数理解析研究所講究録
1503 巻 2006 年 17-23

Some Remarks on Generalized Inverse $*$-Semigroups

島根大学総合理工学部 今岡 輝男 (Teruo Imaoka)
飯貝 高史 (Takashi Iigai)
Department of Mathematics, Shimane University
Matsue, Shimane 690-8504, Japan

1 Preliminaries

A semigroup S with a unary operation $*: S \to S$ is called a *regular $*$-semigroup* if it satisfies the conditions:

(1) $(a^*)^* = a$,

(2) $(ab)^* = b^*a^*$

(3) $aa^*a = a$

for any $a, b \in S$.

Let S be a regular $*$-semigroup. An idempotent e in S is called a *projection* if $e^* = e$. For a subset A of S, denote the set of projections of A by $P(A)$.

A regular $*$-semigroup S is called a *generalized inverse $*$-semigroup* if $E(S)$, the set of idempotents of S, satisfies the identity:

$$x_1x_2x_3x_4 = x_1x_3x_2x_4 \tag{1.1}$$

Such a semigroup is orthodox in the usual sense that $E(S)E(S) \subseteq E(S)$.

Result 1.1 ([12]) *A regular $*$-semigroup S is a generalized inverse $*$-semigroup if, and only if, $P(S)$ satisfies the identity* (1.1).

Let S be a regular $*$-semigroup. For $a, b \in S$, define a relation $\leq$ on S by

$$a \leq b \Leftrightarrow a = eb = bf \text{ for some } e, f \in P(S).$$

Result 1.2 ([5]) *Let a and b be elements of a regular $*$-semigroup S. Then the following are equivalent:*

(i) $a \leq b$.

(ii) $aa^* = ba^*$ *and* $a^*a = b^*a$.

(iii) $aa^* = ab^*$ *and* $a^*a = a^*b$.

(iv) $a = aa^*b = ba^*a$.

Result 1.3 ([4]) *Let S be a regular $*$-semigroup. Then*

(i) $E(S) = P(S)^2$. *In fact, for any $e \in E(S)$, there exist $f, g \in P(S)$ such that $f \mathscr{R} e \mathscr{L} g$ and $e = fg$.*

(ii) *For any $a \in S$ and $e \in P(S)$, $a^*ea \in P(S)$.*

(iii) *For $e, f \in P(S)$, $ef \in P(S)$ if, and only if, $ef = fe$.*

(iv) *Each $\mathscr{L}$-class and each $\mathscr{R}$-class in S contains one and only one projection.*

Result 1.4 ([8]) *Let S be an orthodox semigroup. Then*

$$\sigma = \{(a, b) \in S \times S : eae = ebe \text{ for some } e \in E(S)\}$$

is the minimum group congruence on S.

2 E-unitary generalized inverse $*$-semigroups

PG^*-semigroups

Let (G, X, Y) be a McAlister triple, and let $\{P_\alpha : \alpha \in Y\}$ be a family of disjoint non-empty sets indexed by the elements of Y. Put $P = \bigcup_{\alpha \in Y} P_\alpha$. For each pair α, β of elements of Y where $\alpha \geq \beta$, let $\rho_{\alpha,\beta} : P_\alpha \to P_\beta$ be a mapping such that the following two axioms hold:

(PG*1) $\rho_{\alpha,\alpha}$ is the identity mapping on P_α.

(PG*2) If $\alpha \geq \beta \geq \gamma$ then $\rho_{\alpha,\beta}\rho_{\beta,\gamma} = \rho_{\alpha,\gamma}$.

We call such a quintet $(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ a *PG^*-quintet*.

Proposition 2.1 *Let $(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ be a PG^*-quintet. Then*

$$S = \{(\alpha, g, x_1, x_2) \in Y \times G \times P \times P : g^{-1}\alpha \in Y, x_1 \in P_\alpha, x_2 \in P_{g^{-1}\alpha}\},$$

with multiplication and a unary operation given by

$$(\alpha, g, x_1, x_2)(\beta, h, y_1, y_2) = (\alpha \wedge g\beta, gh, x_1\rho_{\alpha,\alpha\wedge g\beta}, y_2\rho_{h^{-1}\beta,(gh)^{-1}(\alpha\wedge g\beta)}),$$
$$(\alpha, g, x_1, x_2)^* = (g^{-1}\alpha, g^{-1}, x_2, x_1)$$

is an E-unirtary generalized inverse $*$-semigroup.

We say that S is a *PG^*-semigroup* and denoted by $PG^*(G, X, Y, P, \{\rho_{\alpha,\beta}\})$, or simply by $PG^*(G, X, Y, P)$.

We now characterise the *Green's relations* $\mathscr{L}, \mathscr{R}$, the minimum group conguence σ and the natural order $\leq$ on $PG^*(G, X, Y, P, \{\rho_{\alpha,\beta}\})$.

Proposition 2.2 *Let $(\alpha, g, x_1, x_2), (\beta, h, y_1, y_2)$ be elements of $S = PG^*(G, X, Y, P, \{\rho_{\alpha,\beta}\})$.*

(i) $(\alpha, g, x_1, x_2) \leq (\beta, h, y_1, y_2)$ *if, and only if,* $\alpha \leq \beta, g = h, y_1\rho_{\beta,\alpha} = x_1, y_2\rho_{h^{-1}\beta,g^{-1}\alpha} = x_2$.

(ii) $(\alpha, g, x_1, x_2)\, \sigma\, (\beta, h, y_1, y_2)$ *if, and only if,* $g = h$.

(iii) $(\alpha, g, x_1, x_2)\, \mathscr{L}\, (\beta, h, y_1, y_2)$ *if, and only if,* $g^{-1}\alpha = h^{-1}\beta$ *and* $x_2 = y_2$.

(iv) $(\alpha, g, x_1, x_2)\, \mathscr{R}\, (\beta, h, y_1, y_2)$ *if, and only if.* $\alpha = \beta$ *and* $x_1 = y_1$.

Now we have the following theorem.

Theorem 2.3 *The semigroup $PG^*(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ is an E-unitary generalized inverse $*$-semigroup and maximum group homomorphic image isomorphic to G.*

Construction of E-unitary generalized inverse $*$-semigroups

Let S be an E-unitary generalized inverse $*$-semigroup. Put $G = S/\sigma$, and denoted its identity by 1. Since $E(S)$ is a minimum group congruence class of S, $E(S)$ is the identity of G. Let $E(S) \sim \sum\{E_\alpha : \alpha \in Y\}$ be the structure decomposition of $E(S)$, that is $E(S)$ is a semilattice Y of rectangular bands $E_\alpha\, (\alpha \in Y)$. Put $\mathscr{E} = \{E_\alpha : \alpha \in Y\}$. We shall construct PG^*-quintet.

First, we define a relation ρ on $\mathscr{E} \times G$ by

$$(E_\alpha, g)\rho(E_\beta, h) \Leftrightarrow xx^* \in E_\alpha \text{ and } x^*x \in E_\beta \text{ for some } x \in g^{-1}h.$$

Lemma 2.4 *The relation ρ is an equivalence relation on $\mathscr{E} \times G$.*

We shall write $\mathscr{X}$ for $(\mathscr{E} \times G)/\rho$, and denote the ρ-class of $\mathscr{E} \times G$ which contains (E_α, g) by $(E_\alpha, g)\rho$. The following lemmas are immediate.

Lemma 2.5 *For any element $x \in S, xx^* \in E_\alpha, x^*x \in E_\beta$ for some $\alpha, \beta \in Y$. Then*

$$(E_\alpha, 1)\rho(E_\beta, x\sigma) \text{ and } (E_\beta, 1)\rho(E_\alpha, (x\sigma)^{-1}).$$

Lemma 2.6 *Let $\alpha, \beta \in Y$ and $g \in G$ such that $(E_\alpha, g)\rho(E_\beta, g)$. Then $\alpha = \beta$.*

Proposition 2.7 *Let $E_\alpha, E_\beta, E_\gamma \in \mathscr{E}$ and $g, h \in G$. If $\alpha \leq \beta$ and $(E_\beta, g)\rho(E_\gamma, h)$, then there exists $\delta \in Y$ such that $\delta \leq \gamma, (E_\alpha, g)\rho(E_\delta, h)$.*

We define a relation $\leq$ on $\mathscr{X}$ as follows:

$$A \leq B \Leftrightarrow \alpha \leq \beta, (E_\alpha, g) \in A, (E_\beta, g) \in B$$

for some $\alpha, \beta \in Y$ and $g \in G$. The proof of the following is straightforward from Proposition 2.7 and the definition of $\leq$.

Corollary 2.8 *Let $A \leq B$, where $A, B \in \mathscr{X}$. If $(E_\gamma, h) \in B$, then there exists $\delta \in Y$ such that $\delta \leq \gamma$ and $(E_\delta, h) \in A$.*

Lemma 2.9 *The relation $\leq$ is a partial order on $\mathscr{X}$.*

Let

$$\mathscr{Y} = \{(E_\alpha, 1)\rho : \alpha \in Y\}.$$

We define an action of G on $\mathscr{X}$ by order automorphisms. Suppose first that $(E_\alpha, g)\rho(E_\beta, h)$. This means that there exists $x \in g^{-1}h$ such that $xx^* \in E_\alpha, x^*x \in E_\beta$. Let $k \in G$. Then $x \in (kg)^{-1}(kh)$ and so $(E_\alpha, kg)\rho = (E_\beta, kh)\rho$. We can therefore define $\circ : G \times \mathscr{X} \to \mathscr{X}$ by

$$k \circ (E_\alpha, g)\rho = (E_\alpha, kg)\rho.$$

We shall show that the triple $(G, \mathscr{X}, \mathscr{Y})$ form a McAlister triple.

Lemma 2.10 *The mapping $\varphi : Y \to \mathscr{Y}$ defined by $\alpha\varphi = (E_\alpha, 1)\rho$ is an order isomorphism.*

Lemma 2.11 *The mapping $\circ$ is an action of G on $\mathscr{X}$, on the left by order automorphisms.*

Lemma 2.12 *With the above notation:*

(i) *$\mathscr{Y}$ is an order ideal of $\mathscr{X}$.*

(ii) *$G \circ \mathscr{Y} = \mathscr{X}$.*

(iii) *$g \circ \mathscr{Y} \cap \mathscr{Y} \neq \Box$ for all $g \in G$.*

By the lemma above, we have that the triple $(G, \mathscr{X}, \mathscr{Y})$ is a McAlister triple. We shall construct PG^*-quintet by making use of McAlister triple $(G, \mathscr{X}, \mathscr{Y})$ and form the PG^*-semigroup $PG^*(G, \mathscr{X}, \mathscr{Y}, P)$. Put $P_\alpha = P(E_\alpha)$ for each $\alpha \in Y$ and let $P = \bigcup_{\alpha \in Y} P_\alpha$. For each pair α, β of elements of Y where $\alpha \geq \beta$, define the mapping

$$\rho_{\alpha,\beta} : P_\alpha \to P_\beta \text{ by } e\rho_{\alpha,\beta} = efe \text{ where } f \in P_\beta.$$

Lemma 2.13 *With the definition above, $\rho_{\alpha,\beta}$ is a mapping satifying the conditions* (PG*1) *and* (PG*2).

Thus $PG^*(G, \mathscr{X}, \mathscr{Y}, P, \{\rho_{\alpha,\beta}\})$, constructed above, forms a PG^*-semigroup.

Lemma 2.14 *For any $xx^* \in P_\alpha$ and $e \in P_\beta$, $xex^* \in P_{\alpha\wedge(x\sigma)\beta}$.*

Lemma 2.15 *The mapping $\theta : S \to PG^*(G, \mathscr{X}, \mathscr{Y}, P, \{\rho_{\alpha,\beta}\})$ defined by*

$$x\theta = ((E_\alpha, 1)\rho, x\sigma, xx^*, x^*x),$$

where $xx^* \in E_\alpha$, is a $*$-isomorphism.

Now we have the structure of generalized inverse $*$-semigroups.

Proposition 2.16 *A generalized inverse $*$-semigroup is E-unitary if, and only if, it is $*$-isomorphic to some PG^*-semigroup.*

3 The compatibility relations

Let S be a regular $*$-semigroup. For all $s, t \in S$, the *left compatibility relation* is defined by

$$s \sim_l t \Leftrightarrow st^* \in E(S),$$

the *right compatibility relation* is defined by

$$s \sim_r t \Leftrightarrow s^*t \in E(S),$$

and the *compatibility relation*, the intersection of the above two relations, is defined by

$$s \sim t \Leftrightarrow st^*, s^*t \in E(S).$$

It is clear that all three relations are reflexive and symmetric, but none of them need be transitive (see Theorem 3.2 for a characterisation of the generalized inverse $*$-semigroups having a transitive compatility relation). The next lemma describe some of the basic property of these relations.

Lemma 3.1 *Let S be a generalized inverse $*$-semigroup and ρ be any one of the three relations $\sim_l$, $\sim_r$, and $\sim$. Then the following two properties hold.*

(i) *$s \,\rho\, t$ and $u \,\rho\, v$ imply that $su \,\rho\, tv$.*

(ii) *$s \le t, u \le v$ and $t \,\rho\, v$ imply that $s \,\rho\, u$.*

Theorem 3.2 *Let S be a generalized inverse $*$-semigroup. Then the compatibility relation is transitive if, and only if, S is E-unitary.*

Proof Suppose that $\sim$ is transitive. Let $es \in E(S)$, where e is an idempotent. Then $s \sim es$ since elements $s(es)^*$ and s^*es are idempotents. Clearly $es \sim s^*s$, and so, by our assumption that the compatibility relation is transitive, we have that $s \sim s^*s$. But $s(s^*s)^* = s$, so that s is an idempotent.

Conversely, suppose that S is E-unitary and $s \sim t$ and $t \sim u$. Clearly $(s^*t)(t^*u)$ is an idempotent and

$$(s^*t)(t^*u) = s^*u(t^*u)^*(t^*u)$$

But S is E-unitary and so s^*u is an idempotent. Similarly, su^* is an idempotent. Hence $s \sim u$. ∎

Proposition 3.3 *Let S be a regular $*$-semigroup. Then the following are equivalent:*

(i) *The left and right compatibility relations are equal.*

(ii) *For all $s, t \in S$, we have that $st \in E(S)$ if, and only if, $ts \in E(S)$.*

A congruence ρ on an orthodox semigroup S is said to be *idempotent pure* if $a \in S, e \in E(S)$ and $(a, e) \in \rho$ then a is an idempotent.

Proposition 3.4 *Let S be an E-unitary regular $*$-semigroup. Then a congruence ρ is idempotent pure if, and only if, $\rho \subseteq \sim$.*

Proof Let ρ be idempotent pure and let $(a, b) \in \rho$. Then $(ab^*, bb^*) \in \rho$. But ρ is idempotent pure and bb^* is an idempotent. Thus ab^* is an idempotent. Similarly, a^*b is an idempotent. Thus $a \sim b$.

Conversely, let ρ be a congruence contained in the compatibility relation. Let $(a, e) \in \rho$, where e is an idempotent. Then $a \sim e$. Thus $ae^* \in E(S)$. But e^* is an idempotent and so a is an idempotent, since S is E-unitary. ∎

4 Enlargements

We proved in Section 2, that E-unitary generalized inverse $*$-semigroups are essentially isomorphic to the generalized inverse $*$-subsemigroups of PG^*-semigroups. The point is that if X is a meet semilattice, we can form the semigroup $PG^*(G, X, X, P, \{\rho_{\alpha,\beta}\})$, which contains $PG^*(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ as a generalized inverse $*$-subsemigoup. In the following proposition, we shall describe the *abstract* relationship between $PG^*(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ and $PG^*(G, X, X, P, \{\rho_{\alpha,\beta}\})$.

Proposition 4.1 *Let $(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ be a PG^*-quintet, where X is a meet semilattice.*

(i) *The idempotents of $PG^*(G, X, Y, P)$ form an order ideal of $PG^*(G, X, X, P)$.*

(ii) *If $(\alpha, g, x_1, x_2) \in PG^*(G, X, X, P)$ is such that*

$$(\alpha, g, x_1, x_2)^*(\alpha, g, x_1, x_2), (\alpha, g, x_1, x_2)(\alpha, g, x_1, x_2)^* \in PG^*(G, X, Y, P)$$

then $(\alpha, g, x_1, x_2) \in PG^*(G, X, Y, P)$.

(iii) *For each projection $(\alpha, 1, x, x) \in PG^*(G, X, X, P)$ there exists a projection $(\beta, 1, y, y) \in PG^*(G, X, Y, P)$ such that $(\alpha, 1, x, x) \mathscr{D} (\beta, 1, y, y)$.*

On the basis of the above proposition, we make the following definition. Let S be a generalized inverse $*$-subsemigroup of a generalized inverse $*$-semigroup T. We say that T is an *enlargement* of S if the following three axioms hold:

(E1) $E(S)$ is an order ideal of $E(T)$.

(E2) If $t \in T$ and $t^*t, tt^* \in S$ then $t \in S$.

(E3) For every projection $e \in T$ there exists a projection $f \in S$ such that $e \mathscr{D} f$.

The following is easy to prove.

Lemma 4.2 *Let S be a generalized inverse $*$-subsemigroup of T. Then axiom* (E1) *holds if, and only if, S is an order ideal of T.*

We may find a PG^*-representation of an E-unitary generalized inverse $*$-semigroup.

Theorem 4.3 *Let G be a group and X a semilattice, and let S be a generalized inverse $*$-subsemigroup of the generalized inverse $*$-semigroup $PG^*(G, X, X, P, \{\rho_{\alpha,\beta}\})$. Suppose that $PG^*(G. X, X, P, \{\rho_{\alpha,\beta}\})$ is an enlargement of S. Let*

$$Y = \{\alpha \in X : (\alpha, 1, x, y) \in E(S)\} \text{ and } Q = \{x \in P : (\alpha, 1, x, y) \in E(S)\}.$$

Then $(G, X, Y, Q, \{\rho_{\alpha,\beta}\})$ is a PG^*-quintet and $S = PG^*(G, X, Y, Q, \{\rho_{\alpha,\beta}\})$.

5 A Structure Theorem

We can now prove the uniqueness of the PG^*-representation of an E-unitary generalized inverse $*$-semigroup.

Theorem 5.1 *Let $(G, X, Y, P, \{\rho_{\alpha,\beta}\})$ and $(G', X', Y', P', \{\rho'_{\alpha',\beta'}\})$ be two PG^*-quintets. Let $\theta : G \to G'$ be a group isomorphism and let $\psi : X \to X'$ be an order isomorphism such that $\psi|_Y$ is an isomorphism from the semilattice Y onto Y'; now let $\xi : P \to P'$ be a bijection. Suppose also that, for all g in G, α in X and x in P_β,*

$$(g\alpha)\psi = (g\theta)(\alpha\psi),$$
$$(x\rho_{\beta,\gamma})\xi = (x\xi)\rho_{\beta\psi,\gamma\psi},$$

where $\beta, \gamma \in Y$ such that $\beta \geq \gamma$. Then the mapping $\phi : PG^*(G, X, Y, P) \to PG^*(G', X', Y', P')$ defined by

$$(\alpha, g, x, y)\phi = (\alpha\psi, g\theta, x\xi, y\xi)$$

is a $*$-isomorphism. Conversely, every $*$-isomorphism from $PG^*(G, X, Y, P)$ onto $PG^*(G', X', Y', P')$ is of this type.

6 The minimum group congruence

In this subsection, we shall first give an alternative characterization of the minimum group congruence on a generalized inverse $*$-semigroup.

Theorem 6.1 *If S is a generalized inverse $*$-semigroup, then the relation*

$$\sigma = \{(a, b) \in S \times S : eaf = ebf \text{ for some } e, f \in P(S)\}$$

is the minimum group congruence on S.

Idempotent pure congruences, the minimum group congruence and E-unitary generalized inverse $*$-semigroups are all linked by the following result.

Theorem 6.2 *Let S be a generalized inverse $*$-semigroup. Then the following conditions are equivalent:*

(i) *S is E-unitary.*

(ii) $\sim\ = \sigma$.

(iii) *σ is idempotent pure.*

(iv) *$\sigma(e) = E(S)$ for any idempotent e.*

Proof (i) $\Leftrightarrow$ (iv). Immediate.
(i) $\Rightarrow$ (ii). Let $a \sim b$. Then $ab^*, a^*b \in E(S)$. Thus

$$\begin{aligned}(ab^*)(ab^*)^*a(a^*b)(a^*b)^* &= ab^*ba^*aa^*bb^*a \\ &= ab^*ba^*bb^*a \\ &= ab^*(ba^*)(ba^*)bb^*a \\ &= (ab^*)(ab^*)^*b(a^*b)(a^*b)^*.\end{aligned}$$

Hence $a \,\sigma\, b$.

Conversely, suppose $a \sigma b$. Then $eaf = ebf$ for some $e, f \in P(S)$ by Theorem 6.1. Thus we have

$$(ebf)(ebf)^* = eafb^*bb^*e = (eab^*)bfb^*e \in E(S).$$

But bfb^*e is an idempotent. Thus, by (i), $eab^* \in E(S)$. By using (i) again, we obtain $ab^* \in E(S)$ since $e \in E(S)$. Similarly, a^*b is an idempotent.
(ii) $\Rightarrow$ (iii). Let $(a, e) \in \sigma$, where e is an idempotent. Clearly, $e \sim a^*a$. But $\sim = \sigma$ and so $a \sim a^*a$. Hence a is an idempotent.
(iii) $\Rightarrow$ (i). Let $a \in S$ and $e \in E(S)$ such that $ea \in E(S)$. Then $eae = e(eae)e$. Thus, by Result 1.4, $(a, eae) \in \sigma$. But $eae = (ea)e \in E(S)$ and so a is an idempotent since σ is idempotent pure. ∎

References

[1] T. E. Hall, On regular semigroups whose idempotents form a subsemigroup, *Bulletin of the Australian Mathematical Society* **1** *(1969), 195-208.*

[2] P. M. Higgins, *Techniques of semigroup theory*, Oxford University Press, New York, 1992.

[3] J. M. Howie, *Fundamentals of Semigroup Theory*, Academic Press, London, 1995.

[4] T. Imaoka, On fundamental regular *-semigroups, *Memoirs of the Faculty of Science, Shimane University*, **14** *(1980), 19-23.*

[5] T. Imaoka, Prehomomorphisms on regular *-semigroups, *Memoirs of the Faculty of Science. Shimane University*, **15** *(1981), 23-27.*

[6] T. Imaoka, H. Yokoyama and I. Inata, Some remarks on E-unitary regular *-semigroups, *Algebra Colloquium (2)* **3** *(1996), 117-124.*

[7] M. V. Lawson, *Inverse semigroups -The theory of partial symmetries -*, World Scientific, Singapore, 1998.

[8] J. C. Meakin, Congruences on orthodox semigroups II, *Journal of Australian Mathematical Society* **11** *(1972), 259-266.*

[9] M. Petrich, *Inverse semigroups*, John Wiley and Sons, New York, 1984.

[10] M. B. Szendrei, A generalization of McAlister's P-theorem for E-unitary regular semigroups, *Acta Scientiarum Mathematicarum (Szeged) 57 (1987), 229-249.*

[11] M. B. Szendrei, E-unitary regular semigroups, *Proceedings of the Royal Society of Edinburgh* **106A** *(1987), 89-102.*

[12] M. Yamada, Regular semigroups whose idempotents satisfy permutation identities, *Pacific Journal of Mathmatics* **21** *(1967), 371-392.*