})$ for some $q \geq 2$.

Let us recall **Hochster's formula on the Betti numbers**:

$$\beta_{i,j}(I_\Delta) = \sum_{W \subseteq V, \#(W)=j} \dim_k \widetilde{H}_{j-i-2}(\Delta_W; k),$$

where $\widetilde{H}_i(\Delta; k)$ (or simply $\widetilde{H}_i(\Delta)$) denotes the ith reduced simplicial homology group with valued in k. By this formula we have

$$\operatorname{reg} I_\Delta = \max\{r \in \mathbb{Z} : \widetilde{H}_r(\Delta_W) \neq 0 \text{ for some } W \subseteq V\} + 2.$$

In particular, $\operatorname{reg} I_\Delta \leq d+1$.

Now let us reduce Theorem 1.2 to its Alexander dual version. In the proof of Theorem 1.2, we may assume that $c \geq 2$. Moreover,

$$\underline{\text{we suppose that } \operatorname{indeg} I_\Delta = d}$$

for simplicity. Then the **Alexander dual complex** of Δ is defined by

$$\Delta^* = \{F \in 2^V : V \setminus F \notin \Delta\}.$$

This is a simplicial complex on the same vertex set V as that of Δ. For a subset $W = \{i_1, \ldots, i_p\}$ of V, if we put $P_W = (X_{i_1}, \ldots, X_{i_p})S$, then

$$I_\Delta = \bigcap_{F \text{ is a facet of } \Delta} P_{V \setminus F}$$

gives an irredundant primary decomposition of I_Δ. On the other hand, if we put $X^W = X_{i_1} \cdots X_{i_p}$, then we have

$$I_{\Delta^*} = (X^{V \setminus F} : F \text{ is a facet of } \Delta).$$

In particular,

(1) $\operatorname{indeg} I_{\Delta^*} = \operatorname{height} I_\Delta$.
(2) $\beta_{0,q^*}(I_{\Delta^*}) = e(k[\Delta])$, where $q^* = \operatorname{indeg} I_{\Delta^*}$.

Moreover, the following lemma plays a key role in our argument. The latter assertion has been proved in [2] by Eagon and Reiner, and was generalized by the first author and Yanagawa (see [8, Corollary 3.7]).

Lemma 2.1 ([2, 8]). *Let Δ^* be the Alexander dual complex of Δ. For an integer $r \geq 2$, $k[\Delta]$ satisfies (S_r) if and only if I_{Δ^*} satisfies $(N_{*,r})$. In particular, $k[\Delta]$ is Cohen–Macaulay if and only if I_{Δ^*} has linear resolution.*

Remark 2.2. When $r = 1$, Δ is pure if and only if I_{Δ^*} satisfies $(N_{*,1})$, that is, $\operatorname{indeg} I_{\Delta^*} = \operatorname{rt}(I_{\Delta^*})$.

Let Δ^* be the Alexander dual complex of Δ. Then

$$\dim k[\Delta^*] = n - \operatorname{height} k[\Delta^*] = n - \operatorname{indeg} k[\Delta] = n - d = c.$$

Furthermore, $k[\Delta]$ satisfies (S_2) (resp. Δ is pure) if and only if $k[\Delta^*]$ satisfies $(N_{c,2})$ (resp. $(N_{c,1})$) by Lemma 2.1 and Remark 2.2. When this is the case, since I_{Δ^*} is generated by elements of degree c, we have

$$e(k[\Delta]) = \beta_{0,c}(I_{\Delta^*}) = \mu(I_{\Delta^*}) = \binom{n}{c} - f_{c-1}(\Delta^*).$$

Hence

$$e(k[\Delta]) \geq \binom{n}{c} - m \quad \Longleftrightarrow \quad e(k[\Delta^*]) = f_{c-1}(\Delta^*) \leq m.$$

From these observations we have:

Theorem 2.3 (Alexander dual version of Theorem 1.2). *Suppose that one of the following conditions holds:*

(1) $e(k[\Delta]) \leq d$;
(2) $e(k[\Delta]) \leq 2d - 1$ *and* $\operatorname{rt}(I_\Delta) = \operatorname{indeg} I_\Delta = d$;
(3) $e(k[\Delta]) \leq 3d - 2$ *and* I_Δ *satisfies* $(N_{d,2})$.

Then I_Δ *has d-linear resolution.*

In fact, we could prove the following more general assertion:

Theorem 2.4. *Suppose that one of the following conditions holds:*

(1) $e(k[\Delta]) \leq d$;
(2) $e(k[\Delta]) \leq 2d - 1$ *and* $\beta_{0,d+1}(I_\Delta) = 0$;
(3) $e(k[\Delta]) \leq 3d - 2$ *and* $\beta_{1,d+2}(I_\Delta) = 0$.

Then $\operatorname{reg} I_\Delta \leq d$, *that is,* $\widetilde{H}_{d-1}(\Delta) = 0$.

We divide the proof into three cases.

Lemma 2.5. *If* $e(k[\Delta]) \leq d$, *then* $\widetilde{H}_{d-1}(\Delta) = 0$.

Proof (*see* [6]). Assume that there exists a complex Δ such that $e(k[\Delta]) \leq d$, $\widetilde{H}_{d-1}(\Delta) \neq 0$ and $\dim \Delta = d - 1$. Take one Δ whose multiplicity is minimal among the multiplicities of those complexes. Choose any $(d-1)$-facet F of Δ. Then F contains just d subfacets of Δ; say $G_1, \ldots, G_d$. Then G_i is not a **free face**. That is, G_i is contained in at least two facets of Δ. Indeed, if $G = G_i$ is a free face of Δ, then the simplicial complex $\Delta' := \Delta \setminus \{F, G\}$ is homotopy equivalent to Δ and thus $\widetilde{H}_{d-1}(\Delta') \cong \widetilde{H}_{d-1}(\Delta) \neq 0$. This contradicts the minimality of $e(k[\Delta])$ since $e(k[\Delta']) < e(k[\Delta])$.

Thus for each $i \in V$ there exists a $(d-1)$-facet F_i of Δ such that $G_i \subseteq F_i \neq F$. In particular, $F_1, \ldots, F_d, F$ are $(d+1)$ distinct facets of Δ. This is a contradiction. $\square$

Remark 2.6. We have an "algebraic proof" of Lemma 2.5. Namely, we can show that if A is an **F-pure** homogeneous k-algebra with $e(A) \le d$ then $a(A) < 0$. We can also give a direct proof of Theorem 1.2(1) without Alexander dual complexes.

Theorem 2.4(2) follows from the following lemma. Note that $\mathrm{rt}(I_\Delta) \le d$ if and only if $\beta_{0,d+1}(I_\Delta) = 0$.

Lemma 2.7. *If* $e(k[\Delta]) \le 2d-1$ *and* $\mathrm{rt}(I_\Delta) \le d$, *then* $\widetilde{H}_{d-1}(\Delta) = 0$.

Proof (see [6]*).* Put $e = e(k[\Delta])$. Let Δ' be the subcomplex that is spanned by all facets of dimension $d-1$. Replacing Δ with Δ', we may assume that Δ is pure.

We use induction on $d = \dim k[\Delta] \ge 2$. First suppose $d = 2$. The assumption shows that Δ does not contain the boundary complex of a triangle. Hence $\widetilde{H}_1(\Delta) = 0$ since $e(k[\Delta]) \le 3$.

Next suppose that $d \ge 3$, and that the assertion holds for any complex the dimension of which is less than $d-1$. Assume that Δ is a $(d-1)$-dimensional pure complex with $\mathrm{rt}(I_\Delta) \le d$, $e(k[\Delta]) \le 2d-1$ and $\widetilde{H}_{d-1}(\Delta) \ne 0$. Take one Δ whose multiplicity is minimal among the multiplicities of those complexes. Then Δ does not contain any free face by a similar argument as in the proof of the above lemma.

First consider the case of $\underline{\mathrm{rt}(I_\Delta) = d}$. Take a generator $X_1 \cdots X_d$ of I_Δ. Then since each $G_j = \{1, \ldots, \widehat{j}, \ldots, d\} \in \Delta$ is contained in at least two facets, $e(k[\Delta]) \ge 2d$. This is a contradiction.

Next we consider the case of $\underline{\mathrm{rt}(I_\Delta) < d}$. Let $V = [n]$ be the vertex set of Δ. Take the Mayer–Vietoris sequence with respect to $\Delta = \Delta_{V \setminus \{n\}} \cup \mathrm{star}_\Delta\{n\}$ as follows:

$$0 = \widetilde{H}_{d-1}(\Delta_{V\setminus\{n\}}) \oplus \widetilde{H}_{d-1}(\mathrm{star}_\Delta\{n\}) \longrightarrow \widetilde{H}_{d-1}(\Delta) \longrightarrow \widetilde{H}_{d-2}(\mathrm{link}_\Delta\{n\}),$$

where the vanishing in the left-hand side follows from the minimality of $e(k[\Delta])$ since $e(k[\Delta_{V\setminus\{n\}}]) < e(k[\Delta])$. Hence $\widetilde{H}_{d-1}(\Delta) \hookrightarrow \widetilde{H}_{d-2}(\mathrm{link}_\Delta\{n\}) \ne 0$.

Set $\Delta' = \mathrm{link}_\Delta\{n\}$. Then Δ' is a complex on $V \setminus \{n\}$ such that $\dim k[\Delta'] = d-1$ and $\mathrm{rt}(k[\Delta']) \le \mathrm{rt}(k[\Delta]) \le d-1$. One can also easily see $e(k[\Delta_{V\setminus\{n\}}]) \ge 2$, which implies that $e(k[\Delta']) \le 2d-3$. Hence $\widetilde{H}_{d-2}(\Delta') = 0$ by induction hypothesis. This is a contradiction. □

When Δ is pure, we have the following refinement of Lemma 2.5.

Corollary 2.8. *Suppose that* Δ *is pure and* c, $d \ge 2$. *If* $e(k[\Delta]) \le d+1$, *then* $\widetilde{H}_{d-1}(\Delta) = 0$.

Proof. First we prove that $\mathrm{rt}(I_\Delta) \le d$. Suppose not. Since $\mathrm{rt}(I_\Delta) = d+1$, we may assume that $X_1 \cdots X_{d+1}$ is a generator of I_Δ. Then $F_i = \{1, \ldots, \widehat{i}, \ldots, d+1\}$ is a $(d-1)$-facet of Δ for all $i = 1, \ldots, d+1$. Since $n = d+c \ge d+2$, there exists a facet of Δ which contains $\{d+2\}$. Hence $e(k[\Delta]) \ge d+2$, which is a contradiction. Therefore $\mathrm{rt}(I_\Delta) \le d$. Since $e(k[\Delta]) \le d+1 \le 2d-1$, we have $\widetilde{H}_{d-1}(\Delta) = 0$ by Lemma 2.7. □

Our proof in [7] of the following assertion is rather complicated. So we omit the proof and give only its sketch here.

Lemma 2.9. *If* $e(k[\Delta]) \leq 3d-2$ *and* $\beta_{1,d+2}(I_\Delta) = 0$, *then* $\widetilde{H}_{d-1}(\Delta) = 0$.

Proof. We use an induction on $d = \dim\Delta + 1$. When $d = 2$, the assertion easily follows from Hochster's formula on Betti numbers. Suppose $d \geq 3$, and that the assertion of the lemma holds for any complex the dimension of which is less than $d-1$. Assume that there exists a $(d-1)$-dimensional complex Δ such that $e(k[\Delta]) \leq 3d-2$, $\beta_{1,d+2}(I_\Delta) = 0$ and $\widetilde{H}_{d-1}(\Delta) \neq 0$. Take one Δ whose multiplicity is minimal among the multiplicities of those complexes. Put $e = e(k[\Delta]) \geq 2$. If necessary, we may assume that Δ is pure. Then the minimality of the multiplicity implies that Δ does not contain any free face. The assumption $\beta_{1,d+2}(I_\Delta) = 0$ also implies that $\mathrm{rt}(k[\Delta]) \leq d$. We first show the following claim:

Claim 1. Suppose that the following conditions are satisfied:
(1) Δ is pure;
(2) $\mathrm{rt}(I_\Delta) = d$;
(3) Δ does not have any free face;
(4) $\beta_{2,d+2}(k[\Delta]) = 0$.
Then $e(k[\Delta]) \geq 3d-1$.

To see the claim, we may assume that $X_1 \cdots X_d$ is a generator of I_Δ without loss of generality. Put $F := \{1, 2, \ldots, d\} \in 2^V \setminus \Delta$ and $G_i = \{1, \ldots, \widehat{i}, \ldots, d\}$ for each $i = 1, 2, \ldots, d$. Since Δ has no free face, there exist $2d$ facets of Δ whose form are $G_i \cup \{j\}$ for some $j \in V \setminus F$. In particular, if we set

$$U := \{j \in V \setminus F \;:\; \exists G \subseteq F \text{ such that } \#(G) = d-1,\; G \cup \{j\} \in \Delta\},$$

then $\#(U) \geq 2$. Note that there exist no subsets $\{j_1, j_2\}$ of U ($j_1 \neq j_2$) for which the following conditions hold: both $G_i \cup \{j_1\}$ and $G_i \cup \{j_2\}$ are facets of Δ for all $i = 1, 2, \ldots, d$. In fact, we suppose that the assertion does *not* hold. Namely, there exists a subset $\{j_1, j_2\}$ of U for which both $G_i \cup \{j_1\}$ and $G_i \cup \{j_2\}$ are facets of Δ for all $i = 1, 2, \ldots, d$. Set $W = F \cup \{j_1, j_2\}$. Then $\widetilde{H}_{d-1}(\Delta_W) = 0$ since $\#(W) = d+2$ and $\beta_{1,d+2}(I_\Delta) = 0$. Let Δ_1 be a subcomplex of Δ_W spanned by $H \cup \{j_1\}$, $H \cup \{j_2\}$ where $H \in 2^F \setminus \{F\}$, that is, $\Delta_1 = (2^F \setminus \{F\}) * (2^{\{j_1,j_2\}} \setminus \{j_1, j_2\})$. Let Δ_2 be a subcomplex of Δ_W spanned by all facets of Δ_W that contains $\{j_1, j_2\}$. Then $\Delta_W = \Delta_1 \cup \Delta_2$ and $\dim(\Delta_1 \cap \Delta_2) \leq d-2$. Applying Mayer–Vietoris sequence to Δ_W, we get

$$0 = \widetilde{H}_{d-1}(\Delta_1 \cap \Delta_2) \to \widetilde{H}_{d-1}(\Delta_1) \oplus \widetilde{H}_{d-1}(\Delta_2) \to \widetilde{H}_{d-1}(\Delta_W) \to \cdots$$

On the other hand, $\widetilde{H}_{d-1}(\Delta_1) \cong \widetilde{H}_{d-2}(2^F \setminus \{F\}) \cong \widetilde{H}_{d-2}(\mathbf{S}^{d-2}) \cong k \neq 0$. This implies that $\widetilde{H}_{d-1}(\Delta_W) \neq 0$. This is a contradiction.

By the above discussion we can choose $j \in U$ $(d+1 \leq j \leq n)$ and $\ell (1 \leq \ell \leq d-1)$ such that

$$\begin{aligned} F_p &:= \{1, 2, \ldots, \widehat{p}, \ldots, \ell, \ldots, d, j\} \in \Delta \quad \text{for all } p = 1, 2, \ldots, \ell, \\ G_q &:= \{1, 2, \ldots, \ell, \ldots, \widehat{q}, \ldots, d, j\} \notin \Delta \quad \text{for all } q = \ell+1, \ldots, d. \end{aligned}$$

Now let us consider the following subfacets of Δ:

$$H_{p,q} := \{1, \dots, \widehat{p}, \dots, \ell, \ell+1, \dots, \widehat{q}, \dots, d, j\} \quad (1 \le p \le \ell,\ \ell+1 \le q \le d).$$

Since Δ has no free face, $H_{p,q}$ is contained in at least two facets of Δ, but one of those facets cannot be written as in the form $G \cup \{j\}$ where $G \subseteq F$. Counting the number of facets of Δ, we get

$$e(k[\Delta]) \ge 2d + \ell(d-\ell) \ge 2d + (d-1) = 3d-1,$$

as required.

Next, we must show the following claim:

Claim 2. Suppose that $d \ge 3$, $c \ge 3$. Suppose that the following conditions are satisfied:

(1) Δ is pure.
(2) $\mathrm{rt}(I_\Delta) \le d-1$;
(3) Δ does not have any free face;
(4) There exists $y \in V$ such that $\beta_{2,d+1}(k[\mathrm{link}_\Delta\{y\}]) \ne 0$;
(5) $\widetilde{H}_{d-2}(\mathrm{link}_\Delta\{x\}) \ne 0$ holds for all $x \in V$.

Then $e(k[\Delta]) \ge 3d-1$.

We omit a proof of the above claim here because it is technical and long.

Now let us return to the proof of the lemma. By Claim 1, we may assume that $\mathrm{rt}(I_\Delta) < d$. Furthermore, we may assume that $d \ge 3$, $c \ge 3$ and $e \ge 2d$. Let us check the conditions in Claim 2.

Claim 3. $\widetilde{H}_{d-2}(\mathrm{link}_\Delta\{x\}) \ne 0$ holds for all $x \in V$.

Fix $x \in V$. We have $\widetilde{H}_{d-1}(\Delta_{V\setminus\{x\}}) = 0$ by the minimality of $e(k[\Delta])$ and the purity of Δ. Then the assertion follows from the Mayer–Vietoris sequence to $\Delta = \mathrm{star}_\Delta\{x\} \cup \Delta_{V\setminus\{x\}}$.

Claim 4. There exists a vertex $y \in V$ such that $e(k[\Delta_{V\setminus\{y\}}]) \ge 3$.

Now suppose that $e(k[\Delta_{V\setminus\{x\}}]) \le 2$ for all $x \in V$. Then at least $(e-2)$ facets of Δ contains x. Counting the number of vertices which is contained in some facets, we obtain that $ed = e(k[\Delta]) \times d \ge n(e(k[\Delta])-2) = (c+d)(e-2)$. Hence $2(c+d) \ge ce \ge 2cd$, that is, $(c-1)(d-1) \le 1$. This contradicts the assumption that $c \ge 3$ and $d \ge 3$.

Take a vertex $y \in V$ as in Claim 4 and put $\Gamma = \mathrm{link}_\Delta\{y\}$. Since

$$e(k[\Gamma]) = e(k[\mathrm{star}_\Delta\{y\}]) = e(k[\Delta]) - e(k[\Delta_{V\setminus\{y\}}]) \le 3(d-1)-2,$$

if $\beta_{2,d+1}(k[\Gamma]) = 0$, then $\widetilde{H}_{d-2}(\Gamma) = 0$ by the induction hypothesis. But this contradicts Claim 3. Hence $\beta_{2,d+1}(k[\Gamma]) \ne 0$. Therefore $e(k[\Delta]) \ge 3d-1$ by Claim 2 and the lemma is proved. □

3. Examples

We construct some examples of simplicial complexes which satisfy Theorem 1.2 or 2.3.

Example 3.1. Put $F_{i,j} = \{1, 2, \dots, \widehat{i}, \dots, d, j\}$ for each $i = 1, \dots, d; j = d + 1, \dots, n$. For a given integer e with $1 \le e \le cd$, we choose e faces (say, $F_1, \dots, F_e$) from $\{F_{i,j} : 1 \le i \le d, \ d+1 \le j \le n\}$, which is a set of the facets of the simplicial join of $2^{[d]} \setminus \{[d]\}$ and c points.

Let Δ be the simplicial complex spanned by $F_1, \dots, F_e$ and all elements of $\binom{[n]}{d-1}$. Then $k[\Delta]$ is a d-dimensional Stanley–Reisner ring with $\operatorname{indeg} I_\Delta = \operatorname{rt}(I_\Delta) = d$ and $e(k[\Delta]) = e$.

In particular, when $e \le 2d - 1$, $k[\Delta]$ has d-linear resolution by Theorem 2.3. Thus their Alexander dual complexes provide examples satisfying the hypothesis of Theorem 1.2.

The following example shows that the assumption "$e(k[\Delta]) \le 2d - 1$" is optimal in Theorem 2.3(2).

Example 3.2. There exists a complex Δ on $V = [n]$ for which $k[\Delta]$ does *not* have d-linear resolution with $\dim k[\Delta] = \operatorname{indeg} I_\Delta = \operatorname{rt}(I_\Delta) = d$ and $e(k[\Delta]) = 2d$.

In fact, let Δ_0 be a complex on $V_0 = [d+2]$ such that $k[\Delta_0]$ is a complete intersection defined by $(X_1 \cdots X_d, X_{d+1}X_{d+2})$. Let Δ be a complex on V such that

$$\begin{aligned} I_\Delta &= (X_1 \cdots X_d)S + (X_{i_1} \cdots X_{i_{d-2}} X_{d+1} X_{d+2} : 1 \le i_1 < \cdots < i_{d-2} \le d)S \\ &\quad + (X_{j_1} \cdots X_{j_d} : 1 \le j_1 < \cdots < j_d \le n, \ \ j_d \ge d+3)S. \end{aligned}$$

Then $\widetilde{H}_{d-1}(\Delta) \cong \widetilde{H}_{d-1}(\Delta_0) \ne 0$ since $a(k[\Delta_0]) = 0$. Hence $k[\Delta]$ does not have d-linear resolution.

Remark 3.3. The case $n = d + 2$ in the above example is also obtained by considering the case $c = 2$, $e = 2d$ in Example 3.1.

The simplicial complex Δ_0 can be also characterized as a pure complex with Δ such that $e(k[\Delta]) = 2d$, $\operatorname{rt}(I_\Delta) = d$ and $\widetilde{H}_{d-1}(\Delta) \ne 0$.

In fact, if Δ is such a complex, then Δ has no free faces. Let $X_1 \cdots X_d$ be a generator of I_Δ and put $G_i = \{1, \dots, \widehat{i}, \dots, d\}$ for each $i = 1, \dots, d$. Since G_i is not a free face, there exist two distinct points $p_i, p_i' \in V$ such that $F_i := G_i \cup \{p_i\}$, $F_i' := G_i \cup \{p_i'\}$ are facets of Δ. Then $\{F_i, F_i' : i = 1, \dots, d\}$ becomes the set of all facets of Δ. Since $F_1 \setminus \{i\}$ is also not a free face, it is contained in some facet in Δ except F_1. But such a facet must be either F_i or F_i'. Consequently, we may assume that $p_i = p_1$ and $p_i' = p_1'$ for all $i = 1, \dots, d$. Then one can easily see that $\Delta = (2^{[d]} \setminus \{[d]\}) * (2^{\{p,q\}} \setminus \{\{p, q\}\})$ by the purity of Δ and $e(k[\Delta]) = 2d$. In other words, $k[\Delta]$ is a complete intersection of type $(d, 2)$: $k[\Delta] \cong k[X_1, \dots, X_d, Y_1, Y_2]/(X_1 \cdots X_d, \ Y_1Y_2)$.

Using the boundary complex of a stacked d-polytope, let us construct an example which shows the condition "$e(k[\Delta]) \le 3d - 2$" is optimal in Theorem 2.4.

Example 3.4. Let d, n be integers with $d \geq 2$ and $c = n - d \geq 3$. Let Δ_0 be a simplicial complex on $V_0 = [d+3]$ spanned by the following d-subsets of V:

$$\begin{array}{ll} \{1,2,\ldots,\widehat{i},\ldots,d,d+1\}, & i = 2,3,\ldots,d; \\ \{1,\ldots,\widehat{i},\ldots,d,d+2\}, & i = 1,2,\ldots,d; \\ \{2,\ldots,\widehat{i},\ldots,d+1,d+3\}, & i = 2,3,\ldots,d+1. \end{array}$$

Let Δ_1 be a complex defined by $\Delta_1 = \Delta_0 \cup \{\{d+4\},\ldots,\{n\}\}$ (Its geometric realization $|\Delta_1|$ is a disjoint union of $|\Delta_0|$ and $(n-d-3)$ isolated points.). Then Δ_1 is a $(d-1)$-dimensional simplicial complex on $V = [n]$ and $e(k[\Delta_1]) = e(k[\Delta_0]) = 3d - 1$.

Note that Δ_0 can be regarded as the boundary complex of a stacked d-polytope with $d+3$ vertices. Thus the graded minimal free resolution of $k[\Delta_0]$ can be written as in the following shape ([4]):

$$\begin{aligned} 0 \rightarrow\ & S(-d-3) \rightarrow S(-3)^{\beta_{2,3}} \oplus S(-d-1)^{\beta_{2,d+1}} \\ & \rightarrow S(-2)^{\beta_{1,2}} \oplus S(-d)^{\beta_{1,d}} \rightarrow S \rightarrow k[\Delta_0] \rightarrow 0. \end{aligned}$$

In particular, $\beta_{2,d+2}(k[\Delta_1]) = \beta_{2,d+2}(k[\Delta_0]) = 0$, but $\operatorname{reg} k[\Delta_1] = \operatorname{reg} k[\Delta_0] = d$.

Let Δ be the simplicial complex spanned by all facets of Δ_1 and all $(d-1)$-subsets of V. Then $k[\Delta]$ satisfies $(N_{d,2})$ and $e(k[\Delta]) = 3d - 1$, but does not have linear resolution. See also Theorem 2.4.

Using the Alexander dual complex of Δ, one can find a $(d-1)$-dimensional simplicial complex Γ which satisfies (S_2) and $e(k[\Gamma]) = \binom{n}{d} - 3(n-d) + 1$, but it is not Cohen–Macaulay for given integers $d \geq 3$ and $n - d \geq 3$.

The next example shows that it is not enough to assume "*pure and connected in codimension* 1" instead of (S_2) in Theorem 1.2.

Example 3.5. Let $\Delta = (2^{[4]} \setminus \{[4]\}) \cup \operatorname{Span}\{\{1,2,5\},\{3,4,5\}\}$ be a simplicial complex on $V = [5]$. Then $\dim k[\Delta] = 3$, $e(k[\Delta]) = \binom{5}{3} - 3(5-3) + 2 = 6$. Moreover, $k[\Delta]$ is pure and connected in codimension 1, but not (S_2).

Proof. Since $\operatorname{link}_\Delta\{5\}$ is spanned by $\{1,2\}$ and $\{3,4\}$, it is disconnected. Hence $k[\Delta]$ does not satisfy (S_2).

If we put $F_1 = \{1,2,5\}$, $F_2 = \{1,2,4\}$, $F_3 = \{1,2,3\}$, $F_4 = \{2,3,4\}$, $F_5 = \{1,3,4\}$ and $F_6 = \{3,4,5\}$, then $\{F_1,\ldots,F_6\}$ is the set of all facets of Δ such that $\dim F_i \cap F_{i-1} = \dim \Delta - 1 = 1$. Hence Δ is pure and connected in codimension 1. □

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