数理解析研究所講究録
1506巻 2006年 143-166

Higher Than Second-Order Approximations Via Two-Stage Sampling For Selecting From Folded Normal Populations

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ABSTRACT

We consider the problem of calibrating k (≥ 2) channels or signal processors and select the *best* one, that is the one having the largest *signal-to-noise ratio* (SNR). Assuming a Gaussian distribution for the background white noise or clutter, the problem reduces to selection from k (≥ 2) normal populations the one with the largest value of the absolute mean. Assuming unknown means and a common unknown variance, we proceed under Bechhofer's (1954) *indifference-zone formulation.* Since the common variance is unknown, a single-stage procedure guaranteeing a minimum *probability of correct selection* (PCS) would not exist. We revisit a two-stage procedure proposed by Jeyaratnam and Panchapakesan (1998) and in order to reduce oversampling, we propose a modified two-stage selection methodology along the lines of Mukhopadhyay and Duggan (1997). In this paper, we provide asymptotic approximations for both the *average sample size* (ASS) and the minimum PCS for our modified two-stage procedure, successively leading up to significantly higher than second- or third-order terms. This analysis significantly advances the techniques found in Mukhopadhyay (1999).

Keywords: Average sample size; Data analysis; Higher-order approximations; Indifference-zone; Largest SNR; Multiple comparisons; Noise; Preference-zone; Probability of correct selection; Selection methodologies; Signal processing.

1. INTRODUCTION

Let $\Pi_1, \Pi_2, \dots, \Pi_k$ denote k (≥ 2) independent normal populations with *unknown* means $\mu_1, \mu_2, \dots, \mu_k$ respectively, and a common *unknown* variance σ^2, $-\infty < \mu_1, \mu_2, \dots, \mu_k < \infty$, $0 < \sigma < \infty$. Let $\theta_i = |\mu_i|$, $i = 1, \dots, k$. The populations are ordered according to the absolute values of their means. Let $\theta_{[1]} \leq \theta_{[2]} \leq \dots \leq \theta_{[k]}$ denote the ordered θ_i values, however, we assume no prior knowledge about the correspondence between the ordered and the unordered θ_i values.

Our *goal* is to select the population associated with $\theta_{[k]}$, the largest θ_i value. Since the populations have a common variance σ^2, this problem is equivalent to selecting the population associated with the largest θ_i/σ (that is, the largest $|\mu_i|/\sigma$) value, which is the *signal-to-noise ratio* (SNR) well-known in communication theory. In terms of comparing k different electronic devices, our goal is to select the device having the largest SNR. It may be noted that θ_i/σ is the customary *Mahalanobis distance* between the populations Π_i and $N(0, \sigma^2)$. That is, our goal is to identify that population which is most markedly different from the simple white Gaussian noise.

We formulate our problem by using the indifference-zone approach of Bechhofer (1954). Based on sample data, we wish to select one of the k signal processors (populations) and claim it to be the *best*, namely, the one associated with the parameter $\theta_{[k]}$ or equivalently the largest SNR. A *correct selection* (CS) is said to occur if the selected signal processor is indeed the one with mean signal strength $\max_{1\le i\le k}|\mu_i|$, that is $\theta_{[k]}$.

We also require that the *probability of correct selection* (PCS) is at least $P^*, 1/k < P^* < 1$, whenever $\theta_{[k]} - \theta_{[k-1]} \geq \delta > 0$, where δ and P^* are specified in advance by the experimenter. Let us denote the full parameter space

$$\Omega = \{\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k) : \theta_i \geq 0,\ i = 1, \ldots, k\}$$

and a parameter subspace

$$\Omega(\delta) = \{\boldsymbol{\theta} \in \Re^{+k} : \theta_{[k]} - \theta_{[k-1]} \geq \delta\}.$$

The parameter subspace $\Omega(\delta)$ is called the *preference-zone* and its complement $\Omega^c(\delta)$ is called the *indifference-zone.* We can rewrite the minimum PCS requirement for any selection rule under consideration as follows:

$$P(CS) \geq P^* \quad \text{whenever} \quad \boldsymbol{\theta} \in \Omega(\delta). \tag{1.1}$$

In other words, we wish to correctly identify the best signal processor with the minimum preassigned probability P^* whenever $\theta_{[k]}$ is ahead of $\theta_{[k-1]}$ by at least δ amount. Inside the indifference-zone $\Omega^c(\delta)$, that is when $\theta_{[k]} - \theta_{[k-1]} < \delta$, we are not interested to identify the largest SNR because all k SNRs are then judged to be too "small" under practical considerations. This fundamental approach was due to Bechhofer (1954).

2. FORMULATION AND AN EXISTING TWO-STAGE METHODOLOGY

Having recorded n observed signals $X_{i1}, ..., X_{in}$ from the i^{th} channel or processor, $i = 1, 2, ..., k$, let us denote:

$$\begin{gathered}\overline{X}_{in} = n^{-1}\textstyle\sum_{j=1}^{n} X_{ij},\ S_{in}^2 = (n-1)^{-1}\sum_{j=1}^{n}(X_{ij} - \overline{X}_{in})^2, \\ \text{Pooled Sample Variance: } S_n^2 = k^{-1}\textstyle\sum_{i=1}^{k} S_{in}^2 \text{ with the} \\ \text{degree of freedom } \nu = k(n-1).\end{gathered} \tag{2.1}$$

2.1. Known Variance: Fixed-Sample-Size Selection Rule

Rizvi (1971) considered the above selection problem assuming σ^2 to be known. In this case, he proposed a single-stage selection procedure based on a sample of size n from each signal processor. It was shown that the smallest sample size n necessary to satisfy the PCS requirement (1.1) is obtained by solving

$$\begin{gathered}T(\lambda) = P^* \text{ with } \lambda \equiv \lambda_{k,P^*} = \delta\sqrt{n}/\sigma, \text{ where} \\ T(\lambda) = 2(k-1)\textstyle\int_0^\infty \{2\Phi(u) - 1\}^{k-2}\{\Phi(-u+\lambda) + \Phi(-u-\lambda)\}d\Phi(u).\end{gathered} \tag{2.2}$$

Now, having determined n this way, the observed signals X_{ij} would be recorded from the i^{th} channel or processor, $j = 1, 2, ..., n$; $i = 1, 2, ..., k$.

Selection Rule: One would obtain $\overline{X}_{in}$ and $W_i = |\overline{X}_{in}|$, $i = 1, 2, ..., k$, and then select as best the channel or processor that yields the largest W_i. But, note that for given k, P and δ, the determination of the required sample size n from (2.2) would depend on the knowledge of σ.

Rizvi (1971) tabulated λ_{k,P^*} values satisfying the equation $T(\lambda) = P^*$. Table A.8 from Gibbons et al. (1977) also provides some of these λ_{k,P^*} values.

2.2. Unknown Variance: Existing Two-Stage Selection Rule $\mathcal{R}$

When σ remains unknown, no single-stage procedure would guarantee the PCS requirement (1.1). One may refer to Dudewicz (1971). Hence, one would necessarily implement an appropriate two-stage selection rule in the light of Bechhofer et al. (1954) that was developed in the light of Stein's (1945, 1949) pathbreaking two-stage estimation methodologies. In the specific problem of selecting the processor or channel with the best SNR, Jeyaratnam and Panchapakesan (1998) introduced the following two-stage procedure methodology.

Selection Rule $\mathcal{R}$: We initially record m (≥ 2) observed signals $X_{i1}, ..., X_{im}$ from the i^{th} channel or processor and obtain $\overline{X}_{im}$, S^2_{im}, $i = 1, 2, ..., k$ and the pooled sample variance S^2_m whose degree of freedom is $\nu = k(m-1)$. Let us determine

$$N \equiv N(S^2_m) = \max\left\{m, \left\langle \frac{h^2 S^2_m}{\delta^2} \right\rangle\right\}, \tag{2.3}$$

where $\langle y \rangle$ denotes the smallest integer $\geq y$ and $h \equiv h_{k,P^*,m}$ (> 0) is a design constant that is to be made precise shortly. If $N = m$, we need no more observed signals from any processor. Otherwise, that is if $N > m$, then we record a second sample of $(N - m)$ observed signals from each processor or channel. Based on the combined datasets from the two stages of sampling, we define $\overline{X}_{iN} = N^{-1}\sum_{j=1}^{N} X_{ij}$, the over-all mean of N observations from Π_i and let $W_i = |\overline{X}_{iN}|$, $i = 1, 2, ..., k$. Then, we select as best the channel or signal processor that yields the largest W_i.

Now, we show that the selection rule $\mathcal{R}$ from (2.3), with a proper choice of h, satisfies the PCS requirement (1.1). But, first we note the following facts that were given by Rizvi (1971):

(a) For any fixed n, the distribution function of W_i is given by:

$$F_n(w, \theta_i) = \Phi\left(\frac{\sqrt{n}}{\sigma}(w - \theta_i)\right) - \Phi\left(\frac{\sqrt{n}}{\sigma}(-w - \theta_i)\right), \ w \geq 0, \tag{2.4}$$

where $\Phi(\cdot)$ stands for the distribution function of a standard normal variable.

(b) For every fixed n $(\geq m)$, the infimum of $P(CS|N = n$ under rule $\mathcal{R})$ over the preference-zone $\Omega(\delta)$ occurs when $\theta_{[1]} = \cdots = \theta_{[k-1]} = 0$ and $\theta_{[k]} = \delta$. In other words, $\boldsymbol{\theta} = (0, ..., 0, \delta)$ is the *least favorable configuration* (LFC) under the selection rule $\mathcal{R}$ conditionally given $N = n$ whatever n $(\geq m)$.

Since the LFC does not depend on n, it follows that the LFC for $P(CS$ under rule $\mathcal{R})$ is given by $\boldsymbol{\theta} = (0, ..., 0, \delta)$. Recall our $T(\lambda)$ from (2.2) and observe that

$$dT(\lambda)/d\lambda = 2(k-1)\int_0^\infty \{2\Phi(u)-1\}^{k-2}\{\phi(-u+\lambda)-\phi(-u-\lambda)\}d\Phi(u),$$

which is positive since $\phi(-u+\lambda)-\phi(-u-\lambda) > 0$ for all $u > 0$ and $\lambda > 0$. Thus, $T(\lambda)$ is a non-decreasing function in λ (> 0). Hence, we have:

$$\begin{aligned} P_{LFC}(CS|\mathcal{R}) &= P_{\sigma^2}(W_{[k]} \geq W_i,\ i = 1, ..., k-1) \\ &= E_{\sigma^2}\left\{\int_0^\infty F_N^{k-1}(w,0)dF_N(w,\delta)\right\} \\ &= E_{\sigma^2}\left\{T\left(\delta\sqrt{N}/\sigma\right)\right\} && (2.5) \\ &\geq E\left\{T\left(h\sqrt{S_m^2/\sigma^2}\right)\right\} && (2.6) \end{aligned}$$

where the last step follows since it is obvious from (2.3) that $N \geq h^2 S_m^2/\delta^2$.

Let $g_\nu(u)$ be the chi-square density with $\nu = k(m-1)$ degrees of freedom. Thus, for a fixed set of values of k, P and m, the PCS requirement (1.1) is clearly satisfied if we choose h in such a way that

$$\int_0^\infty T\left(h\sqrt{u/\nu}\right) g_\nu(u)du = P^*. \tag{2.7}$$

In other words, suppose that one finds the design constant $h \equiv h_{k,P^*,m}$ satisfying (2.7) and implements Jeyaratnam and Panchapakesan's (1998) selection methodology $\mathcal{R}$ from (2.3). Then, one will have:

$$\min_{\boldsymbol{\theta}\in\Omega(\delta)} P_{LFC}(CS|\mathcal{R}) \geq P^* \text{ whatever be } \sigma^2. \text{ [\textit{Exact Consistency Property}]} \tag{2.8}$$

For its proof, one should refer to Jeyaratnam and Panchapakesan (1998).

The expression

$$n = \lambda^2\sigma^2/\delta^2 \tag{2.9}$$

is regarded as the optimal required fixed number of observed signals from each channel or processor had σ^2 been known. Now, from (2.3), one has

$$E_{\sigma^2}\left[\frac{N}{n}\right] \geq \frac{h^2}{\lambda^2} > 1,$$

which means that on an average the two-stage methodology (2.3) oversamples compared with n. This may not be surprising because the two-stage selection methodology is designed to work when σ^2 is unknown to begin with. What may be a damaging characteristic associated with the two-stage selection methodology is the following property:

$$\lim_{\delta\to 0} E_{\sigma^2}\left[\frac{N}{n}\right] = \frac{h^2}{\lambda^2} > 1. \text{ [\textit{First-Order Asymptotic Inefficiency}]} \tag{2.10}$$

The property (2.10) means that on an average the two-stage methodology (2.3) oversamples compared with n *even asymptotically*, that is, as δ becomes small. This is referred to as the *first-order asymptotic inefficiency* property.

In Table 2, we have included the values of both h and λ plus the oversampling percentages when $m = 10, 20, 30$, $k = 2, 3, 4, 5$, and $P^* = 0.95$. From these oversampling percentage figures, one gets a feel that oversampling can be rather substantial, especially when the pilot sample size m or the number of processors k is small.

Surely, for a fixed k, the oversampling percentages go down as m increases. This is clear from Table 2. But, in field trials, one may feel uneasy to start with m large in the absence of any knowledge about σ^2, because then m itself may overshoot n found in (2.9)! An appropriate choice of m is crucial and a new practical approach via information considerations has been put forward in Mukhopadhyay (2005).

In some field trials, one may reasonably assume that $\sigma > \sigma_\star$ *where* $\sigma_\star(> 0)$ is *known* from one's familiarity and experience in handling the kind of experiment and equipment on hand. Mukhopadhyay and Duggan (1997) first derived asymptotic *second-order characteristics* of a suitably modified Stein's (1945, 1949) two-stage procedure. They obtained asymptotic expansions of the upper and lower bounds for the *average sample size* (ASS) and the *coverage probability* (CP) respectively up to the orders $o(1)$ and $o(\delta^2)$ as $\delta \to 0$. Mukhopadhyay (1999) subsequently obtained asymptotic *third-order characteristics* by proving analogous asymptotic expansions of the upper and lower bounds for the ASS and the CP respectively up to the orders $o(\delta^2)$ and $o(\delta^4)$ as $\delta \to 0$. Later, Aoshima and Takada (2000) gave asymptotic expansions of both the ASS and the CP themselves up to the orders $o(1)$ and $o(\delta^2)$, respectively, as $\delta \to 0$. Those asymptotic expansions given by Aoshima and Takada (2000) are referred to as the *second-order asymptotic efficiency* property and as the *second-order asymptotic consistency* property, respectively. Such methodologies were successfully extended and implemented in solving some interesting problems in multiple comparisons by Aoshima and Aoki (2000), and Aoshima and Mukhopadhyay (2002), among others.

In the sections that follow, we assume that $\sigma > \sigma_\star$ *where* $\sigma_\star(> 0)$ is *known* in the context of our best SNR selection problem and accordingly modify the two-stage selection methodology $\mathcal{R}$ of Jeyaratnam and Panchapakesan (1998) from (2.3). Then, we provide asymptotic expansions of the upper and lower bounds for the ASS and the PCS under the LFC respectively up to the orders $o(\delta^6)$ and $o(\delta^8)$ as $\delta \to 0$. This advances the techniques of Mukhopadhyay (1999) in obtaining significantly sharper rates of convergences! Then, we also provide an approach to come up with the optimal choice of a design constant in lieu of h determined from (2.7).

3. A MODIFIED TWO-STAGE METHODOLOGY WITH ASYMPTOTIC APPROXIMATIONS

Under the assumption that $\sigma > \sigma_\star$ where $\sigma_\star(> 0)$ is known, we modify the original two-stage selection procedure $\mathcal{R}$ as follows. Clearly, the optimal fixed-sample-size n from (2.9) will exceed $\lambda^2\sigma_\star^2/\delta^2$ and hence we ought to choose the pilot sample size $m \approx \lambda^2\sigma_\star^2/\delta^2$.

Modified Selection Rule $\mathcal{R}_*$: We let

$$m = \max\left\{m_0, \left\langle \frac{\lambda^2\sigma_*^2}{\delta^2} \right\rangle\right\}, \tag{3.1}$$

where $m_0(\geq 2)$ is a fixed integer and λ is the constant defined by (2.2). According as (3.1), we initially record m observed signals $X_{i1}, ..., X_{im}$ from the i^{th} channel or processor and obtain $\overline{X}_{im}, S_{im}^2$, $i = 1, 2, ..., k$ and the pooled sample variance S_m^2 whose degree of freedom is $\nu = k(m-1)$. Then, with a proper choice of h (>0), we let

$$N = \max\left\{m, \left\langle \frac{h^2 S_m^2}{\delta^2} \right\rangle\right\}, \tag{3.2}$$

and implement this two-stage selection procedure as we did in the case of (2.3). Based on the combined datasets from the two stages of sampling, we define $\overline{X}_{iN} = N^{-1}\sum_{j=1}^{N} X_{ij}$, the over-all mean of N observations from Π_i and let $W_i = |\overline{X}_{iN}|$, $i = 1, 2, ..., k$. Then, we select as best the channel or signal processor that yields the largest W_i.

Notice that the distribution of the sample size N would involve the parameter σ^2 alone. Also, the random variables $I(N = b)$ and $\overline{X}_{ib}$ are obviously independent for all fixed integers $b \geq m$, where $I(\cdot)$ stands for the indicator function of $(\cdot)$. Hence, we have from (2.5) that

$$P_{LFC}(CS|\mathcal{R}_*) = E_{\sigma^2}\left[Q(N/n)\right], \tag{3.3}$$

with

$$Q(x) = T(\lambda\sqrt{x}), x > 0. \text{ Denote } Q^{(s)}(x) = d^sQ(x)/dx^s,\ s = 1, 2, ... \tag{3.4}$$

Note that $Q(1) = P^*$.

The following theorem provides the *fifth-order approximation* for the ASS associated with the modified two-stage selection procedure. Recall that an expansion for $E_{\sigma^2}[N-n]$ up to $o(1)$ is referred to as a *second-order* expansion. This constitutes a non-trivial extension of what is found in Mukhopadhyay and Duggan (1997, *second-order approximation*) and Mukhopadhyay (1999, *third-order approximation*).

Theorem 1. (Fifth-order approximation for ASS) *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), we have as $\delta \to 0$:*

$$\begin{aligned} a_1\rho + a_2\rho^2 n^{-1} + a_3\rho^3 n^{-2} + a_4\rho^4 n^{-3} + o(\delta^6) \leq E_{\sigma^2}[N-n] \leq (1 + a_1\rho) \\ + a_2\rho^2 n^{-1} + a_3\rho^3 n^{-2} + a_4\rho^4 n^{-3} + o(\delta^6), \end{aligned}$$

where $n = \lambda^2\sigma^2/\delta^2$, $\rho = \sigma^2/\sigma_*^2$, and the coefficients $a_1, ..., a_4$ are defined in (B.2).

Its proof and those of the following theorems are successively outlined in Appendix.

Remark 1. For the second-order approximation, we can obtain that

i) $E_{\sigma^2}[N-n] = a_1\rho + \frac{1}{2} + o(1)$ [*Second-Order Asymptotic Efficiency*],

ii) $P_{LFC}(CS|\mathcal{R}_*) = P^* + n^{-1}\left\{Q^{(1)}(1)\left(a_1\rho + \frac{1}{2}\right) + Q^{(2)}(1)k^{-1}\rho\right\} + o(\delta^2)$
[*Second-Order Asymptotic Consistency*],

along the lines of Aoshima and Takada (2000).

The following theorem provides the *third-order approximation* for the PCS under the LFC. This result is slightly different from an analogous third-order approximation found in Mukhopadhyay (1999). A subtle difference arises because in the Taylor expansion given by (B.19), we have actually evaluated $E_{\sigma^2}[n^{-3}(N-n)^3]$ whereas Mukhopadhyay (1999) used bounds for it.

Theorem 2. (Third-order approximation for PCS) *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), we have as $\delta \to 0$:*

$$P^* + \sum_{i=1}^{2} \alpha_i n^{-i} + o(\delta^4) \le P_{LFC}(CS|\mathcal{R}_*) \le P^* + \sum_{i=1}^{2} \beta_i n^{-i} + o(\delta^4),$$

where $n = \lambda^2\sigma^2/\delta^2$, $\rho = \sigma^2/\sigma_*^2$, and

$$\begin{aligned}
\alpha_1 &= Q^{(1)}(1)a_1\rho + Q^{(2)}(1)\left(1 + k^{-1}\rho\right), \\
\alpha_2 &= Q^{(1)}(1)a_2\rho^2 + Q^{(2)}(1)\left(\tfrac{1}{2} + a_1\rho + \left(k^{-1} + 2a_1k^{-1} + \tfrac{1}{2}a_1^2\right)\rho^2\right) \\
&\quad + Q^{(3)}(1)\left(\tfrac{1}{2}k^{-1}\rho + \tfrac{4}{3}k^{-2}\rho^2 + a_1k^{-1}\rho^2\right) + \tfrac{1}{2}Q^{(4)}(1)k^{-2}\rho^2,
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= Q^{(1)}(1)(1 + a_1\rho) + Q^{(2)}(1)\left(-1 + k^{-1}\rho\right), \\
\beta_2 &= Q^{(1)}(1)a_2\rho^2 + Q^{(2)}(1)\left(k^{-1} + 2a_1k^{-1} + \tfrac{1}{2}a_1^2\right)\rho^2 \\
&\quad + Q^{(3)}(1)\left(\tfrac{1}{2}k^{-1}\rho + \tfrac{4}{3}k^{-2}\rho^2 + a_1k^{-1}\rho^2\right) + \tfrac{1}{2}Q^{(4)}(1)k^{-2}\rho^2,
\end{aligned}$$

with (a_1, a_2) defined in (B.2).

The following theorem gives the *fourth-order approximation* for the PCS under the LFC which is more involved than what we have stated in Theorem 2.

Theorem 3. (Fourth-order approximation for PCS) *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), we have as $\delta \to 0$:*

$$P^* + \sum_{i=1}^{3} \alpha_i n^{-i} + o(\delta^6) \le P_{LFC}(CS|\mathcal{R}_*) \le P^* + \sum_{i=1}^{3} \beta_i n^{-i} + o(\delta^6),$$

where $n = \lambda^2\sigma^2/\delta^2$, $\rho = \sigma^2/\sigma_*^2$ and

$$\begin{aligned}
\alpha_1 &= Q^{(1)}(1)a_1\rho + Q^{(2)}(1)\left(1 + k^{-1}\rho\right) - Q^{(3)}(1) + \tfrac{2}{3}Q^{(4)}(1), \\
\alpha_2 &= Q^{(1)}(1)a_2\rho^2 + Q^{(2)}(1)\left(\tfrac{1}{2} + a_1\rho + \left(k^{-1} + 2a_1k^{-1} + \tfrac{1}{2}a_1^2\right)\rho^2\right) \\
&\quad + Q^{(3)}(1)\left(-\tfrac{1}{2} - a_1\rho + k^{-1}\left(\tfrac{4}{3}k^{-1} + a_1\right)\rho^2\right) \\
&\quad + Q^{(4)}(1)\left(\tfrac{1}{2} + k^{-1}\rho + \tfrac{1}{2}k^{-2}\rho^2 + a_1\rho\right),
\end{aligned}$$

$$
\begin{aligned}
\alpha_3 = {} & Q^{(1)}(1)a_3\rho^3 \\
& +Q^{(2)}(1)\left(a_2\rho^2 + \left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1}+a_2\right)\right)\rho^3\right) \\
& +Q^{(3)}(1)\left\{-a_2\rho^2 + \left(k^{-1}a_1\left(1+4k^{-1}\right) + 2k^{-1}a_1^2 + \tfrac{1}{6}a_1^3\right.\right. \\
& \quad \left.\left. + k^{-1}\left(\tfrac{8}{3}k^{-1} + a_2\right)\right)\rho^3\right\} \\
& +Q^{(4)}(1)\left\{\tfrac{1}{6} + \tfrac{1}{2}k^{-1}\rho + \tfrac{1}{2}a_1\rho + k^{-1}\left(\tfrac{4}{3}k^{-1}+a_1\right)\rho^2 + a_2\rho^2\right. \\
& \quad \left. + \left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^2 + k^{-1}\left(k^{-1} + 2k^{-2} + \tfrac{10}{3}k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^3\right\} \\
& +Q^{(5)}(1)\left(\tfrac{1}{4}k^{-2}\rho^2 + \tfrac{4}{3}k^{-3}\rho^3 + \tfrac{1}{2}k^{-2}a_1\rho^3\right) + \tfrac{1}{6}Q^{(6)}(1)k^{-3}\rho^3,
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 = {} & Q^{(1)}(1)\left(1+a_1\rho\right) + Q^{(2)}(1)\left(-1+k^{-1}\rho\right) + Q^{(3)}(1) - \tfrac{2}{3}Q^{(4)}(1), \\
\beta_2 = {} & Q^{(1)}(1)a_2\rho^2 + Q^{(2)}(1)\left(k^{-1} + 2a_1k^{-1} + \tfrac{1}{2}a_1^2\right)\rho^2 \\
& +Q^{(3)}(1)\left\{\tfrac{1}{2} + k^{-1}\rho + a_1\rho + k^{-1}\left(\tfrac{4}{3}k^{-1}+a_1\right)\rho^2\right\} \\
& +Q^{(4)}(1)\left(-\tfrac{1}{2} - k^{-1}\rho + \tfrac{1}{2}k^{-2}\rho^2 - a_1\rho\right), \\
\beta_3 = {} & Q^{(1)}(1)a_3\rho^3 + Q^{(2)}(1)\left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1}+a_2\right)\right)\rho^3 \\
& +Q^{(3)}(1)\left\{\tfrac{1}{6} + \tfrac{1}{2}a_1\rho + \left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^2 + a_2\rho^2\right. \\
& \quad \left. + \left(k^{-1}a_1\left(1+4k^{-1}\right) + 2k^{-1}a_1^2 + \tfrac{1}{6}a_1^3 + \tfrac{8}{3}k^{-2} + k^{-1}a_2\right)\rho^3\right\} \\
& +Q^{(4)}(1)\left\{-\tfrac{1}{6} - \tfrac{1}{2}a_1\rho - \left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^2 - a_2\rho^2\right. \\
& \quad \left. + k^{-1}\left(k^{-1} + 2k^{-2} + \tfrac{10}{3}k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^3\right\} \\
& +Q^{(5)}(1)\left(\tfrac{1}{4}k^{-2}\rho^2 + \tfrac{4}{3}k^{-3}\rho^3 + \tfrac{1}{2}a_1k^{-2}\rho^3\right) + \tfrac{1}{6}Q^{(6)}(1)k^{-3}\rho^3,
\end{aligned}
$$

with (a_1, a_2, a_3) defined in (B.2).

The following theorem gives *fifth-order approximation* for the PCS under the LFC which is more involved than what we have stated in Theorem 3.

Theorem 4. (Fifth-order approximation for PCS) *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), we have as $\delta \to 0$:*

$$
P^* + \sum_{i=1}^{4} \alpha_i n^{-i} + o(\delta^8) \le P_{LFC}(CS|\mathcal{R}_*) \le P^* + \sum_{i=1}^{4} \beta_i n^{-i} + o(\delta^8),
$$

where $n = \lambda^2\sigma^2/\delta^2$, $\rho = \sigma^2/\sigma_*^2$ and

$$
\begin{aligned}
\alpha_1 = {} & Q^{(1)}(1)a_1\rho + Q^{(2)}(1)\left(1+k^{-1}\rho\right) - Q^{(3)}(1) + \tfrac{2}{3}Q^{(4)}(1) \\
& -\tfrac{1}{3}Q^{(5)}(1) + \tfrac{2}{15}Q^{(6)}(1), \\
\alpha_2 = {} & Q^{(1)}(1)a_2\rho^2 + Q^{(2)}(1)\left(\tfrac{1}{2} + a_1\rho + \left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^2\right) \\
& +Q^{(3)}(1)\left(-\tfrac{1}{2} - a_1\rho + k^{-1}\left(\tfrac{4}{3}k^{-1}+a_1\right)\rho^2\right) \\
& +Q^{(4)}(1)\left(\tfrac{1}{2} + k^{-1}\rho + \tfrac{1}{2}k^{-2}\rho^2 + a_1\rho\right) + Q^{(5)}(1)\left(-\tfrac{1}{3} - k^{-1}\rho - \tfrac{2}{3}a_1\rho\right) \\
& +Q^{(6)}(1)\left(\tfrac{1}{6} + \tfrac{2}{3}k^{-1}\rho + \tfrac{1}{3}a_1\rho\right),
\end{aligned}
$$

$$
\begin{aligned}
\alpha_3 = {} & Q^{(1)}(1)a_3\rho^3 \\
& +Q^{(2)}(1)\left\{a_2\rho^2 + \left(k^{-1} + k^{-1}a_1{}^2 + 2k^{-1}a_2 + a_1\left(2k^{-1} + a_2\right)\right)\rho^3\right\} \\
& +Q^{(3)}(1)\left\{-a_2\rho^2 + \left(a_1\left(k^{-1} + 4k^{-2}\right) + 2k^{-1}a_1^2 + \tfrac{1}{6}a_1^3\right.\right. \\
& \quad \left.\left. + \tfrac{8}{3}k^{-2} + k^{-1}a_2\right)\rho^3\right\} \\
& +Q^{(4)}(1)\left\{\tfrac{1}{6} + \tfrac{1}{2}k^{-1}\rho + \tfrac{1}{2}a_1\rho + \left(\tfrac{4}{3}k^{-2} + a_2 + k^{-1} + 3k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^2\right. \\
& \quad \left. + k^{-1}\left(k^{-1} + 2k^{-2} + \tfrac{10}{3}k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^3\right\} \\
& +Q^{(5)}(1)\left\{-\tfrac{1}{6} - \tfrac{1}{2}k^{-1}\rho - \tfrac{1}{2}a_1\rho + k^{-2}\left(\tfrac{4}{3}k^{-1} + \tfrac{1}{2}a_1\right)\rho^3\right. \\
& \quad \left. - \left(\tfrac{4}{3}k^{-2} + 3k^{-1}a_1 + k^{-1} + \tfrac{1}{2}a_1^2 + \tfrac{2}{3}a_2\right)\rho^2\right\} \\
& +Q^{(6)}(1)\left\{\tfrac{1}{9} + \tfrac{1}{2}k^{-1}\rho + \tfrac{1}{2}k^{-2}\rho^2 + \tfrac{1}{6}k^{-3}\rho^3 + \tfrac{1}{3}a_1\rho + k^{-1}\left(\tfrac{4}{3}k^{-1} + a_1\right)\rho^2\right. \\
& \quad \left. + \left(\tfrac{2}{3}k^{-1} + \tfrac{4}{3}k^{-1}a_1 + \tfrac{1}{3}a_1^2\right)\rho^2 + \tfrac{1}{3}a_2\rho^2\right\}, \\
\alpha_4 = {} & Q^{(1)}(1)a_4\rho^4 + Q^{(2)}(1)\left\{a_3\rho^3 + \left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2\left(1 + a_1\right)\right.\right. \\
& \quad \left.\left. + \tfrac{1}{2}a_2^2 + 2k^{-1}a_3 + a_1\left(2k^{-1} + a_3\right)\right)\rho^4\right\} \\
& +Q^{(3)}(1)\left\{-a_3\rho^3 + \left(k^{-1}a_1^3 + a_1^2\left(2k^{-1} + 4k^{-2} + \tfrac{1}{2}a_2\right)\right.\right. \\
& \quad \left.\left. + k^{-1}a_1\left(1 + 8k^{-1} + 4a_2\right) + k^{-1}\left(4k^{-1} + a_2 + 4k^{-1}a_2 + a_3\right)\right)\rho^4\right\} \\
& +Q^{(4)}(1)\left\{\tfrac{1}{24} + \tfrac{1}{6}a_1\rho + \left(\tfrac{1}{2}k^{-1} + k^{-1}a_1 + \tfrac{1}{4}a_1^2 + \tfrac{1}{2}a_2\right)\rho^2\right. \\
& \quad + \left(4k^{-2}a_1 + \tfrac{1}{6}a_1^3 + 3k^{-1}(a_1 + a_1^2 + a_2) + a_1a_2 + k^{-1} + \tfrac{8}{3}k^{-2} + a_3\right)\rho^3 \\
& \quad + \left(a_1^2\left(\tfrac{1}{2}k^{-1} + 7k^{-2}\right) + k^{-1}a_1^3 + \tfrac{1}{24}a_1^4 + k^{-1}a_1\left(\tfrac{20}{3}k^{-1} + 8k^{-2} + a_2\right)\right. \\
& \quad \left.\left. + k^{-2}\left(\tfrac{3}{2} + 6k^{-1} + \tfrac{10}{3}a_2\right)\right)\rho^4\right\} \\
& +Q^{(5)}(1)\left\{-\tfrac{1}{24} - \tfrac{1}{6}a_1\rho - \left(\tfrac{1}{2}k^{-1} + k^{-1}a_1 + \tfrac{1}{4}a_1^2 + \tfrac{1}{2}a_2\right)\rho^2\right. \\
& \quad - \left(k^{-1} + a_1\left(3k^{-1} + 4k^{-2} + a_2\right) + 3k^{-1}a_1^2 + \tfrac{1}{6}a_1^3 + 3k^{-1}a_2\right. \\
& \quad \left. + \tfrac{8}{3}k^{-2} + \tfrac{2}{3}a_3\right)\rho^3 + k^{-1}\left(a_1\left(k^{-1} + \tfrac{26}{3}k^{-2}\right) + \tfrac{8}{3}k^{-1}a_1^2 + \tfrac{1}{6}a_1^3\right. \\
& \quad \left.\left. + k^{-1}\left(4k^{-1} + \tfrac{16}{5}k^{-2} + \tfrac{1}{2}a_2\right)\right)\rho^4\right\} \\
& +Q^{(6)}(1)\left\{\tfrac{1}{24} + \tfrac{1}{6}\left(a_1 + k^{-1}\right)\rho + \left(\tfrac{11}{12}k^{-2} + \tfrac{3}{2}k^{-1}a_1 + \tfrac{1}{2}k^{-1} + \tfrac{1}{4}a_1^2 + \tfrac{1}{3}a_2\right)\rho^2\right. \\
& \quad + \left(\tfrac{2}{3}k^{-1} + \tfrac{11}{3}k^{-2} + \tfrac{10}{3}k^{-3} + \tfrac{7}{3}k^{-1}a_1 + \tfrac{47}{6}k^{-2}a_1 + \tfrac{19}{6}k^{-1}a_1^2 + \tfrac{1}{6}a_1^3\right. \\
& \quad \left.\left. + \tfrac{7}{3}k^{-1}a_2 + \tfrac{2}{3}a_1a_2 + \tfrac{1}{3}a_3\right)\rho^3 + k^{-2}\left(\tfrac{1}{2}k^{-1} + \tfrac{26}{9}k^{-2} + \tfrac{7}{3}k^{-1}a_1 + \tfrac{1}{4}a_1^2\right)\rho^4\right\} \\
& +Q^{(7)}(1)\left(\tfrac{1}{12}k^{-3}\rho^3 + \tfrac{2}{3}k^{-4}\rho^4 + \tfrac{1}{6}k^{-3}a_1\rho^4\right) + \tfrac{1}{24}Q^{(8)}(1)k^{-4}\rho^4,
\end{aligned}
$$

$$
\begin{aligned}
\beta_1 = {} & Q^{(1)}(1)(1 + a_1\rho) + Q^{(2)}(1)\left(-1 + k^{-1}\rho\right) + Q^{(3)}(1) - \tfrac{2}{3}Q^{(4)}(1) \\
& +\tfrac{1}{3}Q^{(5)}(1) - \tfrac{2}{15}Q^{(6)}(1), \\
\beta_2 = {} & Q^{(1)}(1)a_2\rho^2 + Q^{(2)}(1)\left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^2 \\
& +Q^{(3)}(1)\left(\tfrac{1}{2} + k^{-1}\rho + a_1\rho + k^{-1}\left(\tfrac{4}{3}k^{-1} + a_1\right)\rho^2\right) \\
& +Q^{(4)}(1)\left(-\tfrac{1}{2} - k^{-1}\rho + \tfrac{1}{2}k^{-2}\rho^2 - a_1\rho\right) + Q^{(5)}(1)\left(\tfrac{1}{3} + k^{-1}\rho + \tfrac{2}{3}a_1\rho\right) \\
& +Q^{(6)}(1)\left(-\tfrac{1}{6} - \tfrac{2}{3}k^{-1}\rho - \tfrac{1}{3}a_1\rho\right),
\end{aligned}
$$

$$
\begin{aligned}
\beta_3 = {} & Q^{(1)}(1)a_3\rho^3 + Q^{(2)}(1)\left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1} + a_2\right)\right)\rho^3 \\
& + Q^{(3)}(1)\{\tfrac{1}{6} + \tfrac{1}{2}a_1\rho + \left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2 + a_2\right)\rho^2 \\
& \quad + \left(a_1\left(k^{-1} + 4k^{-2}\right) + 2k^{-1}a_1^2 + \tfrac{1}{6}a_1^3 + \tfrac{8}{3}k^{-2} + k^{-1}a_2\right)\rho^3\} \\
& + Q^{(4)}(1)\{-\tfrac{1}{6} - \tfrac{1}{2}a_1\rho - \left(k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2 + a_2\right)\rho^2 \\
& \quad + k^{-1}\left(k^{-1} + 2k^{-2} + \tfrac{10}{3}k^{-1}a_1 + \tfrac{1}{2}a_1^2\right)\rho^3\} \\
& + Q^{(5)}(1)\{\tfrac{1}{6} + \tfrac{1}{2}k^{-1}\rho + \tfrac{1}{2}a_1\rho + k^{-2}\left(\tfrac{4}{3}k^{-1} + \tfrac{1}{2}a_1\right)\rho^3 \\
& \quad + \left(\tfrac{11}{6}k^{-2} + a_1k^{-1} + k^{-1} + 2k^{-1}a_1 + \tfrac{1}{2}a_1^2 + \tfrac{2}{3}a_2\right)\rho^2\} \\
& + Q^{(6)}(1)\{-\tfrac{1}{9} - \tfrac{1}{2}k^{-1}\rho - \tfrac{1}{2}k^{-2}\rho^2 + \tfrac{1}{6}k^{-3}\rho^3 - \tfrac{1}{3}a_1\rho \\
& \quad - \left(\tfrac{4}{3}k^{-2} + a_1k^{-1} + \tfrac{2}{3}k^{-1} + \tfrac{4}{3}k^{-1}a_1 + \tfrac{1}{3}a_1^2 + \tfrac{1}{3}a_2\right)\rho^2\}, \\
\beta_4 = {} & Q^{(1)}(1)a_4\rho^4 + Q^{(2)}(1)\{k^{-1} + k^{-1}a_1^2 + a_2\left(2k^{-1} + 2k^{-1}a_1\right) \\
& \quad + \tfrac{1}{2}a_2^2 + 2k^{-1}a_3 + a_1\left(2k^{-1} + a_3\right)\}\rho^4 \\
& + Q^{(3)}(1)\{\tfrac{1}{2}a_2\rho^2 + \left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1} + a_2\right) + a_3\right)\rho^3 \\
& \quad + \left(k^{-1}{a_1}^3 + {a_1}^2\left(2k^{-1} + 4k^{-2} + \tfrac{1}{2}a_2\right) + k^{-1}a_1\left(1 + 8k^{-1} + 4a_2\right)\right. \\
& \quad \left. + k^{-1}\left(4k^{-1} + a_2 + 4k^{-1}a_2 + a_3\right)\right)\rho^4\} \\
& + Q^{(4)}(1)\{-\tfrac{1}{2}a_2\rho^2 - \left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1} + a_2\right) + a_3\right)\rho^3 \\
& \quad + \left({a_1}^2\left(\tfrac{1}{2}k^{-1} + 7k^{-2}\right) + k^{-1}a_1^3 + \tfrac{1}{24}a_1^4 + k^{-1}a_1\left(\tfrac{20}{3}k^{-1} + 8k^{-2} + a_2\right)\right. \\
& \quad \left. + k^{-2}\left(\tfrac{3}{2} + 6k^{-1} + \tfrac{10}{3}a_2\right)\right)\rho^4\} \\
& + Q^{(5)}(1)\{\tfrac{1}{24} + \tfrac{1}{6}k^{-1}\rho + \tfrac{1}{6}a_1\rho + \left(\tfrac{2}{3}k^{-2} + \tfrac{3}{2}k^{-1}a_1 + \tfrac{1}{2}a_2 + \tfrac{1}{2}k^{-1} + \tfrac{1}{4}a_1^2\right)\rho^2 \\
& \quad + \left(k^{-1} + \tfrac{11}{3}k^{-2} + 2k^{-3} + 3k^{-1}a_1 + \tfrac{22}{3}k^{-2}a_1 + \tfrac{7}{2}k^{-1}a_1^2 + \tfrac{1}{6}a_1^3\right. \\
& \quad \left. + 3k^{-1}a_2 + a_1a_2 + \tfrac{2}{3}a_3\right)\rho^3 + k^{-1}\left(\tfrac{1}{6}a_1^3 + \tfrac{8}{3}k^{-1}a_1^2 + a_1\left(k^{-1} + \tfrac{26}{3}k^{-2}\right)\right. \\
& \quad \left. + k^{-1}\left(4k^{-1} + \tfrac{16}{5}k^{-2} + \tfrac{1}{2}a_2\right)\right)\rho^4\} \\
& + Q^{(6)}(1)\{-\tfrac{1}{24} - \tfrac{1}{6}k^{-1}\rho - \tfrac{1}{6}a_1\rho \\
& \quad - \left(\tfrac{1}{2}k^{-1} + k^{-1}a_1 + \tfrac{1}{4}a_1^2 + \tfrac{1}{3}a_2 + k^{-1}\left(\tfrac{2}{3}k^{-1} + \tfrac{1}{2}a_1\right)\right)\rho^2 \\
& \quad - \left(\tfrac{2}{3}k^{-1} - \tfrac{11}{3}k^{-2} - 2k^{-3} - \tfrac{7}{3}k^{-1}a_1 - \tfrac{22}{3}k^{-2}a_1 - \tfrac{19}{6}k^{-1}a_1^2 - \tfrac{1}{6}a_1^3\right. \\
& \quad \left. - \tfrac{7}{3}k^{-1}a_2 - \tfrac{2}{3}a_1a_2 - \tfrac{1}{3}a_3\right)\rho^3 + k^{-2}\left(\tfrac{1}{2}k^{-1} + \tfrac{26}{9}k^{-2} + \tfrac{7}{3}k^{-1}a_1 + \tfrac{1}{4}a_1^2\right)\rho^4\} \\
& + Q^{(7)}(1)\{\tfrac{1}{12}k^{-3}\rho^3 + \tfrac{2}{3}k^{-4}\rho^4 + \tfrac{1}{6}k^{-3}a_1\rho^4\} + \tfrac{1}{24}Q^{(8)}(1)k^{-4}\rho^4,
\end{aligned}
$$

with (a_1, a_2, a_3, a_4) *defined in (B.2).*

Remark 2. It should be noted that the average of the coefficients of order n^{-i}, namely $(\alpha_i + \beta_i)/2$, remains same among three approximations found in Theorems 2-4 for the PCS under the LFC. In view of Remark 1 (ii), it is shown that $|P_{LFC}(CS|\mathcal{R}_*) - \{P^* + \frac{1}{2}(\alpha_1 + \beta_1)n^{-1}\}| = o(\delta^2)$. In Section 4, we investigate the role of the expression $P^* + \sum_{i=1}^{4} \frac{1}{2}(\alpha_i + \beta_i)n^{-i}$ as an approximation for the PCS under the LFC.

4. AN APPROXIMATION FOR THE OPTIMAL DESIGN CONSTANT

The question we address now is this: Can we find a suitable design constant h_* so that the associated modified two-stage selection methodology $\mathcal{R}_*$ from (3.1)-(3.2) would have higher-order *asymptotic consistency* property, namely $P_{LFC}(CS|\mathcal{R}_*) = P^* + o(\delta^8)$ as $\delta \to 0$? In view of Remark 2, we feel that we should be able to come up with such h_*, which will then be referred to as an "optimal" choice given the sharpest rate of convergence available at this point. The following results give precise statements of what is at stake.

Theorem 5. (Optimal design constant) *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), we have as $\delta \to 0$ that*

$$P^* + \textstyle\sum_{i=1}^4 \frac{1}{2}(\alpha_i + \beta_i)n^{-i} = P^* + o(\delta^8) \tag{4.1}$$

if the design constant h^2 used in (3.2) can be asymptotically expanded as follows:

$$h^2 = \lambda^2\,(1 + a_1 m^{-1} + a_2 m^{-2} + a_3 m^{-3} + a_4 m^{-4}) + o(m^{-4}). \tag{4.2}$$

Here, m comes from (3.1), and $\rho = \sigma^2/\sigma_*^2$, $Q_j = Q^{(j)}(1)/Q^{(1)}(1)$, $j = 2, ..., 8$, with $a_i \equiv a_i(\rho)$, $i = 1, ..., 4$, defined as:

$$
\begin{aligned}
a_1(\rho) = {} & -\tfrac{1}{2}\rho^{-1} - k^{-1}Q_2, \\
a_2(\rho) = {} & -(8\rho^2)^{-1}Q_2 + (k\rho)^{-1}Q_2 - k^{-1}Q_2 + 2k^{-2}{Q_2}^2 - \tfrac{1}{2}k^{-2}{Q_2}^3 \\
& -\tfrac{4}{3}k^{-2}Q_3 + k^{-2}Q_2Q_3 - \tfrac{1}{2}k^{-2}Q_4, \\
a_3(\rho) = {} & -k^{-1}Q_2 - \tfrac{1}{4}(k\rho^2)^{-1}Q_2 + (k\rho)^{-1}Q_2 + \tfrac{1}{4}(k\rho^2)^{-1}Q_2^2 + 4k^{-2}Q_2^2 \\
& -3(k^2\rho)^{-1}Q_2^2 - \tfrac{1}{8}(k\rho^2)^{-1}Q_2^3 - k^{-2}Q_2^3 + (k^2\rho)^{-1}Q_2^3 - 5k^{-3}Q_2^3 \\
& +3k^{-3}Q_2^4 - \tfrac{1}{2}k^{-3}Q_2^5 - \tfrac{8}{3}k^{-2}Q_3 + 2(k^2\rho)^{-1}Q_3 + \tfrac{1}{4}(k\rho^2)^{-1}Q_2Q_3 \\
& +2k^{-2}Q_2Q_3 - 2(k^2\rho)^{-1}Q_2Q_3 + \tfrac{20}{3}k^{-3}Q_2Q_3 - \tfrac{22}{3}k^{-3}Q_2^2Q_3 \\
& +\tfrac{5}{3}k^{-3}Q_2^3Q_3 + \tfrac{4}{3}k^{-3}Q_3^2 - k^{-3}Q_2Q_3^2 - \tfrac{1}{8}(k\rho^2)^{-1}Q_4 - k^{-2}Q_4 \\
& +(k^2\rho)^{-1}Q_4 - 2k^{-3}Q_4 + \tfrac{13}{3}k^{-3}Q_2Q_4 - k^{-3}Q_2^2Q_4 + \tfrac{1}{2}k^{-3}Q_3Q_4 \\
& -\tfrac{4}{3}k^{-3}Q_5 + \tfrac{1}{2}k^{-3}Q_2Q_5 - \tfrac{1}{6}k^{-3}Q_6, \\
a_4(\rho) = {} & -k^{-1}Q_2 - \tfrac{1}{4}(k\rho^2)^{-1}Q_2 + (k\rho)^{-1}Q_2 - \tfrac{1}{8}(k\rho^3)^{-1}Q_2^2 + \tfrac{1}{4}(k\rho^2)^{-1}Q_2^2 \\
& +\tfrac{3}{2}(k^2\rho^2)^{-1}Q_2^2 + 6k^{-2}Q_2^2 - 6(k^2\rho)^{-1}Q_2^2 - \tfrac{1}{128}\rho^{-4}Q_2^3 + \tfrac{1}{8}(k\rho^3)^{-1}Q_2^3 \\
& -\tfrac{3}{2}k^{-2}Q_2^3 - \tfrac{1}{8}(k\rho^2)^{-1}Q_2^3 - \tfrac{3}{2}(k^2\rho^2)^{-1}Q_2^3 + 10(k^3\rho)^{-1}Q_2^3 \\
& +2(k^2\rho)^{-1}Q_2^3 - 15k^{-3}Q_2^3 + \tfrac{3}{4}(k^2\rho^2)^{-1}Q_2^4 - \tfrac{15}{2}(k^3\rho)^{-1}Q_2^4 \\
& +14k^{-4}Q_2^4 + 9k^{-3}Q_2^4 - \tfrac{3}{16}(k^2\rho^2)^{-1}Q_2^5 - \tfrac{3}{2}k^{-3}Q_2^5 + \tfrac{3}{2}(k^3\rho)^{-1}Q_2^5 \\
& -14k^{-4}Q_2^5 + 5k^{-4}Q_2^6 - \tfrac{5}{8}k^{-4}Q_2^7 - 4k^{-2}Q_3 - (k^2\rho^2)^{-1}Q_3 \\
& +4(k^2\rho)^{-1}Q_3 + \tfrac{1}{64}\rho^{-4}Q_2Q_3 - \tfrac{1}{4}(k\rho^3)^{-1}Q_2Q_3 + \tfrac{1}{4}(k\rho^2)^{-1}Q_2Q_3 \\
& +3k^{-2}Q_2Q_3 + 2(k^2\rho^2)^{-1}Q_2Q_3 - 4(k^2\rho)^{-1}Q_2Q_3 + 20k^{-3}Q_2Q_3 \\
& -\tfrac{40}{3}(k^3\rho)^{-1}Q_2Q_3 - 22k^{-3}Q_2^2Q_3 - 28k^{-4}Q_2^2Q_3 + 5k^{-3}Q_2^3Q_3 \\
& -\tfrac{5}{3}(k^2\rho^2)^{-1}Q_2^2Q_3 + \tfrac{55}{3}(k^3\rho)^{-1}Q_2^2Q_3 + \tfrac{5}{8}(k^2\rho^2)^{-1}Q_2^3Q_3 \\
& -5(k^3\rho)^{-1}Q_2^3Q_3 + 42k^{-4}Q_2^3Q_3 - \tfrac{58}{3}k^{-4}Q_2^4Q_3 + \tfrac{35}{12}k^{-4}Q_2^5Q_3 \\
& +\tfrac{1}{6}(k^2\rho^2)^{-1}Q_3^2 + 4k^{-3}Q_3^2 - \tfrac{10}{3}(k^3\rho)^{-1}Q_3^2 + \tfrac{16}{3}k^{-4}Q_3^2 \\
& -\tfrac{3}{8}(k^2\rho^2)^{-1}Q_2Q_3^2 - 3k^{-3}Q_2Q_3^2 + 3(k^3\rho)^{-1}Q_2Q_3^2 - \tfrac{176}{9}k^{-4}Q_2Q_3^2 \\
& +\tfrac{50}{3}k^{-4}{Q_2}^2Q_3^2 - \tfrac{11}{3}k^{-4}Q_2^3Q_3^2 - \tfrac{4}{3}k^{-4}Q_3^3 + k^{-4}Q_2Q_3^3 - \tfrac{1}{384}\rho^{-4}Q_4 \\
& +\tfrac{1}{8}(k\rho^3)^{-1}Q_4 - \tfrac{1}{8}(k\rho^2)^{-1}Q_4 - \tfrac{3}{2}k^{-2}Q_4 - \tfrac{3}{4}(k^2\rho^2)^{-1}Q_4 + 2(k^2\rho)^{-1}Q_4 \\
& -6k^{-3}Q_4 + 4(k^3\rho)^{-1}Q_4 + \tfrac{11}{12}(k^2\rho^2)^{-1}Q_2Q_4 + 13k^{-3}Q_2Q_4
\end{aligned}
$$

$$
\begin{aligned}
&-\tfrac{65}{6}(k^3\rho)^{-1}Q_2Q_4 + 12k^{-4}Q_2Q_4 - \tfrac{3}{8}(k^2\rho^2)^{-1}Q_2^2Q_4 - 3k^{-3}Q_2^2Q_4 \\
&+3(k^3\rho)^{-1}Q_2^2Q_4 - \tfrac{76}{3}k^{-4}Q_2^2Q_4 + 12k^{-4}Q_2^3Q_4 - \tfrac{43}{24}k^{-4}Q_2^4Q_4 \\
&+\tfrac{3}{16}(k^2\rho^2)^{-1}Q_3Q_4 + \tfrac{3}{2}k^{-3}Q_3Q_4 - \tfrac{3}{2}(k^3\rho)^{-1}Q_3Q_4 + \tfrac{76}{9}k^{-4}Q_3Q_4 \\
&-\tfrac{38}{3}k^{-4}Q_2Q_3Q_4 + \tfrac{13}{4}k^{-4}Q_2^2Q_3Q_4 - \tfrac{1}{2}k^{-4}Q_3^2Q_4 + \tfrac{5}{3}k^{-4}Q_4^2 \\
&-\tfrac{5}{8}k^{-4}Q_2Q_4^2 - \tfrac{1}{6}(k^2\rho^2)^{-1}Q_5 - 4k^{-3}Q_5 + \tfrac{10}{3}(k^3\rho)^{-1}Q_5 - \tfrac{16}{5}k^{-4}Q_5 \\
&+\tfrac{3}{16}(k^2\rho^2)^{-1}Q_2Q_5 + \tfrac{3}{2}k^{-3}Q_2Q_5 - \tfrac{3}{2}(k^3\rho)^{-1}Q_2Q_5 + \tfrac{34}{3}k^{-4}Q_2Q_5 \\
&-6k^{-4}Q_2^2Q_5 + \tfrac{11}{12}k^{-4}Q_2^3Q_5 + 2k^{-4}Q_3Q_5 - k^{-4}Q_2Q_3Q_5 - \tfrac{1}{2}k^{-3}Q_6 \\
&+\tfrac{1}{4}k^{-4}Q_4Q_5 - \tfrac{1}{16}(k^2\rho^2)^{-1}Q_6 + \tfrac{1}{2}(k^3\rho)^{-1}Q_6 - \tfrac{26}{9}k^{-4}Q_6 + \tfrac{8}{3}k^{-4}Q_2Q_6 \\
&-\tfrac{5}{12}k^{-4}{Q_2}^2Q_6 + \tfrac{1}{6}k^{-4}Q_3Q_6 - \tfrac{2}{3}k^{-4}Q_7 + \tfrac{1}{6}k^{-4}Q_2Q_7 - \tfrac{1}{24}k^{-4}Q_8.
\end{aligned}
$$

Remark 3. The specific design constant h satisfying (2.7) was used in (2.3) so that one could claim the *exact consistency* property found in (2.8). This specific design constant h^2 can also be asymptotically expanded in the following form:

$$
\begin{gathered}
\lambda^2\left(1 + a_{1E}m^{-1} + a_{2E}m^{-2} + a_{3E}m^{-3} + a_{4E}m^{-4}\right) + O(m^{-5}), \text{ with} \\
a_{1E} = ak^{-1},\ a_{2E} = ak^{-1} + bk^{-2},\ a_{3E} = ak^{-1} + 2bk^{-2} + ck^{-3}, \\
\text{and } a_{4E} = ak^{-1} + 3bk^{-2} + 3ck^{-3} + dk^{-4},
\end{gathered}
\tag{4.3}
$$

with the coefficients $a, b, c,$ and d the same as those in Lemma 1 in Section B.1. From Remark 1, one has as $\delta \to 0$:

i) $E_{\sigma^2}[N-n] = -k^{-1}Q_2\rho + \frac{1}{2} + o(1)$ for (2.7), $\quad -k^{-1}Q_2\rho + o(1)$ for (4.2),
ii) $P_{LFC}(CS|\mathcal{R}_*) = P^* + \frac{1}{2}n^{-1}Q^{(1)}(1) + o(\delta^2)$ for (2.7), $\quad P^* + o(\delta^2)$ for (4.2).

It may also be noted that $\lim_{\rho\to\infty} a_i(\rho) = a_{iE},\ i = 1, ..., 4$. In view of Theorem 5, we can see that the specific design constant h satisfying (2.7) that was used in (2.3) to claim the exact consistency property is *optimal* in our sense when σ^2 is large or σ_*^2 is poorly set.

Now, since the optimal design constant given by Theorem 5 involves the unknown parameter σ^2 through $\rho\ (= \sigma^2/\sigma_*^2)$, we give an approximate (asymptotic) optimal design constant $\hat{h}$ involving S_m^2 instead of σ^2. The following theorem shows that the modified two-stage selection procedure $\mathcal{R}_*$ from (3.1)-(3.2) with such estimated design constant $\hat{h}$ would also enjoy the *asymptotic consistency* property as $\delta \to 0$ at the same rate as $o(\delta^8)$ that was given in Theorem 5.

Theorem 6. (Asymptotic optimal design constant) *Consider $\rho\ (= \sigma^2/\sigma_*^2)$ and suppose that we estimate it with $\hat{\rho} = S_m^2/\sigma_*^2$ where S_m^2 is the pooled sample variance based on the pilot data from all k channels or signal processors. Define $a_i(\hat{\rho})$ where, that is to replace ρ with $\hat{\rho}$ in the expressions of $a_i(\rho),\ i = 1, 2, 3, 4$, which were given in Theorem 5. Now, for the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), we have as $\delta \to 0$ that*

$$
P^* + \sum_{i=1}^{4} \tfrac{1}{2}(\alpha_i + \beta_i)n^{-i} = P^* + o(\delta^8) \tag{4.4}
$$

when the design constant h^2 used in (3.2) is replaced by $\hat{h}^2$ that

$$
\hat{h}^2 = \lambda^2\left(1 + \hat{a}_1m^{-1} + \hat{a}_2m^{-2} + \hat{a}_3m^{-3} + \hat{a}_4m^{-4}\right). \tag{4.5}
$$

Here, m comes from (3.1), and $\hat{a}_i$, $i = 1, ..., 4$, are defined as:

$$
\begin{aligned}
\hat{a}_1 &= a_1(\hat{\rho}),\\
\hat{a}_2 &= a_2(\hat{\rho}) - (k\hat{\rho})^{-1}Q_2,\\
\hat{a}_3 &= a_3(\hat{\rho}) + \tfrac{1}{4}(k\hat{\rho}^2)^{-1}Q_2 - \tfrac{1}{2}(k\hat{\rho}^2)^{-1}Q_2^2 - (k\hat{\rho})^{-1}Q_2 + 3(k^2\hat{\rho})^{-1}Q_2^2\\
&\quad -(k^2\hat{\rho})^{-1}Q_2^3 - 2(k^2\hat{\rho})^{-1}Q_3 + 2(k^2\hat{\rho})^{-1}Q_2Q_3 - (k^2\hat{\rho})^{-1}Q_4,\\
\hat{a}_4 &= a_4(\hat{\rho}) - \tfrac{1}{8}(k\hat{\rho}^3)^{-1}Q_2^3 + \tfrac{1}{4}(k\hat{\rho}^3)^{-1}Q_2Q_3 - \tfrac{1}{8}(k\hat{\rho}^3)^{-1}Q_4\\
&\quad +\tfrac{1}{4}(k\hat{\rho}^2)^{-1}Q_2 + \tfrac{3}{2}(k^2\hat{\rho}^2)^{-1}Q_2 - \tfrac{1}{2}(k\hat{\rho}^2)^{-1}Q_2^2 - 2(k^2\hat{\rho}^2)^{-1}Q_2^2\\
&\quad +(k^2\hat{\rho}^2)^{-1}Q_2^3 - (k^2\hat{\rho}^2)^{-1}Q_2^4 + (k^2\hat{\rho}^2)^{-1}Q_3 - \tfrac{7}{4}(k^2\hat{\rho}^2)^{-1}Q_2Q_3\\
&\quad +2(k^2\hat{\rho}^2)^{-1}Q_2^2Q_3 + \tfrac{3}{4}(k^2\hat{\rho}^2)^{-1}Q_4 - (k^2\hat{\rho}^2)^{-1}Q_2Q_4 - (k\hat{\rho})^{-1}Q_2\\
&\quad +6(k^2\hat{\rho})^{-1}Q_2^2 - 2(k^2\hat{\rho})^{-1}Q_2^3 - 10(k^3\hat{\rho})^{-1}Q_2^3 + \tfrac{15}{2}(k^3\hat{\rho})^{-1}Q_2^4\\
&\quad -\tfrac{3}{2}(k^3\hat{\rho})^{-1}Q_2^5 - 4(k^2\hat{\rho})^{-1}Q_3 + 4(k^2\hat{\rho})^{-1}Q_2Q_3 + \tfrac{40}{3}(k^3\hat{\rho})^{-1}Q_2Q_3\\
&\quad -\tfrac{55}{3}(k^3\hat{\rho})^{-1}Q_2^2Q_3 + 5(k^3\hat{\rho})^{-1}Q_2^3Q_3 + \tfrac{10}{3}(k^3\hat{\rho})^{-1}Q_3^2 - 3(k^3\hat{\rho})^{-1}Q_2Q_3^2\\
&\quad -2(k^2\hat{\rho})^{-1}Q_4 - 4(k^3\hat{\rho})^{-1}Q_4 + \tfrac{65}{6}(k^3\hat{\rho})^{-1}Q_2Q_4 - 3(k^3\hat{\rho})^{-1}Q_2^2Q_4\\
&\quad +\tfrac{3}{2}(k^3\hat{\rho})^{-1}Q_3Q_4 - \tfrac{10}{3}(k^3\hat{\rho})^{-1}Q_5 + \tfrac{3}{2}(k^3\hat{\rho})^{-1}Q_2Q_5 - \tfrac{1}{2}(k^3\hat{\rho})^{-1}Q_6.
\end{aligned}
$$

APPENDIX A: TABLES FOR EVALUATING THE APPROXIMATIONS

A.1. Evaluations of Approximations

Table 1. Values of $Q^{(i)}(1)$, $i = 1, ..., 8$, and $Q_i = Q^{(i)}(1)/Q^{(1)}(1)$, $i = 2, ..., 8$

k	2	3	4	5
		$P^* = 0.95$		
$Q^{(1)}(1)$	0.110433	0.128363	0.138330	0.145240
$Q^{(2)}(1)$	-0.251373	-0.331680	-0.379901	-0.415017
$Q^{(3)}(1)$	0.593235	0.862970	1.029791	1.154240
$Q^{(4)}(1)$	-1.459467	-2.261786	-2.728646	-3.064282
$Q^{(5)}(1)$	3.757703	5.973744	6.942852	7.451249
$Q^{(6)}(1)$	-10.14030	-15.90426	-16.33991	-14.78087
$Q^{(7)}(1)$	28.64187	42.69261	32.12851	11.59280
$Q^{(8)}(1)$	-84.36110	-115.5652	-30.60812	106.1388
Q_2	-2.276250	-2.583910	-2.746334	-2.857460
Q_3	5.371897	6.722862	7.444437	7.947124
Q_4	-13.21586	-17.62018	-19.72559	-21.09807
Q_5	34.02700	46.53774	50.19041	51.30302
Q_6	-91.82307	-123.9003	-118.1225	-101.7686
Q_7	259.3597	332.5917	232.2594	79.81823
Q_8	-763.9121	-900.2970	-221.2684	730.7824

Table 1. (Continued)

k	2	3	4	5
		$P^* = 0.90$		
$Q^{(1)}(1)$	0.155875	0.188437	0.205769	0.217630
$Q^{(2)}(1)$	-0.251706	-0.357508	-0.418285	-0.461816
$Q^{(3)}(1)$	0.423512	0.683192	0.832424	0.938673
$Q^{(4)}(1)$	-0.745775	-1.315539	-1.595362	-1.761299
$Q^{(5)}(1)$	1.377195	2.553347	2.846286	2.765058
$Q^{(6)}(1)$	-2.664714	-4.996568	-4.326876	-2.196113
$Q^{(7)}(1)$	5.383323	9.859685	3.716734	-8.422874
$Q^{(8)}(1)$	-11.29565	-19.62083	9.589869	62.92136
Q_2	-1.614792	-1.897227	-2.032784	-2.122022
Q_3	2.716993	3.625567	4.045422	4.313159
Q_4	-4.784426	-6.981306	-7.753152	-8.093088
Q_5	8.835226	13.55011	13.83240	12.70531
Q_6	-17.09515	-26.51580	-21.02778	-10.09104
Q_7	34.53606	52.32341	18.06261	-38.70271
Q_8	-72.46586	-104.1239	46.60492	289.1207

Recall the specially defined function $Q(x)$ from (3.4) and its s^{th}-order derivative denoted by $Q^{(s)}(x) \equiv d^sQ(x)/dx^s, s = 1, ..., 8$. Also, we have denoted $Q_i \equiv Q^{(i)}(1)/Q^{(1)}(1), i = 2, ..., 8$. In evaluating the upper and lower bounds found in the Theorems 1-4, one would first need $Q^{(s)}(1)$ and Q_s for $s = 1, ..., 8$. In Table 1, we have provided values of these expressions when $k = 2, 3, 4, 5$ and $P^* = 0.90, 0.95$.

A.2. Design Constant, Its Role, and Optimality

Table 2. Values of the design constants λ_{k,P^*} from (2.2), $h_{k,P^*,m}$ from (2.7) and its approximation error $(h - r^{1/2}\lambda)$, within parentheses, given by Lemma 1, and the oversampling percentage when $P^* = 0.95$

m	$k = 2$ $\lambda = 2.756050$	$k = 3$ $\lambda = 3.079897$	$k = 4$ $\lambda = 3.258250$	$k = 5$ $\lambda = 3.380284$
10	$h = 2.939961$	$h = 3.233331$	$h = 3.386455$	$h = 3.490330$
	(0.000135453)	(0.000079433)	(0.0000487757)	(0.0000339652)
	$h^2/\lambda^2 \approx 1.14$	$h^2/\lambda^2 \approx 1.10$	$h^2/\lambda^2 \approx 1.08$	$h^2/\lambda^2 \approx 1.07$
	Oversampling 14%	Oversampling 10%	Oversampling 8%	Oversampling 7%
20	$h = 2.840716$	$h = 3.151045$	$h = 3.317990$	$h = 3.431726$
	(4.21803×10^{-6})	(3.00188×10^{-6})	(1.78956×10^{-6})	(8.11535×10^{-7})
	$h^2/\lambda^2 \approx 1.06$	$h^2/\lambda^2 \approx 1.05$	$h^2/\lambda^2 \approx 1.04$	$h^2/\lambda^2 \approx 1.03$
	Oversampling 6%	Oversampling 5%	Oversampling 4%	Oversampling 3%
30	$h = 2.811036$	$h = 3.126206$	$h = 3.297192$	$h = 3.413850$
	(5.31065×10^{-7})	(7.00935×10^{-7})	(5.28897×10^{-7})	$(-7.39785 \times 10^{-8})$
	$h^2/\lambda^2 \approx 1.04$	$h^2/\lambda^2 \approx 1.03$	$h^2/\lambda^2 \approx 1.02$	$h^2/\lambda^2 \approx 1.02$
	Oversampling 4%	Oversampling 3%	Oversampling 2%	Oversampling 2%

Recall that the constant $\lambda \equiv \lambda_{k,P^*}$ from (2.2) was needed for the determination of $n \equiv \lambda^2\sigma^2/\delta^2$, the optimal fixed sample size from each channel. On the other hand, $h \equiv h_{k,P^*,m}$

from (2.7) was required to implement the selection methodology $\mathcal{R}$ from (2.3). Table 2 provides the values of λ_{k,P^*} and $h_{k,P^*,m}$ for $k = 2,3,4,5$, $P^* = 0.95$, and $m = 10,20,30$. On the second line of each block in Table 2, we have also provided the value of $h - r^{1/2}\lambda$ within parenthesis when r was determined via Lemma 1 from Section B.1. We had already remarked about the ratio h^2/λ^2 both in and underneath (2.10). It should be obvious from Table 2 that the magnitude of oversampling relative to n can be quite substantial, especially when k or m or both are small.

APPENDIX B: ADDITIONAL LEMMAS AND SELECTED OUTLINES OF PROOFS OF THEOREMS

In this appendix, we do not show all the details of proofs. Rather, we systematically state a number of lemmas in order to provide road-maps for some of the crucial expansions needed for numerous intermediate links. This will hopefully help readers to connect the dots in order to come up with full-blown proofs of our theorems stated in Sections 3 and 4. We approach this way not because the formal proofs are trivial extensions of known results. We do this to keep the length of our presentation within reason.

B.1. Additional Lemmas

The following result is a generalization of Lemma 1 in Aoshima and Takada (2000). Its proof is omitted.

Lemma 1. *Suppose that $Q(x)$ is a distribution function with $x > 0$ such that there exists constants $c_i > 0$, $|\alpha_i| < \infty$, $i = 1, \ldots, s$, for which*

(i) $Q^{(10)}(x)$ is continuous at $x = 1$;
(ii) $|Q^{(10)}(x)| \le \sum_{i=1}^{s} c_i x^{\alpha_i}$ in a neighborhood of $x = 1$.

Then, the constant r (> 0) such that $E\left[Q\left(r\chi_\nu^2/\nu\right)\right] = Q(1)$ is asymptotically given by

$$r = 1 + a\nu^{-1} + b\nu^{-2} + c\nu^{-3} + d\nu^{-4} + O(\nu^{-5}), \tag{B.1}$$

where χ_ν^2 is a chi-square random variable with ν degrees of freedom, $Q_i = Q^{(i)}(1)/Q^{(1)}(1)$, $i = 2, \ldots, 8$, and

$$\begin{aligned}
a &= -Q_2, \\
b &= 2Q_2^2 - \tfrac{1}{2}Q_2^3 - \tfrac{4}{3}Q_3 + Q_2Q_3 - \tfrac{1}{2}Q_4, \\
c &= -5Q_2^3 + 3Q_2^4 - \tfrac{1}{2}Q_2^5 + \tfrac{20}{3}Q_2Q_3 - \tfrac{22}{3}Q_2^2Q_3 + \tfrac{5}{3}Q_2^3Q_3 + \tfrac{4}{3}Q_3^2 \\
&\quad -Q_2Q_3^2 - 2Q_4 + \tfrac{13}{3}Q_2Q_4 - Q_2^2Q_4 + \tfrac{1}{2}Q_3Q_4 - \tfrac{4}{3}Q_5 + \tfrac{1}{2}Q_2Q_5 - \tfrac{1}{6}Q_6, \\
d &= 14Q_2^4 - 14Q_2^5 + 5Q_2^6 - \tfrac{5}{8}Q_2^7 - 28Q_2^2Q_3 + 42Q_2^3Q_3 - \tfrac{58}{3}Q_2^4Q_3 \\
&\quad +\tfrac{35}{12}Q_2^5Q_3 + \tfrac{16}{3}Q_3^2 - \tfrac{176}{9}Q_2Q_3^2 + \tfrac{50}{3}Q_2^2Q_3^2 - \tfrac{11}{3}Q_2^3Q_3^2 - \tfrac{4}{3}Q_3^3 \\
&\quad +Q_2Q_3^3 + 12Q_2Q_4 - \tfrac{76}{3}Q_2^2Q_4 + 12Q_2^3Q_4 - \tfrac{43}{24}Q_2^4Q_4 + \tfrac{76}{9}Q_3Q_4 \\
&\quad -\tfrac{38}{3}Q_2Q_3Q_4 + \tfrac{13}{4}Q_2^2Q_3Q_4 - \tfrac{1}{2}Q_3^2Q_4 + \tfrac{5}{3}Q_4^2 - \tfrac{5}{8}Q_2Q_4^2 - \tfrac{16}{5}Q_5 \\
&\quad +\tfrac{34}{3}Q_2Q_5 - 6Q_2^2Q_5 + \tfrac{11}{12}Q_2^3Q_5 + 2Q_3Q_5 - Q_2Q_3Q_5 + \tfrac{1}{4}Q_4Q_5 - \tfrac{26}{9}Q_6 \\
&\quad +\tfrac{8}{3}Q_2Q_6 - \tfrac{5}{12}Q_2^2Q_6 + \tfrac{1}{6}Q_3Q_6 - \tfrac{2}{3}Q_7 + \tfrac{1}{6}Q_2Q_7 - \tfrac{1}{24}Q_8.
\end{aligned}$$

Now, we let $r = h^2/\lambda^2$ with λ and h coming from (3.1) and (3.2) respectively. Let us formally write

$$r = 1 + a_1 m^{-1} + a_2 m^{-2} + a_3 m^{-3} + a_4 m^{-4} + o(m^{-4}), \tag{B.2}$$

where the coefficients $a_1, ..., a_4$ are free from m and δ.

We note that the function $Q(x)$ defined in (3.4) satisfies the sufficient conditions (i) and (ii) described in Lemma 1. Now, utilizing (2.6) and Lemma 1, it follows that if the selection procedure $\mathcal{R}$ has the exact consistency property, then we must have:

$$\begin{gathered} r = 1 + a_1 m^{-1} + a_2 m^{-2} + a_3 m^{-3} + a_4 m^{-4} + O(m^{-5}), \text{ with} \\ a_1 = ak^{-1},\ a_2 = ak^{-1} + bk^{-2},\ a_3 = ak^{-1} + 2bk^{-2} + ck^{-3}, \\ \text{and } a_4 = ak^{-1} + 3bk^{-2} + 3ck^{-3} + dk^{-4}. \end{gathered}$$

The following two lemmas can be proved using techniques similar to those found in Mukhopadhyay and Duggan (1997).

Lemma 2. *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), consider the stopping variable N. Then, we have as $\delta \to 0$:*

$$P_{\sigma^2}(N = m) = O(\eta^\kappa),$$

where $\eta = \rho^{-1}\exp(1-\rho^{-1})$ with $\rho = \sigma^2/\sigma_\star^2$ so that ρ^{-1} is a positive proper fraction. Here, κ is a positive number that may be chosen as large as possible.

Lemma 3. *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), consider the stopping variable N. Then, we have as $\delta \to 0$:*

(1) $n^{-1/2}(N-n) \xrightarrow{\mathcal{L}} N(0, 2\rho)$ as $\delta \to 0$;

(2) $n^{-1}(N-n)^2$ is uniformly integrable for $0 < \delta < \delta_0$ with sufficiently small δ_0,

where $n = \lambda^2\sigma^2/\delta^2$ and $\rho = \sigma^2/\sigma_\star^2$.

The following two lemmas are very crucial in our investigations for deriving asymptotic expansions of various characteristics beyond second- and third-order. The required techniques are substantially more involved than what one finds in Mukhopadhyay and Duggan (1997) or Mukhopadhyay (1999).

Lemma 4. *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), consider the stopping variable N. Then, we have as $\delta \to 0$:*

(1) $E_{\sigma^2}[n^{-1}(N-n)] = n^{-1}(\frac{1}{2} + a_1\rho) + o(n^{-1})$,

(2) $E_{\sigma^2}[n^{-2}(N-n)^2] = n^{-1}(2k^{-1}\rho) + o(n^{-3/2})$,

(3) $E_{\sigma^2}[n^{-3}(N-n)^3] = n^{-2}(3k^{-1}\rho + 6a_1k^{-1}\rho^2 + 8k^{-2}\rho^2) + o(n^{-2})$,

(4) $E_{\sigma^2}[n^{-4}(N-n)^4] = n^{-2}(12k^{-2}\rho^2) + o(n^{-5/2})$,

(5) $E_{\sigma^2}[n^{-5}(N-n)^5] = n^{-3}(30k^{-2}\rho^2 + 60a_1k^{-2}\rho^3 + 160k^{-3}\rho^3) + o(n^{-3})$,

(6) $E_{\sigma^2}[n^{-6}(N-n)^6] = n^{-3}(120k^{-3}\rho^3) + o(n^{-7/2})$,

(7) $E_{\sigma^2}[n^{-7}(N-n)^7] = n^{-4}(420k^{-3}\rho^3 + 840a_1k^{-3}\rho^4 + 3360k^{-4}\rho^4) + o(n^{-4})$,

(8) $E_{\sigma^2}[n^{-8}(N-n)^8] = n^{-4}(1680k^{-4}\rho^4) + o(n^{-4})$,

where $n = \lambda^2\sigma^2/\delta^2$, $\rho = \sigma^2/\sigma_\star^2$, *and* a_1 *is given by (B.2).*

Proof of Lemma 4. Here, we provide only a sketch. Let us write that

$$N = \frac{h^2 S_m^2}{\delta^2} + \left(\left\langle \frac{h^2 S_m^2}{\delta^2} \right\rangle - \frac{h^2 S_m^2}{\delta^2}\right) + \left(N - \left\langle \frac{h^2 S_m^2}{\delta^2} \right\rangle\right). \tag{B.3}$$

We note that $h^2S_m^2/\delta^2 = nr\chi_\nu^2/\nu$ with $r = h^2/\lambda^2$ and $\nu = k(m-1)$. Now, using techniques similar to those in Hall (1981) and Aoshima and Takada (2000), we can claim that $U \equiv \langle h^2S_m^2/\delta^2\rangle - h^2S_m^2/\delta^2$ is asymptotically distributed as $U(0,1)$. Also, with $D \equiv N - \langle h^2S_m^2/\delta^2\rangle$ and using Lemma 2, we can show that $E_{\sigma^2}(D^g)$ converges to zero at an arbitrarily fast rate, whatever be g (> 0) fixed.

Now, without any loss of generality, let δ (> 0) converge to zero through a sequence such that $\lambda^2\sigma_\star^2/\delta^2$ remains an integer and we may write $m = \lambda^2\sigma_\star^2/\delta^2$. From (B.2), we can express $r = 1 + A$ where

$$A = \rho n^{-1}(a_1 + a_2\rho n^{-1} + a_3\rho^2 n^{-2} + a_4\rho^3 n^{-3}) + o(n^{-4}). \tag{B.4}$$

Let $Y = \chi_\nu^2/\nu - 1$. Then, from (B.3), we have:

$$n^{-1}(N-n) = A + (1+A)Y + n^{-1}U + n^{-1}D. \tag{B.5}$$

Noting that $E(Y^{2s-1}) = O(\nu^{-s})$ and $E(Y^{2s}) = O(\nu^{-s})$, $s = 1, 2, \ldots$, it follows that

$$\begin{aligned} &E_{\sigma^2}[n^{-s}(N-n)^s] \\ &= \begin{cases} E(Y^s) + sn^{-1}(a_1\rho + \frac{1}{2})E(Y^{s-1}) + o(n^{-(s+1)/2}), & s = 1, 3, \ldots \\ E(Y^s) + o(n^{-(s+1)/2}), & s = 2, 4, \ldots \end{cases} \end{aligned} \tag{B.6}$$

Some simple calculations would now yield the results. □

Lemma 5. *For the modified selection procedure $\mathcal{R}_*$ from (3.1)-(3.2), consider the stopping variable N. Then, we have as $\delta \to 0$:*

$$L_s \le E_{\sigma^2}[n^{-s}(N-n)^s] \le U_s, \quad s = 1, 2, \ldots, 8,$$

where $n = \lambda^2\sigma^2/\delta^2$, $\rho = \sigma^2/\sigma_\star^2$, $(a_1, \ldots, a_4)$ *are given by (B.2), and*

$$\begin{aligned} L_1 &= a_1\rho n^{-1} + a_2\rho^2 n^{-2} + a_3\rho^3 n^{-3} + a_4\rho^4 n^{-4} + o(n^{-4}), \\ U_1 &= (a_1\rho + 1)n^{-1} + a_2\rho^2 n^{-2} + a_3\rho^3 n^{-3} + a_4\rho^4 n^{-4} + o(n^{-4}); \end{aligned} \tag{B.7}$$

$$\begin{aligned} L_2 = {} & (-2 + 2k^{-1}\rho)\,n^{-1} + (2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2 n^{-2} \\ & +2\left\{k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\,(2k^{-1} + a_2)\right\}\rho^3 n^{-3} \\ & + \left\{4k^{-1}a_2\,(1 + a_1) + a_2^2 + 2k^{-1} + 2k^{-1}a_1^2 + 4k^{-1}a_3 \right.\\ & \left. + \, 2a_1\,(2k^{-1} + a_3)\right\}\rho^4 n^{-4} + o(n^{-4}), \\ U_2 = {} & (2 + 2k^{-1}\rho)\,n^{-1} + \left\{1 + 2a_1\rho + (2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2\right\}n^{-2} \\ & + \left\{2a_2\rho^2 + 2\,(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\,(2k^{-1} + a_2))\,\rho^3\right\}n^{-3} \\ & + \left\{2a_3\rho^3 + (4k^{-1}a_2\,(1 + a_1) + a_2^2 + 2k^{-1} + 2k^{-1}a_1^2 + 4k^{-1}a_3 \right.\\ & \left. + \, 2a_1\,(2k^{-1} + a_3))\,\rho^4\right\}n^{-4} + o(n^{-4}); \end{aligned} \tag{B.8}$$

$$\begin{aligned}
L_3 = & -6n^{-1} + \{-3 - 6a_1\rho + 2k^{-1}(4k^{-1} + 3a_1)\rho^2\} n^{-2} \\
& + \{-6a_2\rho^2 + (6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 \\
& \quad + 2k^{-1}(8k^{-1} + 3a_2))\rho^3\} n^{-3} \\
& + \{-6a_3\rho^3 + 3(2k^{-1}a_1^3 + 2k^{-1}(4k^{-1} + a_2 + 4k^{-1}a_2 + a_3) \\
& \quad + 2k^{-1}a_1(1 + 8k^{-1} + 4a_2) + a_1{}^2(4k^{-1} + 8k^{-2} + a_2))\rho^4\} n^{-4} + o(n^{-4}), \\
U_3 = & \; 6n^{-1} + \{3 + 6k^{-1}\rho + 6a_1\rho + 2k^{-1}(4k^{-1} + 3a_1)\rho^2\} n^{-2} \\
& + \{1 + 3a_1\rho + 3(2k^{-1} + 4k^{-1}a_1 + a_1{}^2)\rho^2 + 6a_2\rho^2 \\
& \quad + (a_1(6k^{-1} + 24k^{-2}) + 12k^{-1}a_1^2 + a_1^3 + 16k^{-2} + 6k^{-1}a_2)\rho^3\} n^{-3} \\
& + \{(3a_2\rho^2 + 6(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2) + a_3)\rho^3 \\
& \quad + 3(2k^{-1}a_1^3 + a_1{}^2(4k^{-1} + 8k^{-2} + a_2) + 2k^{-1}a_1(1 + 8k^{-1} + 4a_2) \\
& \quad + 2k^{-1}(4k^{-1} + a_2 + 4k^{-1}a_2 + a_3))\rho^4\} n^{-4} + o(n^{-4});
\end{aligned} \tag{B.9}$$

$$\begin{aligned}
L_4 = & -16n^{-1} + (-12 - 24k^{-1}\rho + 12k^{-2}\rho^2 - 24a_1\rho) n^{-2} \\
& + \{-4 - 12a_1\rho - 12(2k^{-1} + 4k^{-1}a_1 + a_1^2 + 2a_2)\rho^2 \\
& + 4k^{-1}(6k^{-1} + 12k^{-2} + 20k^{-1}a_1 + 3a_1^2)\rho^3\} n^{-3} \\
& + \{-12a_2\rho^2 - 24(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2) + a_3)\rho^3 \\
& \quad + (12k^{-1}a_1^2(1 + 14k^{-1}) + 8k^{-1}a_1(4k^{-1}(5 + 6k^{-1}) + 3a_2) \\
& \quad + 24k^{-1}a_1^3 + a_1^4 + 4k^{-2}(9 + 36k^{-1} + 20a_2))\rho^4\} n^{-4} + o(n^{-4}), \\
U_4 = & \; 16n^{-1} + (12 + 24k^{-1}\rho + 12k^{-2}\rho^2 + 24a_1\rho) n^{-2} \\
& + \{4 + 12k^{-1}\rho + 12a_1\rho + 8k^{-1}(4k^{-1} + 3a_1)\rho^2 \\
& \quad + 12(2k^{-1} + 4k^{-1}a_1 + a_1^2 + 2a_2)\rho^2 \\
& \quad + 4k^{-1}(6k^{-1}(1 + 2k^{-1}) + 20k^{-1}a_1 + 3a_1^2)\rho^3\} n^{-3} \\
& + \{1 + 4a_1\rho + 6(2k^{-1} + 4k^{-1}a_1 + a_1^2 + 2a_2)\rho^2 \\
& \quad + 24(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\rho^3 \\
& \quad + 4(a_1(6k^{-1} + 24k^{-2}) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}(8k^{-1} + 3a_2))\rho^3 \\
& \quad + (12k^{-1}a_1^2(1 + 14k^{-1}) + 8k^{-1}a_1(4k^{-1}(5 + 6k^{-1}) + 3a_2) + 24k^{-1}a_1^3 \\
& \quad + 4k^{-2}(9 + 36k^{-1} + 20a_2) + a_1^4)\rho^4 + 24a_3\rho^3\} n^{-4} + o(n^{-4});
\end{aligned} \tag{B.10}$$

$$\begin{aligned}
L_5 = & -40n^{-1} + (-40 - 120k^{-1}\rho - 80a_1\rho) n^{-2} \\
& + \{-20 - 60k^{-1}\rho - 60a_1\rho - 40(4k^{-2} + 3k^{-1}a_1 + 2a_2)\rho^2 \\
& \quad - 60(2k^{-1} + 4k^{-1}a_1 + a_1^2)\rho^2 + 20k^{-2}(8k^{-1} + 3a_1)\rho^3\} n^{-3} \\
& + \{-5 - 20a_1\rho - 30(2k^{-1} + 4k^{-1}a_1 + a_1^2 + 2a_2)\rho^2 \\
& \quad - 120(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\rho^3 \\
& \quad - 20(6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}(8k^{-1} + 3a_2) \\
& \quad + 4a_3)\rho^3 + 4k^{-1}(10k^{-1}a_1(3 + 26k^{-1}) + 80k^{-1}a_1^2 + 5a_1^3 \\
& \quad + 3k^{-1}(8k^{-1}(5 + 4k^{-1}) + 5a_2))\rho^4\} n^{-4} + o(n^{-4}), \\
U_5 = & \; 40n^{-1} + (40 + 120k^{-1}\rho + 80a_1\rho) n^{-2} \\
& + \{20 + 60k^{-1}\rho + 60k^{-2}\rho^2 + 60a_1\rho + 40k^{-1}(4k^{-1} + 3a_1)\rho^2 + 80a_2\rho^2 \\
& \quad + 60(2k^{-1} + 4k^{-1}a_1 + a_1^2)\rho^2 + 20k^{-2}(8k^{-1} + 3a_1)\rho^3\} n^{-3} \\
& + \{5 + 20k^{-1}\rho + 20a_1\rho + 20k^{-1}(4k^{-1} + 3a_1)\rho^2 + 60a_2\rho^2 \\
& \quad + 30(2k^{-1} + 4k^{-1}a_1 + a_1^2)\rho^2 + 20k^{-1}(6k^{-1} + 12k^{-2} + 20k^{-1}a_1 \\
& \quad + 3a_1^2)\rho^3 + 120(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\rho^3 \\
& \quad + 20(6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}(8k^{-1} + 3a_2))\rho^3 \\
& \quad + 4k^{-1}(10k^{-1}a_1(3 + 26k^{-1}) + 80k^{-1}a_1^2 + 5a_1^3 \\
& \quad + 3k^{-1}(40k^{-1} + 32k^{-2} + 5a_2))\rho^4 + 80a_3\rho^3\} n^{-4} + o(n^{-4});
\end{aligned} \tag{B.11}$$

$$
\begin{aligned}
L_6 = &-96n^{-1} + (-120 - 480k^{-1}\rho - 240a_1\rho)\, n^{-2} \\
&+ \{-80 - 360k^{-1}\rho - 360k^{-2}\rho^2 + 120k^{-3}\rho^3 - 240a_1\rho - 240a_2\rho^2 \\
&\quad -240k^{-1}(4k^{-1} + 3a_1)\,\rho^2 - 240\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2\}\, n^{-3} \\
&+ \{-30 - 120k^{-1}\rho - 120a_1\rho - 120k^{-1}(4k^{-1} + 3a_1)\,\rho^2 \\
&\quad -180\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2 - 240a_2\rho^2 \\
&\quad -120k^{-1}(6k^{-1}(1 + 2k^{-1}) + 20k^{-1}a_1 + 3a_1^2)\,\rho^3 \\
&\quad -480\,(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\,\rho^3 \\
&\quad -120\,(6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}(8k^{-1} + 3a_2) \\
&\quad + 2a_3)\,\rho^3 + 20k^{-2}(2k^{-1}(9 + 52k^{-1}) + 84k^{-1}a_1 + 9a_1^2)\,\rho^4\}\, n^{-4} \\
&\quad +o(n^{-4}), \qquad\qquad \text{(B.12)} \\
U_6 = &\ 96n^{-1} + (120 + 480k^{-1}\rho + 240a_1\rho)\, n^{-2} \\
&+ \{80 + 360k^{-1}\rho + 240a_1\rho + 360k^{-2}\rho^2 + 240k^{-1}(4k^{-1} + 3a_1)\,\rho^2 \\
&\quad + 240\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2 + 240a_2\rho^2 + 120k^{-3}\rho^3\}\, n^{-3} \\
&+ \{30 + 120k^{-1}\rho + 180k^{-2}\rho^2 + 120a_1\rho + 120k^{-1}(4k^{-1} + 3a_1)\,\rho^2 \\
&\quad +120k^{-2}(8k^{-1} + 3a_1)\,\rho^3 + 180\,(2k^{-1} + 4k^{-1}a_1 + a_1{}^2)\,\rho^2 \\
&\quad +240a_2\rho^2 + 120k^{-1}(6k^{-1}(1 + 2k^{-1}) + 20k^{-1}a_1 + 3a_1^2)\,\rho^3 \\
&\quad +120\,(6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}(8k^{-1} + 3a_2) \\
&\quad + 2a_3)\,\rho^3 + 480\,(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\,\rho^3 \\
&\quad + 20k^{-2}(2k^{-1}(9 + 52k^{-1}) + 84k^{-1}a_1 + 9a_1^2)\,\rho^4\}\, n^{-4} + o(n^{-4});
\end{aligned}
$$

$$
\begin{aligned}
L_7 = &-224n^{-1} + (-336 - 1680k^{-1}\rho - 672a_1\rho)\, n^{-2} \\
&+ \{-280 - 1680k^{-1}\rho - 2520k^{-2}\rho^2 - 840a_1\rho - 672a_2\rho^2 \\
&\quad - 1120k^{-1}(4k^{-1} + 3a_1)\,\rho^2 - 840\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2\}\, n^{-3} \\
&+ \{-140 - 840k^{-1}\rho - 1260k^{-2}\rho^2 - 560a_1\rho + 840k^{-3}(4k^{-1} + a_1)\,\rho^4 \\
&\quad -840k^{-1}(4k^{-1} + 3a_1)\,\rho^2 - 840k^{-2}(8k^{-1} + 3a_1)\,\rho^3 \\
&\quad -840\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2 - 840a_2\rho^2 \\
&\quad -840k^{-1}(6k^{-1}(1 + 2k^{-1}) + 20k^{-1}a_1 + 3a_1^2)\,\rho^3 \\
&\quad -1680\,(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\,\rho^3 \\
&\quad -560\,(6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 \\
&\quad + 2k^{-1}(8k^{-1} + 3a_2))\,\rho^3 - 672a_3\rho^3\}\, n^{-4} + o(n^{-4}), \qquad\qquad \text{(B.13)} \\
U_7 = &\ 224n^{-1} + (336 + 1680k^{-1}\rho + 672a_1\rho)\, n^{-2} \\
&+ \{280 + 1680k^{-1}\rho + 2520k^{-2}\rho^2 + 840a_1\rho + 672a_2\rho^2 \\
&\quad + 1120k^{-1}(4k^{-1} + 3a_1)\,\rho^2 + 840\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2\}\, n^{-3} \\
&+ \{140 + 840k^{-1}\rho + 1260k^{-2}\rho^2 + 840k^{-3}\rho^3 + 560a_1\rho \\
&\quad +840k^{-3}(4k^{-1} + a_1)\,\rho^4 + 840k^{-1}(4k^{-1} + 3a_1)\,\rho^2 \\
&\quad +840k^{-2}(8k^{-1} + 3a_1)\,\rho^3 + 840\,(2k^{-1} + 4k^{-1}a_1 + a_1^2)\,\rho^2 \\
&\quad +840k^{-1}(6k^{-1}(1 + 2k^{-1}) + 20k^{-1}a_1 + 3a_1^2)\,\rho^3 \\
&\quad +840a_2\rho^2 + 1680\,(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1(2k^{-1} + a_2))\,\rho^3 \\
&\quad +560\,(6k^{-1}a_1(1 + 4k^{-1}) + 12k^{-1}a_1^2 + a_1^3 \\
&\quad + 2k^{-1}(8k^{-1} + 3a_2))\,\rho^3 + 672a_3\rho^3\}\, n^{-4} + o(n^{-4});
\end{aligned}
$$

$$
\begin{aligned}
L_8 = & -512n^{-1} + \left(-896 - 5376k^{-1}\rho - 1792a_1\rho\right) n^{-2} \\
& + \{-896 - 6720k^{-1}\rho - 13440k^{-2}\rho^2 - 2688a_1\rho - 1792a_2\rho^2 \\
& \quad - 4480k^{-1}\left(4k^{-1} + 3a_1\right)\rho^2 - 2688\left(2k^{-1} + 4k^{-1}a_1 + a_1^2\right)\rho^2\} n^{-3} \\
& + \{-560 - 4480k^{-1}\rho - 2240a_1\rho - 10080k^{-2}\rho^2 - 2688a_2\rho^2 \\
& \quad -4480k^{-1}\left(4k^{-1} + 3a_1\right)\rho^2 - 3360\left(2k^{-1} + 4k^{-1}a_1 + a_1^2\right)\rho^2 \\
& \quad -4480k^{-1}\left(6k^{-1}\left(1 + 2k^{-1}\right) + 20k^{-1}a_1 + 3a_1^2\right)\rho^3 \\
& \quad -5376\left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1} + a_2\right)\right)\rho^3 \\
& \quad -2240\left(6k^{-1}a_1\left(1 + 4k^{-1}\right) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}\left(8k^{-1} + 3a_2\right)\right)\rho^3 \\
& \quad - 6720k^{-2}\left(9k^{-1} + 3a_1\right)\rho^3 - 1792a_3\rho^3 + 1680k^{-4}\rho^4\} n^{-4} + o(n^{-4}), \\
U_8 = & \ 512n^{-1} + \left(896 + 5376k^{-1}\rho + 1792a_1\rho\right) n^{-2} \\
& + \{896 + 6720k^{-1}\rho + 13440k^{-2}\rho^2 + 2688a_1\rho + 1792a_2\rho^2 \\
& \quad + 4480k^{-1}\left(4k^{-1} + 3a_1\right)\rho^2 + 2688\left(2k^{-1} + 4k^{-1}a_1 + a_1^2\right)\rho^2\} n^{-3} \\
& + \{560 + 4480k^{-1}\rho + 10080k^{-2}\rho^2 + 6720k^{-3}\rho^3 + 1680k^{-4}\rho^4 + 2240a_1\rho \\
& \quad +4480k^{-1}\left(4k^{-1} + 3a_1\right)\rho^2 + 3360\left(2k^{-1} + 4k^{-1}a_1 + a_1^2\right)\rho^2 \\
& \quad +4480k^{-1}\left(6k^{-1}\left(1 + 2k^{-1}\right) + 20k^{-1}a_1 + 3a_1^2\right)\rho^3 \\
& \quad +2688a_2\rho^2 + 5376\left(k^{-1} + k^{-1}a_1^2 + 2k^{-1}a_2 + a_1\left(2k^{-1} + a_2\right)\right)\rho^3 \\
& \quad +2240\left(6k^{-1}a_1\left(1 + 4k^{-1}\right) + 12k^{-1}a_1^2 + a_1^3 + 2k^{-1}\left(8k^{-1} + 3a_2\right)\right)\rho^3 \\
& \quad +6720k^{-2}\left(8k^{-1} + 3a_1\right)\rho^3 + 1792a_3\rho^3\} n^{-4} + o(n^{-4}).
\end{aligned}
\tag{B.14}
$$

Proof of Lemma 5. Let us define

$$
T = \max\left\{m, \frac{h^2 S_m^2}{\delta^2}\right\}. \tag{B.15}
$$

Note from Lemma 2 that $m^s P_{\sigma^2}(T = m)$ converges to zero at an arbitrarily fast rate, whatever be s (> 0) fixed. Hence, we can show that

$$
E_{\sigma^2}[n^{-s}(T - n)^s] = E_{\sigma^2}[(rY + A)^s] + o(n^{-4}) \tag{B.16}
$$

as $\delta \to 0$, where $Y = \chi_\nu^2/\nu - 1$ and A was defined by (B.4). Since $T \le N \le T + 1$, in view of (3.2) and (B.15), one observes

$$
L_{s0} \le n^{-s}(N - n)^s \le U_{s0}
$$

with

$$
L_{s0} = T^s n^{-s} - s(T+1)^{s-1} n^{1-s} + \tfrac{1}{2}s(s-1)T^{s-2}n^{2-s} - \cdots + (-1)^s, \tag{B.17}
$$

$$
U_{s0} = (T+1)^s n^{-s} - sT^{s-1}n^{1-s} + \tfrac{1}{2}s(s-1)(T+1)^{s-2}n^{2-s} - \cdots + (-1)^s. \tag{B.18}
$$

Now, by combining (B.17)-(B.18), (B.16), as well as $E_{\sigma^2}[L_{s0}]$ and $E_{\sigma^2}[U_{s0}]$ yield the desired results. □

Lemma 6. *For the modified selection procedure $\mathcal{R}_*$ from* (3.1)-(3.2), consider the stopping variable N. Then, we have as $\delta \to 0$:

$$
E_{\sigma^2}[n^{-8}(N - n)^8 Q^{(8)}(W)] = 1680k^{-4}\rho^4 Q^{(8)}(1)n^{-4} + o(n^{-4}),
$$

where W is a random variable between Nn^{-1} and 1, with $n = \lambda^2\sigma^2/\delta^2$, and $\rho = \sigma^2/\sigma_*^2$.

Proof of Lemma 6. Note that $Q(x)$ meets the conditions (i)-(ii) in Lemma 1. Then, the result can be obtained in a way similar to Lemma 3.1 in Mukhopadhyay (1999). □

B.2. Proofs of Theorems

Proof of Theorem 1. The result is obtained from (B.7) in a straightforward fashion. □

Proofs of Theorems 2-4. From (3.3), recall that $P_{LFC}(CS|\mathcal{R}_*) = E_{\sigma^2}[Q(N/n)]$. Now, with some suitable random variable W between 1 and Nn^{-1}, we can write:

$$\begin{aligned} &E_{\sigma^2}\left[Q\left(\tfrac{N}{n}\right)\right] \\ &= P^* + Q^{(1)}(1)E_{\sigma^2}\left[\tfrac{N-n}{n}\right] + \tfrac{1}{2}Q^{(2)}(1)E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^2\right] \\ &+ \tfrac{1}{6}Q^{(3)}(1)E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^3\right] + \tfrac{1}{24}Q^{(4)}(1)E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^4\right] \\ &+ \tfrac{1}{120}Q^{(5)}(1)E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^5\right] + \tfrac{1}{720}Q^{(6)}(1)E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^6\right] \\ &+ \tfrac{1}{5040}Q^{(7)}(1)E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^7\right] + \tfrac{1}{40320}E_{\sigma^2}\left[\left(\tfrac{N-n}{n}\right)^8 Q^{(8)}(W)\right]. \end{aligned} \tag{B.19}$$

In what follows, sometimes we write E_s instead of $E_{\sigma^2}[n^{-s}(N-n)^s]$, $s = 3, 4, ..., 8$, which are given by Lemma 4, parts (3) through (8). Also, recall the expressions of L_s and U_s, $s = 1, 2, ..., 6$, which are given by (B.7) through (B.12) in Lemma 5. Now, note that $Q^{(2)}(1)$, $Q^{(4)}(1)$ and $Q^{(6)}(1)$ are all negative. Thus, in view of Lemma 6, as $\delta \to 0$, we obtain the following results.

$$\begin{aligned} &\textit{Third-Order Approximation:}\ P^* + L_1Q^{(1)}(1) + \tfrac{1}{2}U_2Q^{(2)}(1) \\ &+\tfrac{1}{6}E_3Q^{(3)}(1) + \tfrac{1}{24}E_4Q^{(4)}(1) + o(\delta^4) \le E_{\sigma^2}\left[Q\left(\tfrac{N}{n}\right)\right] \le P^* \\ &+U_1Q^{(1)}(1) + \tfrac{1}{2}L_2Q^{(2)}(1) + \tfrac{1}{6}E_3Q^{(3)}(1) + \tfrac{1}{24}E_4Q^{(4)}(1) + o(\delta^4); \end{aligned} \tag{B.20}$$

$$\begin{aligned} &\textit{Fourth-Order Approximation:}\ P^* + L_1Q^{(1)}(1) + \tfrac{1}{2}U_2Q^{(2)}(1) \\ &+\tfrac{1}{6}L_3Q^{(3)}(1) + \tfrac{1}{24}U_4Q^{(4)}(1) + \tfrac{1}{120}E_5Q^{(5)}(1) + \tfrac{1}{720}E_6Q^{(6)}(1) \\ &+o(\delta^6) \le E_{\sigma^2}\left[Q\left(\tfrac{N}{n}\right)\right] \le P^* + U_1Q^{(1)}(1) + \tfrac{1}{2}L_2Q^{(2)}(1) \\ &+\tfrac{1}{6}U_3Q^{(3)}(1) + \tfrac{1}{24}L_4Q^{(4)}(1) + \tfrac{1}{120}E_5Q^{(5)}(1) + \tfrac{1}{720}E_6Q^{(6)}(1) + o(\delta^6); \end{aligned} \tag{B.21}$$

$$\begin{aligned} &\textit{Fifth-Order Approximation:}\ P^* + L_1Q^{(1)}(1) + \tfrac{1}{2}U_2Q^{(2)}(1) \\ &+\tfrac{1}{6}L_3Q^{(3)}(1) + \tfrac{1}{24}U_4Q^{(4)}(1) + \tfrac{1}{120}L_5Q^{(5)}(1) + \tfrac{1}{720}U_6Q^{(6)}(1) \\ &+\tfrac{1}{5040}E_7Q^{(7)}(1) + \tfrac{1}{40320}E_8Q^{(8)}(1) + o(\delta^8) \le E_{\sigma^2}\left[Q\left(\tfrac{N}{n}\right)\right] \\ &\le P^* + U_1Q^{(1)}(1) + \tfrac{1}{2}L_2Q^{(2)}(1) + \tfrac{1}{6}U_3Q^{(3)}(1) + \tfrac{1}{24}L_4Q^{(4)}(1) \\ &+\tfrac{1}{120}U_5Q^{(5)}(1) + \tfrac{1}{720}L_6Q^{(6)}(1) + \tfrac{1}{5040}E_7Q^{(7)}(1) + \tfrac{1}{40320}E_8Q^{(8)}(1) + o(\delta^8). \end{aligned} \tag{B.22}$$

Then, tedious calculations would yield the results. □

Proof of Theorem 5. In Theorem 4, when one solves simultaneously $\alpha_i + \beta_i = 0$ to find a_i, $i = 1, ..., 4$, the solutions turn out to be $a_i \equiv a_i(\rho)$, $i = 1, ..., 4$, as stated. □

Proof of Theorem 6. From Theorem 5, we estimate

$$a_1(\rho) \equiv -\tfrac{1}{2}\sigma_\star^2\sigma^{-2} - k^{-1}Q_2 \ \text{ by }\ \hat{a}_1 \equiv a_1(\hat{\rho}) = -\tfrac{1}{2}\sigma_\star^2 S_m^{-2} - k^{-1}Q_2$$

where S_m^2 is the pooled sample variance based on the pilot observations from all k channels. From the proofs of Lemmas 4 and 5, recall that we wrote $Y \equiv \chi_\nu^2/\nu - 1$. Then, we can rewrite

$$\hat{a}_1 = -\tfrac{1}{2}\rho^{-1}(1+Y)^{-1} - k^{-1}Q_2, \text{ with } \rho = \sigma^2/\sigma_\star^2.$$

Let us plug

$$A = \rho n^{-1}(\hat{a}_1 + a_2\rho n^{-1} + a_3\rho^2 n^{-2} + a_4\rho^3 n^{-3}) + o(n^{-4}) \tag{B.23}$$

in (B.5) and (B.16) and follow along the line of proof of Theorem 4. Now, for the average of both the upper and lower bounds given by Theorem 4, we can conclude:

$$P^* + \textstyle\sum_{i=1}^4 \frac{1}{2}(\alpha_i + \beta_i)n^{-i} = P^* + \sum_{i=2}^4 \frac{1}{2}(\tilde{\alpha}_i + \tilde{\beta}_i)n^{-i} + o(\delta^8) \text{ as } \delta \to 0, \tag{B.24}$$

where $(\tilde{\alpha}_i, \tilde{\beta}_i)$, $i = 2,3,4$, are the modified coefficients in this case.

Now, by solving $\tilde{\alpha}_2 + \tilde{\beta}_2 = 0$ for a_2, we have a modification of the optimal design constant as follows:

$$\begin{aligned}\tilde{a}_2(\rho) = &-k^{-1}Q_2 + 2k^{-2}Q_2 - \tfrac{1}{2}k^{-2}Q_2^3 - \tfrac{4}{3}k^{-2}Q_3 + k^{-2}Q_2Q_3 \\ &-\tfrac{1}{2}k^{-2}Q_4 - \tfrac{1}{8}\rho^{-2}Q_2,\end{aligned} \tag{B.25}$$

where $\rho^2 = (\sigma^2/\sigma_\star^2)^2$. So, we estimate $\tilde{a}_2(\rho)$ by $\hat{a}_2 \equiv \tilde{a}_2(\hat{\rho})$ which is found by replacing the last term in (B.25) with $-\frac{1}{8}\sigma_\star^4 S_m^{-4} Q_2$. Next, we plug

$$A = \rho n^{-1}(\hat{a}_1 + \hat{a}_2\rho n^{-1} + a_3\rho^2 n^{-2} + a_4\rho^3 n^{-3}) + o(n^{-4}) \tag{B.26}$$

and follow the same routine as described above. Now, for the average of both the upper and lower bounds given by Theorem 4, we can conclude:

$$P^* + \textstyle\sum_{i=1}^4 \frac{1}{2}(\alpha_i + \beta_i)n^{-i} = P^* + \sum_{i=3}^4 \frac{1}{2}(\tilde{\alpha}_i + \tilde{\beta}_i)n^{-i} + o(\delta^8) \text{ as } \delta \to 0,$$

where $(\tilde{\alpha}_i, \tilde{\beta}_i)$, $i = 3,4$, are updated modified-coefficients.

Now, by solving $\tilde{\alpha}_3 + \tilde{\beta}_3 = 0$ for a_3, we have an updated modification of the optimal design constant as follows:

$$\begin{aligned}\tilde{a}_3(\rho) = &-k^{-1}Q_2 + 4k^{-2}Q_2^2 - k^{-2}Q_2^3 - 5k^{-3}Q_2^3 + 3k^{-3}Q_2^4 - \tfrac{1}{2}k^{-3}Q_2^5 \\ &-\tfrac{8}{3}k^{-2}Q_3 + 2k^{-2}Q_2Q_3 + \tfrac{20}{3}k^{-3}Q_2Q_3 - \tfrac{22}{3}k^{-3}Q_2^2Q_3 + \tfrac{5}{3}k^{-3}Q_2^3Q_3 \\ &+\tfrac{4}{3}k^{-3}Q_3^2 - k^{-3}Q_2Q_3^2 - k^{-2}Q_4 - 2k^{-3}Q_4 + \tfrac{13}{3}k^{-3}Q_2Q_4 \\ &-k^{-3}Q_2^2Q_4 + \tfrac{1}{2}k^{-3}Q_3Q_4 - \tfrac{4}{3}k^{-3}Q_5 + \tfrac{1}{2}k^{-3}Q_2Q_5 - \tfrac{1}{6}k^{-3}Q_6 \\ &+\rho^{-2}\left(-\tfrac{1}{8}k^{-1}Q_2^3 - \tfrac{1}{4}k^{-1}Q_2^2 + \tfrac{1}{4}k^{-1}Q_2Q_3 - \tfrac{1}{8}k^{-1}Q_4\right).\end{aligned} \tag{B.27}$$

Hence, we estimate $\tilde{a}_3(\rho)$ by $\hat{a}_3 \equiv \tilde{a}_3(\hat{\rho})$ by replacing the last term in (B.27) with

$$\sigma_\star{}^4 S_m^{-4}\left(-\tfrac{1}{8}k^{-1}Q_2^3 - \tfrac{1}{4}k^{-1}Q_2^2 + \tfrac{1}{4}k^{-1}Q_2Q_3 - \tfrac{1}{8}k^{-1}Q_4\right).$$

Next, we define

$$A = \rho n^{-1}(\hat{a}_1 + \hat{a}_2\rho n^{-1} + \hat{a}_3\rho^2 n^{-2} + a_4\rho^3 n^{-3}) + o(n^{-4}), \tag{B.28}$$

and follow the same routine. Now, for the average of both the upper and lower bounds given by Theorem 4, we can conclude:

$$P^* + \sum_{i=1}^{4} \tfrac{1}{2}(\alpha_i + \beta_i)n^{-i} = P^* + \tfrac{1}{2}(\tilde{\alpha}_4 + \tilde{\beta}_4)n^{-4} + o(\delta^8) \ \text{ as } \ \delta \to 0,$$

where $(\tilde{\alpha}_4, \tilde{\beta}_4)$ are updated modified-coefficients.

Again, by solving $\tilde{\alpha}_4 + \tilde{\beta}_4 = 0$ for a_4, we have an updated modification of the optimal design constant as follows:

$$\begin{aligned}
&\tilde{a}_4(\rho) \\
&\quad = -k^{-1}Q_2 + 6k^{-2}Q_2^2 - \tfrac{3}{2}k^{-2}Q_2^3 - 15k^{-3}{Q_2}^3 + 9k^{-3}Q_2^4 + 14k^{-4}Q_2^4 \\
&\quad -\tfrac{3}{2}k^{-3}Q_2^5 - 14k^{-4}Q_2^5 + 5k^{-4}Q_2^6 - \tfrac{5}{8}k^{-4}Q_2^7 - 4k^{-2}Q_3 + 3k^{-2}Q_2Q_3 \\
&\quad +20k^{-3}Q_2Q_3 - 22k^{-3}Q_2^2Q_3 - 28k^{-4}Q_2^2Q_3 + 5k^{-3}Q_2^3Q_3 + 42k^{-4}Q_2^3Q_3 \\
&\quad -\tfrac{58}{3}k^{-4}Q_2^4Q_3 + \tfrac{35}{12}k^{-4}Q_2^5Q_3 + 4k^{-3}Q_3^2 + \tfrac{16}{3}k^{-4}Q_3^2 - 3k^{-3}Q_2Q_3^2 \\
&\quad -\tfrac{176}{9}k^{-4}Q_2Q_3^2 + \tfrac{50}{3}k^{-4}Q_2^2Q_3^2 - \tfrac{11}{3}k^{-4}Q_2^3Q_3^2 - \tfrac{4}{3}k^{-4}Q_3^3 + k^{-4}Q_2Q_3^3 \\
&\quad -\tfrac{3}{2}k^{-2}Q_4 - 6k^{-3}Q_4 + 13k^{-3}Q_2Q_4 + 12k^{-4}Q_2Q_4 - 3k^{-3}Q_2^2Q_4 \\
&\quad -\tfrac{76}{3}k^{-4}Q_2^2Q_4 + 12k^{-4}Q_2^3Q_4 - \tfrac{43}{24}k^{-4}Q_2^4Q_4 + \tfrac{3}{2}k^{-3}Q_3Q_4 + \tfrac{76}{9}k^{-4}Q_3Q_4 \\
&\quad -\tfrac{38}{3}k^{-4}Q_2Q_3Q_4 + \tfrac{13}{4}k^{-4}Q_2^2Q_3Q_4 - \tfrac{1}{2}k^{-4}Q_3^2Q_4 + \tfrac{5}{3}k^{-4}Q_4^2 - \tfrac{5}{8}k^{-4}Q_2Q_4^2 \qquad \text{(B.29)} \\
&\quad -4k^{-3}Q_5 - \tfrac{16}{5}k^{-4}Q_5 + \tfrac{3}{2}k^{-3}Q_2Q_5 + \tfrac{34}{3}k^{-4}Q_2Q_5 - k^{-4}Q_2Q_3Q_5 \\
&\quad -6k^{-4}Q_2^2Q_5 + \tfrac{11}{12}k^{-4}Q_2^3Q_5 + 2k^{-4}Q_3Q_5 + \tfrac{1}{4}k^{-4}Q_4Q_5 - \tfrac{1}{2}k^{-3}Q_6 \\
&\quad -\tfrac{26}{9}k^{-4}Q_6 + \tfrac{8}{3}k^{-4}Q_2Q_6 - \tfrac{5}{12}k^{-4}Q_2^2Q_6 + \tfrac{1}{6}k^{-4}Q_3Q_6 - \tfrac{2}{3}k^{-4}Q_7 \\
&\quad +\tfrac{1}{6}k^{-4}Q_2Q_7 - \tfrac{1}{24}k^{-4}Q_8 + \rho^{-2}\left(\tfrac{3}{2}k^{-2}Q_2 - \tfrac{1}{8}k^{-1}\rho^{-1}Q_2^2 - \tfrac{1}{4}k^{-1}Q_2^2\right) \\
&\quad +\rho^{-2}\Big\{\left(-\tfrac{1}{2}k^{-2}Q_2^2 - \tfrac{1}{128}\rho^{-2}Q_2^3 - \tfrac{1}{8}k^{-1}Q_2^3 - \tfrac{1}{2}k^{-2}Q_2^3 - \tfrac{1}{4}k^{-2}Q_2^4 - \tfrac{3}{16}k^{-2}Q_2^5\right) \\
&\quad + \left(\tfrac{1}{64}\rho^{-2}Q_2Q_3 + \tfrac{1}{4}k^{-1}Q_2Q_3 + \tfrac{1}{4}k^{-2}Q_2Q_3 + \tfrac{1}{3}k^{-2}Q_2^2Q_3 + \tfrac{5}{8}k^{-2}Q_2^3Q_3\right) \\
&\quad + \left(\tfrac{1}{6}k^{-2}Q_3^2 - \tfrac{3}{8}k^{-2}Q_2Q_3^2 - \tfrac{1}{384}\rho^{-2}Q_4 - \tfrac{1}{8}k^{-1}Q_4 - \tfrac{1}{12}k^{-2}Q_2Q_4\right) \\
&\quad + \left(-\tfrac{3}{8}k^{-2}Q_2^2Q_4 + \tfrac{3}{16}k^{-2}Q_3Q_4 - \tfrac{1}{6}k^{-2}Q_5 + \tfrac{3}{16}k^{-2}Q_2Q_5 - \tfrac{1}{16}k^{-2}Q_6\right)\Big\}.
\end{aligned}$$

Hence, we estimate $\tilde{a}_4(\rho)$ by $\hat{a}_4 \equiv \tilde{a}_4(\hat{\rho})$, that is, by replacing $\rho^2 \equiv (\sigma^2/\sigma_\star^2)^2$ with $S_m^4\sigma_\star^{-4}$ throughout the last term in (B.29). Note that $\widehat{a}_4$ is a consistent estimator of $\tilde{a}_4(\rho)$. The result follows after much tedious simplifications. □

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