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Calculation of Harish-Chandra homomorphism for certain non-central elements in the enveloping algebras of classical Lie algebras

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1 Introduction

E_{ij} を $\mathfrak{gl}_N = \operatorname{End}\mathbb{C}^N$ の行列単位として $F_{ij} = \frac{1}{2}(E_{ij} - E_{N+1-j,N+1-i})$ とすると，F_{ij} は $\mathbb{C}$-線形空間として $\mathfrak{gl}_N$ の部分 Lie 環 $\mathfrak{g} := \mathfrak{o}_N$ を張る．$U(\mathfrak{g})$ を係数とする N 次正方行列 $F_{\pi^\natural} = (F_{ij}) \in M(N; U(\mathfrak{g}))$ を考える．$U(\mathfrak{g})$ の中心を $Z(\mathfrak{g})$ とすると，任意の $q(x) \in \mathbb{C}[x]$ について $\operatorname{Trace} q(F_{\pi^\natural}) \in Z(\mathfrak{g})$ であるが，短期共同研究のすぐ後，参加者の一部の間で $\operatorname{Trace} q(F_{\pi^\natural})$ の Harish-Chandra 同型 γ による像である

(1)　$\gamma(\operatorname{Trace} q(F_{\pi^\natural}))$ の計算

が既にされているか等についての e-mail による議論があった．これは Gould の結果 [Go1] の特別なものであることを知っていたが (その他にも (1) に対するいろいろな結果があることを上記の議論中に教わった)，その際 [Go1] を見直してみたところ，面白い側面に気付いた．それは，$\mathfrak{a}^*$ ($\mathfrak{a}$ は $\mathfrak{g}$ の Cartan 部分環) 上の多項式関数とみたときの $\gamma(\operatorname{Trace} q(F_{\pi^\natural})) \in S(\mathfrak{a})$ をスカラー一般 Verma 加群 $M_\Theta(\lambda)$ のスカラーパラメータ λ が作る $\mathfrak{a}^*$ のアフィン部分空間に制限すると，Gould の公式中の各項・各因子が非自明にキャンセルした大変単純化された公式 (“Gould の公式の退化版”) が元々の Gould の公式と同じ方法により得られる，というものである (§4).

一方，$M_\Theta(\lambda)$ の最小多項式 $q_{\pi^\natural, M_\Theta(\lambda)}(x)$ についての研究結果 ([Os], [OOs]) を精密化するために

(3)　任意の $q(x) \in \mathbb{C}[x]$ に対して $(Q_{ij}) = q(F_{\pi^\natural})$ としたときの各 $\gamma(Q_{ij})$ の計算,

(4)　特に，$q(x) = q_{\pi^\natural, M_\Theta(\lambda)}(x)$ のときの各 $\gamma(Q_{ij})$ の計算,

を数年来試みてきたが，最近になってようやく Harish-Chandra 同型の small 表現への一般化 ([O]) を応用して，(3) (従って (4)) を少なくとも $S(\mathfrak{a})$ 内で閉じた形で与えることができた (§3).

最小多項式 $q_{\pi^\natural, M_\Theta(\lambda)}(x) \in \mathbb{C}[x]$ は x に $F_{\pi^\natural}$ を代入したときの各成分が $\operatorname{Ann} M_\Theta(\lambda)$ に属するような最小次数のモニック多項式であり，[Os] で導入された．$q_{\pi^\natural, M_\Theta(\lambda)}(x)$ の倍

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元 $q(x)$ と $M_\Theta(\lambda)$ のパラメータ λ に対して $U(\mathfrak{g})$ の両側イデアル $I_{q(x)}(\lambda)$ が自然に定まり，$\operatorname{Ann} M(\lambda_\Theta) \subset I_{q(x)}(\lambda) \subset \operatorname{Ann} M_\Theta(\lambda)$ が成り立つ．ここで $M(\lambda_\Theta)$ は $M_\Theta(\lambda)$ を商加群として持つ Verma 加群である．[Os] では

(5) generic な λ に対して $q_{\pi^\natural, M_\Theta(\lambda)}(x)$ に一致する x, λ の多項式 $q_\Theta(\mathfrak{g}; x, \lambda)$,

(6) $M_\Theta(\lambda) = M(\lambda_\Theta)/I_{q_\Theta(\mathfrak{g};x,\lambda)}(\lambda)M(\lambda_\Theta)$ が成り立つための λ の十分条件,

が与えられた．(6) は積分幾何等への応用があり重要である ([Os], [OOs] 参照)．今回これらを強めた以下が得られた：

(5^+) 各 λ に対する $q_{\pi^\natural, M_\Theta(\lambda)}(x)$ の明示公式,

(6^+) $M_\Theta(\lambda) = M(\lambda_\Theta)/I_{q_{\pi^\natural, M_\Theta(\lambda)}(x)}(\lambda)M(\lambda_\Theta)$ が成り立つための λ の必要十分条件.

(5^+) については §2 の諸定理で完全な結果を述べる．本稿では証明の一部を省いたが，これは [BJ] の結果を主に用いるもので，本稿の主要テーマである Harish-Chandra 準同型の計算とはあまり関係がないからである．(6^+) は Lemma 5.1 により $S(\mathfrak{a})$ の左イデアル

$$\gamma\Bigl(I_{q_{\pi^\natural, M_\Theta(\lambda)}(x)}(\lambda)\Bigr) = \sum_{ij} S(\mathfrak{a})\gamma(Q_{ij}) + \sum_{D \in S(\mathfrak{a})^W} S(\mathfrak{a})\bigl(D - \gamma(D)(\lambda+\rho)\bigr)$$

$$\text{(ここで } \bigl(Q_{ij}\bigr) = q_{\pi^\natural, M_\Theta(\lambda)}(F_{\pi^\natural})\text{)}$$

の零点の決定することに帰着する．実は Q_{ii} ($i \le \frac{N}{2}$) に対する (3) の結果はそのままでは複雑すぎてこの問題に使うことはできないのだが，(4) を弱めた

(4^-) $q(x) = q_{\pi^\natural, M_\Theta(\lambda)}(x)$ のときの各 $\gamma(Q_{ij}) \bmod \sum_{D \in S(\mathfrak{a})^W} S(\mathfrak{a})\bigl(D - \gamma(D)(\lambda+\rho)\bigr)$

を $i = j \le \frac{N}{2}$ の場合にうまく計算することができ (Proposition 5.3)，これで上記の零点が完全に分かる．(4^-) は "Gould の公式の退化版" が持つ不思議な対称性と (3) を使って計算される．その際，いくつかの有理関数の等式 (5.11), (5.12), (5.22), (5.23), (5.24) が鍵となるが，これらの等式はいずれも底空間の Zariski dense な部分集合の各点で evaluate することにより確かめられる．各等式に対する Zariski dense な部分集合は表現論的にその等式が成立するとできるところであるが，1 つの $\mathfrak{g}$ に対する表現論だけではなく，無数の異なる Lie 環の表現論が絡んでいるところに注目してほしい．

ここまで $\mathfrak{g} = \mathfrak{o}_N$ としてきたが，ここで述べたことは他の古典型複素 Lie 環にも当てはまる．$\mathfrak{gl}_n$ に対する (5^+), (6^+) はすでに [Os] で得られているので，本稿では $\mathfrak{g} = \mathfrak{o}_{2n+1}, \mathfrak{sp}_n, \mathfrak{o}_{2n}$ を主な対象とする．

2 Generalized Verma modules and minimal polynomials

Let $\mathfrak{g}$ be a complex reductive Lie algebra, $\mathfrak{a}$ a fixed Cartan subalgebra of $\mathfrak{g}$, $\Sigma(\mathfrak{g})$ the root system for $(\mathfrak{a}, \mathfrak{g})$, and $\Sigma(\mathfrak{g})^+$ a fixed positive system of $\Sigma(\mathfrak{g})$. Let $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ be the nilpotent Lie subalgebras of $\mathfrak{g}$ corresponding to $\Sigma(\mathfrak{g})^+$ and $-\Sigma(\mathfrak{g})^+$, respectively.

We thus have the triangular decomposition $\mathfrak{g} = \bar{\mathfrak{n}} + \mathfrak{a} + \mathfrak{n}$. Let $\Psi(\mathfrak{g})$ be the basis of $\Sigma(\mathfrak{g})^+$ (the set of simple roots). As usual, ρ is the half sum of positive roots and if $\mathfrak{l}$ is a Lie algebra, $U(\mathfrak{l})$ (resp. $S(\mathfrak{l})$) denotes the universal enveloping algebra (resp. the symmetric algebra) of $\mathfrak{l}$. Associated to a subset Θ of $\Psi(\mathfrak{g})$, define the following Lie subalgebras of $\mathfrak{g}$:

$$\begin{aligned}
\mathfrak{a}_\Theta &= \{H \in \mathfrak{a};\ \alpha(H) = 0 \text{ for } \alpha \in \Theta\},\\
\mathfrak{g}_\Theta &= \{X \in \mathfrak{g};\ [H, X] = 0 \text{ for } H \in \mathfrak{a}_\Theta\},\\
\mathfrak{m}_\Theta &= [\mathfrak{g}_\Theta, \mathfrak{g}_\Theta],\\
\mathfrak{p}_\Theta &= \mathfrak{g}_\Theta + \mathfrak{n},\\
\mathfrak{n}_\Theta &= \text{the nilpotent radical of } \mathfrak{p}_\Theta,\\
\bar{\mathfrak{n}}_\Theta &= \text{the nilpotent radical of } \mathfrak{g}_\Theta + \bar{\mathfrak{n}}.
\end{aligned}$$

Moreover, if $\lambda \in \mathfrak{a}_\Theta^*$ then the character λ_Θ of $\mathfrak{p}_\Theta$ is defined by $\mathfrak{p}_\Theta = \mathfrak{a}_\Theta \oplus (\mathfrak{m}_\Theta + \mathfrak{n}_\Theta) \xrightarrow{\text{projection}} \mathfrak{a}_\Theta \xrightarrow{\lambda} \mathbb{C}$. Put

$$\begin{aligned}
J_\Theta(\lambda) &= \sum_{X \in \mathfrak{p}_\Theta} U(\mathfrak{g})(X - \lambda_\Theta(X)),\\
M_\Theta(\lambda) &= U(\mathfrak{g})/J_\Theta(\lambda).
\end{aligned}$$

The left $U(\mathfrak{g})$-module $M_\Theta(\lambda)$ is called a *scalar generalized Verma module.* If $\Theta = \emptyset$, we also use the simple symbols $J(\lambda)$ and $M(\lambda)$ for $J_\emptyset(\lambda)$ and $M_\emptyset(\lambda)$. Note that in this case, $M(\lambda)$ is a *Verma module* with highest weight $\lambda \in \mathfrak{a}^*$.

Suppose (π, V) is a faithful finite-dimensional representation of $\mathfrak{g}$ such that the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\operatorname{End} V \times \operatorname{End} V$ defined by $\langle X, Y \rangle = \operatorname{Trace}(XY)$ is non-degenerate on $\pi(\mathfrak{g}) \times \pi(\mathfrak{g})$. Then $\langle \cdot, \cdot \rangle$ defines a natural projection $p : \operatorname{End} V \to \pi(\mathfrak{g})$. Via the natural identifications $\pi(\mathfrak{g}) \simeq \mathfrak{g} \subset U(\mathfrak{g})$ and $(\operatorname{End} V)^* \simeq \operatorname{End} V^*$, we identify p with an element $F_\pi \in U(\mathfrak{g}) \otimes \operatorname{End} V^*$. Note that $U(\mathfrak{g}) \otimes \operatorname{End} V^*$ has the same algebra structure with the algebra $M(\dim V; U(\mathfrak{g}))$ of $U(\mathfrak{g})$-coefficient square matrices of size $\dim V$. The following notion of *minimal polynomials* is defined in [Os] (see also [OOs]).

Definition 2.1 (minimal polynomial). Let $\Theta \subset \Psi(\mathfrak{g})$ and $\lambda \in \mathfrak{a}_\Theta^*$. The minimal polynomial $q_{\pi, M_\Theta(\lambda)}(x) \in \mathbb{C}[x]$ for the pair $(\pi, M_\Theta(\lambda))$ is defined to be a monic polynomial satisfying $q_{\pi, M_\Theta(\lambda)}(F_\pi) \in (\operatorname{Ann} M_\Theta(\lambda)) \otimes \operatorname{End} V^*$ with the minimal degree.

The existence and the uniqueness of $q_{\pi, M_\Theta(\lambda)}(x)$ are assured in [Os]. As stated in §1, if $\mathfrak{g}$ is a classical Lie algebra and π is its natural representation, the explicit formula of $q_{\pi, M_\Theta(\lambda)}(x)$ for a generic λ is obtained in [Os]. We shall now give a precise description of it together with some related results by [OOs].

Suppose $\ell = 1, 2, \ldots$ and V_ℓ is an ℓ-dimensional vector space with basis $\{v_1, \ldots, v_\ell\}$. Let $\{v_1^*, \ldots, v_\ell^*\}$ be the dual basis of $\{v_1, \ldots, v_\ell\}$ and put $E_{ij} = v_i \otimes v_j^* \in \operatorname{End} V_\ell$. The classical Lie algebras are by definition

$$(2.1)\qquad \begin{cases} \mathfrak{gl}_\ell = \operatorname{End} V_\ell, \\ \mathfrak{o}_\ell = \mathbb{C}\text{-span of } \{E_{ij} - E_{\ell+1-j,\ell+1-i} \in \operatorname{End} V_\ell\} \quad (\ell > 1), \\ \mathfrak{sp}_\ell = \mathbb{C}\text{-span of} \\ \qquad \{E_{ij} - \operatorname{sgn}\left((i-\ell-\frac{1}{2})(j-\ell-\frac{1}{2})\right) E_{2\ell+1-j,2\ell+1-i} \in \operatorname{End} V_{2\ell}\}. \end{cases}$$

(Throughout the paper we use such realizations.) If $\mathfrak{g} = \mathfrak{gl}_\ell$, we always assume $\mathfrak{a}, \mathfrak{n}$, and $\bar{\mathfrak{n}}$ are the Lie subalgebras spanned by $\{E_{ii}\}$, $\{E_{ij};\, i < j\}$, and $\{E_{ij};\, i > j\}$, respectively. If $\mathfrak{g} = \mathfrak{o}_\ell$ or $\mathfrak{sp}_\ell$, we assume $\mathfrak{a}, \mathfrak{n}$, and $\bar{\mathfrak{n}}$ are the intersections of those for $\mathfrak{gl}_\ell$ or $\mathfrak{gl}_{2\ell}$ with $\mathfrak{g}$. The natural representation $\pi^\natural$ of $\mathfrak{g} = \mathfrak{gl}_\ell, \mathfrak{o}_\ell$, or $\mathfrak{sp}_\ell$ is by definition the representation on V_ℓ or $V_{2\ell}$ coming from the inclusion map $\mathfrak{gl}_\ell \xrightarrow{\sim} \operatorname{End} V_\ell$, $\mathfrak{o}_\ell \hookrightarrow \operatorname{End} V_\ell$, or $\mathfrak{sp}_\ell \hookrightarrow \operatorname{End} V_{2\ell}$. In $\pi^\natural$ each v_i is a weight vector. Let e_i denote the weight of v_i. Thus for $n = 1, 2, \ldots,$

$$(2.2)\qquad \begin{cases} (A_{n-1}) & \Psi(\mathfrak{gl}_n) = \{e_1 - e_2, \ldots, e_{n-1} - e_n\}, \\ (B_n) & \Psi(\mathfrak{o}_{2n+1}) = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_n\}, \\ (C_n) & \Psi(\mathfrak{sp}_n) = \{e_1 - e_2, \ldots, e_{n-1} - e_n, 2e_n\}, \\ (D_n) & \Psi(\mathfrak{o}_{2n}) = \{e_1 - e_2, \ldots, e_{n-1} - e_n, e_{n-1} + e_n\}. \end{cases}$$

For an increasing sequence of integers

$$(2.3)\qquad \Pi : n_0 = 0 < n_1 < \cdots < n_L = n,$$

we define

$$(2.4)\qquad \Theta_\Pi = \bigcup_{k=1}^{L} \bigcup_{n_{k-1} < i < n_k} \{e_i - e_{i+1}\} \subset \Psi(\mathfrak{gl}_n), \Psi(\mathfrak{o}_{2n+1}), \Psi(\mathfrak{sp}_n), \text{or } \Psi(\mathfrak{o}_{2n}),$$

and if $\mathfrak{g} = \mathfrak{o}_{2n+1}, \mathfrak{sp}_n$ $(n \geq 1)$, or $\mathfrak{o}_{2n}$ $(n \geq 2)$, we also define

$$(2.5)\qquad \overline{\Theta}_\Pi = \Theta_\Pi \cup \{\text{the } n\text{-th simple root of (2.2)}\} \subset \Psi(\mathfrak{o}_{2n+1}), \Psi(\mathfrak{sp}_n), \text{or } \Psi(\mathfrak{o}_{2n}).$$

An element $\lambda \in \mathfrak{a}^*_{\Theta_\Pi}$ is identified with $(\lambda_1, \ldots, \lambda_L) \in \mathbb{C}^L$ by the relation $\lambda_{\Theta_\Pi}|_{\mathfrak{a}} = \sum_{k=1}^{L} \lambda_k \left(\sum_{n_{k-1} < i \leq n_k} e_i\right)$. The similar identification $\mathfrak{a}^* \simeq \mathbb{C}^n$ is also used. In this paper, whenever we work with $\overline{\Theta}_\Pi$ for $\mathfrak{o}_{2n}$, we assume $e_{n-1} - e_n \in \overline{\Theta}_\Pi$, or equivalently $n_{L-1} < n - 1$. This assumption is justifiable because if τ is an outer automorphism of $\mathfrak{o}_{2n}$ corresponding to the transposition of $e_{n-1} - e_n$ and $e_{n-1} + e_n$ in the Dynkin diagram then $(\tau \circ \pi^\natural \circ \tau^{-1}, V_{2n}) \simeq (\pi^\natural, V_{2n})$. Accordingly, in all cases an element $\lambda \in \mathfrak{a}^*_{\overline{\Theta}_\Pi}$ is identified with $(\lambda_1, \ldots, \lambda_{L-1}) \in \mathbb{C}^{L-1}$ by $\lambda_{\overline{\Theta}_\Pi}|_{\mathfrak{a}} = \sum_{k=1}^{L-1} \lambda_k \left(\sum_{n_{k-1} < i \leq n_k} e_i\right)$.

Definition 2.2. Define the following polynomials in x with parameter $\lambda \in \mathfrak{a}^*_{\Theta_\Pi}$ or $\mathfrak{a}^*_{\overline{\Theta}_\Pi}$:

$$q_{\Theta_\Pi}(\mathfrak{gl}_n; x, \lambda) = \prod_{k=1}^{L} \left(x + \lambda_k - n_{k-1}\right),$$

$$q_{\Theta_\Pi}(\mathfrak{o}_{2n+1};x,\lambda)=\Big(x-\frac{n}{2}\Big)\prod_{k=1}^{L}\Big(x+\frac{\lambda_k}{2}-\frac{n_{k-1}}{2}\Big)\Big(x-\frac{\lambda_k}{2}-\frac{2n-n_k}{2}\Big),$$

$$q_{\overline{\Theta}_\Pi}(\mathfrak{o}_{2n+1};x,\lambda)=\Big(x-\frac{n_{L-1}}{2}\Big)\prod_{k=1}^{L-1}\Big(x+\frac{\lambda_k}{2}-\frac{n_{k-1}}{2}\Big)\Big(x-\frac{\lambda_k}{2}-\frac{2n-n_k}{2}\Big),$$

$$q_{\Theta_\Pi}(\mathfrak{sp}_n;x,\lambda)=\prod_{k=1}^{L}\Big(x+\frac{\lambda_k}{2}-\frac{n_{k-1}}{2}\Big)\Big(x-\frac{\lambda_k}{2}-\frac{2n-n_k+1}{2}\Big),$$

$$q_{\overline{\Theta}_\Pi}(\mathfrak{sp}_n;x,\lambda)=\Big(x-\frac{n_{L-1}}{2}\Big)\prod_{k=1}^{L-1}\Big(x+\frac{\lambda_k}{2}-\frac{n_{k-1}}{2}\Big)\Big(x-\frac{\lambda_k}{2}-\frac{2n-n_k+1}{2}\Big),$$

$$q_{\Theta_\Pi}(\mathfrak{o}_{2n};x,\lambda)=\prod_{k=1}^{L}\Big(x+\frac{\lambda_k}{2}-\frac{n_{k-1}}{2}\Big)\Big(x-\frac{\lambda_k}{2}-\frac{2n-n_k-1}{2}\Big),$$

$$q_{\overline{\Theta}_\Pi}(\mathfrak{o}_{2n};x,\lambda)=\Big(x-\frac{n_{L-1}}{2}\Big)\prod_{k=1}^{L-1}\Big(x+\frac{\lambda_k}{2}-\frac{n_{k-1}}{2}\Big)\Big(x-\frac{\lambda_k}{2}-\frac{2n-n_k-1}{2}\Big).$$

Theorem 2.3 ([Os]). *Suppose $\mathfrak{g}=\mathfrak{gl}_n$. Then the minimal polynomial $q_{\pi^\natural,M_{\Theta_\Pi}(\lambda)}(x)$ equals $q_{\Theta_\Pi}(\mathfrak{gl}_n;x,\lambda)$ for any $\lambda\in\mathfrak{a}^*_{\Theta_\Pi}$.*

Theorem 2.4 ([Os], [OOs]). *Suppose $\mathfrak{g}=\mathfrak{o}_{2n+1},\mathfrak{sp}_n$, or $\mathfrak{o}_{2n}$, and $\Theta=\Theta_\Pi$ or $\overline{\Theta}_\Pi$. (If $\mathfrak{g}=\mathfrak{o}_{2n}$ and $\Theta=\overline{\Theta}_\Pi$, we assume $n_{L-1}<n-1$.) Then the minimal polynomial $q_{\pi^\natural,M_\Theta(\lambda)}(x)$ divides $q_\Theta(\mathfrak{g};x,\lambda)$ for any $\lambda\in\mathfrak{a}^*_\Theta$. Moreover, if each root of $q_\Theta(\mathfrak{g};x,\lambda)\in\mathbb{C}[x]$ is simple, then the two polynomials coincide.*

Theorem 2.5 ([Os], [OOs]). *Suppose $\mathfrak{g}=\mathfrak{o}_{2n}$, $e_{n-1}-e_n\notin\Theta_\Pi$ and $\lambda\in\mathfrak{a}^*_{\Theta_\Pi}$ is of the form $(\lambda_1,\dots,\lambda_{L-1},0)$. Then $n_{L-1}=n-1$ and $q_{\Theta_\Pi}(\mathfrak{o}_{2n};x,\lambda)$ has the double root $x=\frac{n-1}{2}$. In this case, $q_{\pi^\natural,M_{\Theta_\Pi}(\lambda)}(x)$ divides $\frac{1}{x-\frac{n-1}{2}}q_{\Theta_\Pi}(\mathfrak{o}_{2n};x,\lambda)$. Also, if each root of $\frac{1}{x-\frac{n-1}{2}}q_{\Theta_\Pi}(\mathfrak{o}_{2n};x,\lambda)\in\mathbb{C}[x]$ is simple, then*

$$q_{\pi^\natural,M_{\Theta_\Pi}(\lambda)}(x)=\frac{1}{x-\frac{n-1}{2}}q_{\Theta_\Pi}(\mathfrak{o}_{2n};x,\lambda). \tag{2.6}$$

Compared with Theorem 2.4 or Theorem 2.5, Theorem 2.3 is strong since it determines the minimal polynomials for all parameters. As a matter of fact, it is not so difficult to strengthen part of Theorem 2.4 and Theorem 2.5 at this level.

Theorem 2.6. (i) *Suppose $\mathfrak{g}=\mathfrak{o}_{2n+1}$, and $\Theta=\Theta_\Pi$ or $\overline{\Theta}_\Pi$. Then $q_{\pi^\natural,M_\Theta(\lambda)}(x)=q_\Theta(\mathfrak{o}_{2n+1};x,\lambda)$ for any $\lambda\in\mathfrak{a}^*_\Theta$.*
(ii) *Suppose $\mathfrak{g}=\mathfrak{sp}_n$. Then $q_{\pi^\natural,M_{\Theta_\Pi}(\lambda)}(x)=q_{\Theta_\Pi}(\mathfrak{sp}_n;x,\lambda)$ for any $\lambda\in\mathfrak{a}^*_{\Theta_\Pi}$.*
(iii) *Suppose $\mathfrak{g}=\mathfrak{o}_{2n}$ and $\Theta=\Theta_\Pi$ or $\overline{\Theta}_\Pi$. (If $\Theta=\overline{\Theta}_\Pi$, we assume $n_{L-1}<n-1$.) Then $\deg_x q_{\pi^\natural,M_\Theta(\lambda)}(x)\geq 2L-1$. Therefore, $q_{\pi^\natural,M_{\overline{\Theta}_\Pi}(\lambda)}(x)=q_{\overline{\Theta}_\Pi}(\mathfrak{o}_{2n};x,\lambda)$ for any $\lambda\in\mathfrak{a}^*_{\overline{\Theta}_\Pi}$. Furthermore, in* Theorem 2.5, *the equality* (2.6) *holds even when $\frac{1}{x-\frac{n-1}{2}}q_{\Theta_\Pi}(\mathfrak{o}_{2n};x,\lambda)\in\mathbb{C}[x]$ has a double root.*

Proof. Regard $U(\mathfrak{g})$ as a filtered algebra by the standard filtration $\{U_d(\mathfrak{g})\}$ and $J_\Theta(\lambda)$ as a filtered $U(\mathfrak{g})$-module by the filtration $\{J_\Theta(\lambda) \cap U_d(\mathfrak{g})\}$. Then $\operatorname{gr} J_\Theta(\lambda) = S(\mathfrak{g})\mathfrak{p}_\Theta$ where $\operatorname{gr} J_\Theta(\lambda)$ is the graded $S(\mathfrak{g})$-module associated to $J_\Theta(\lambda)$. Denote the symmetric bilinear form $\langle \pi^\natural(\cdot), \pi^\natural(\cdot)\rangle$ on $\mathfrak{g} \times \mathfrak{g}$ simply by $\langle \cdot, \cdot \rangle$ and identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through it. Since $\langle \cdot, \cdot \rangle$ has $\operatorname{ad}(\mathfrak{g})$-invariance, it is easy to see $\langle X, Y \rangle = 0$ for $X \in \mathfrak{n}_\Theta$ and $Y \in \mathfrak{p}_\Theta$. For $D \in U_d(\mathfrak{g})$ $(d = 0, 1, 2, \ldots)$, let $\sigma_d(D)$ be its image under the composition map $U_d(\mathfrak{g}) \to U_d(\mathfrak{g})/U_{d-1}(\mathfrak{g}) \to S(\mathfrak{g})$. Since $\operatorname{Ann} M_\Theta(\lambda) \subset J_\Theta(\lambda)$, $\sigma_d(D)(X) = 0$ for $D \in \operatorname{Ann} M_\Theta(\lambda) \cap U_d(\mathfrak{g})$ and $X \in \mathfrak{n}_\Theta$. We first show that if $\tilde{d} = \deg_x q_{\pi^\natural, M_\Theta(\lambda)}(x)$ then $\pi^\natural(X)^{\tilde{d}} = 0$ for any $X \in \mathfrak{n}_\Theta$. For this purpose, suppose $\tilde{d} = \deg_x q_{\pi^\natural, M_\Theta(\lambda)}(x)$ and take an arbitrary $X \in \mathfrak{n}_\Theta$. Let V_N be the representation space of $\pi^\natural$. Since $F_{\pi^\natural} = \sum_{i,j} p(E_{ij}) \otimes (v_i^* \otimes v_j)$, the image of $q_{\pi^\natural, M_\Theta(\lambda)}(F_{\pi^\natural})$ under the composition map

$$U_{\tilde{d}}(\mathfrak{g}) \otimes \operatorname{End} V_N^* \xrightarrow{\sigma_{\tilde{d}} \otimes 1_{\operatorname{End} V_N^*}} S(\mathfrak{g}) \otimes \operatorname{End} V_N^* \xrightarrow{(\text{evaluation at } X) \otimes 1_{\operatorname{End} V_N^*}} \operatorname{End} V_N^*$$

is $\left(\sum_{i,j} \langle p(E_{ij}), X \rangle \otimes (v_i^* \otimes v_j)\right)^{\tilde{d}} = \left(\sum_{i,j} \langle E_{ij}, \pi^\natural(X) \rangle \otimes (v_i^* \otimes v_j)\right)^{\tilde{d}} = ({}^t\pi^\natural(X))^{\tilde{d}}$ where ${}^t\pi^\natural(X) \in \operatorname{End} V_N^*$ is the transposed map of $\pi^\natural(X) \in \operatorname{End} V_N$. But since $q_{\pi^\natural, M_\Theta(\lambda)}(F_{\pi^\natural}) \in \operatorname{Ann} M_\Theta(\lambda) \otimes \operatorname{End} V_N^*$, we get $({}^t\pi^\natural(X))^{\tilde{d}} = 0$, which shows our claim.

Next, we examine each case separately using the matrix expression for $\operatorname{End} V_N$ with respect to the basis $\{v_i\}$. Put $n'_k = n_k - n_{k-1}$ $(k = 1, \ldots, L)$ and

$$\tilde{\mathbf{1}}_{st} = \overset{\longleftarrow\; t \;\longrightarrow}{s\updownarrow\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}} \qquad (s, t = 1, 2, \ldots).$$

(i) <u>Case $\mathfrak{g} = \mathfrak{o}_{2n+1}$ with $\Theta = \Theta_{\mathrm{II}}$.</u> Take

$$X = \begin{array}{c} \\ n'_1 \\ n'_2 \\ \\ n'_L \\ 1 \\ n'_L \\ \\ n'_2 \\ n'_1 \end{array}\!\!\begin{array}{c} \begin{array}{ccccccccc} n'_1 & n'_2 & & n'_L & 1 & n'_L & & n'_2 & n'_1 \end{array} \\ \begin{pmatrix} 0 & \tilde{\mathbf{1}}_{n'_1 n'_2} & & & & & & & \\ & 0 & \ddots & & & & & 0 & \\ & & \ddots & \ddots & & & & & \\ & & & 0 & \tilde{\mathbf{1}}_{n'_L 1} & & & & \\ & & & & 0 & -\tilde{\mathbf{1}}_{1 n'_L} & & & \\ & & & & & 0 & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & 0 & & & & & & 0 & -\tilde{\mathbf{1}}_{n'_2 n'_1} \\ & & & & & & & & 0 \end{pmatrix} \end{array} \in \mathfrak{n}_{\Theta_{\mathrm{II}}}.$$

Then $\pi^\natural(X)^{2L} \neq 0$. Hence $\deg_x q_{\pi^\natural, M_{\Theta_{\mathrm{II}}}(\lambda)}(x) \geq 2L + 1 = \deg_x q_{\Theta_{\mathrm{II}}}(\mathfrak{o}_{2n+1}; x, \lambda)$ for any $\lambda \in \mathfrak{a}^*_{\Theta_{\mathrm{II}}}$.

Case $\mathfrak{g} = \mathfrak{o}_{2n+1}$ with $\Theta = \overline{\Theta}_\Pi$. Take

$$X = \begin{array}{c} \\ n_1' \\ n_2' \\ \\ n_{L-1}' \\ 2n_L'+1 \\ n_{L-1}' \\ \\ n_2' \\ n_1' \end{array} \begin{array}{c} \begin{array}{ccccccccc} n_1' & n_2' & & n_{L-1}' & 2n_L'+1 & n_{L-1}' & & n_2' & n_1' \end{array} \\ \begin{pmatrix} 0 & \tilde{\mathbf{1}}_{n_1'n_2'} & & & & & & & 0 \\ & 0 & \ddots & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & 0 & \tilde{\mathbf{1}}_{n_{L-1}',2n_L'+1} & & & & \\ & & & & 0 & -\tilde{\mathbf{1}}_{2n_L'+1,n_{L-1}'} & & & \\ & & & & & 0 & \ddots & & \\ & & & & & & \ddots & \ddots & \\ & & & & & & & 0 & -\tilde{\mathbf{1}}_{n_2'n_1'} \\ 0 & & & & & & & & 0 \end{pmatrix} \end{array}$$

$\in \mathfrak{n}_{\overline{\Theta}_\Pi}$. Then $\pi^\natural(X)^{2L-2} \neq 0$. Hence $\deg_x q_{\pi^\natural, M_{\overline{\Theta}_\Pi}(\lambda)}(x) \geq 2L-1 = \deg_x q_{\overline{\Theta}_\Pi}(\mathfrak{o}_{2n+1}; x, \lambda)$ for any $\lambda \in \mathfrak{a}^*_{\overline{\Theta}_\Pi}$.

(ii) Case $\mathfrak{g} = \mathfrak{sp}_n$ with $\Theta = \Theta_\Pi$. Take

$$X = \begin{array}{c} \\ n_1' \\ \\ n_{L-1}' \\ n_L' \\ n_L' \\ \\ n_2' \\ n_1' \end{array} \begin{array}{c} \begin{array}{ccccccc} n_1' & n_2' & & n_L' & n_L' & n_{L-1}' & n_1' \end{array} \\ \begin{pmatrix} 0 & \tilde{\mathbf{1}}_{n_1'n_2'} & & & & & 0 \\ & \ddots & \ddots & & & & \\ & & \ddots & \tilde{\mathbf{1}}_{n_{L-1}'n_L'} & & & \\ & & & 0 & \tilde{\mathbf{1}}_{n_L'n_L'} & & \\ & & & & 0 & -\tilde{\mathbf{1}}_{n_L'n_{L-1}'} & \\ & & & & & \ddots & \ddots \\ & & & & & & \ddots \quad -\tilde{\mathbf{1}}_{n_2'n_1'} \\ 0 & & & & & & 0 \end{pmatrix} \end{array} \in \mathfrak{n}_{\Theta_\Pi}.$$

Then $\pi^\natural(X)^{2L-1} \neq 0$. Hence $\deg_x q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x) \geq 2L = \deg_x q_{\Theta_\Pi}(\mathfrak{sp}_n; x, \lambda)$ for any $\lambda \in \mathfrak{a}^*_{\Theta_\Pi}$.

(iii) Case $\mathfrak{g} = \mathfrak{o}_{2n}$ with $\Theta = \Theta_\Pi$ or $\overline{\Theta}_\Pi$. Take

$$X = \begin{array}{c|ccccccccc|}
 & n'_1 & n'_2 & & n'_{L-1} & 2n'_L & n'_{L-1} & & n'_2 & n'_1 \\
\hline
n'_1 & 0 & \tilde{1}_{n'_1 n'_2} & & & & & & & 0 \\
n'_2 & & 0 & \ddots & & & & & & \\
 & & & \ddots & \ddots & & & & & \\
n'_{L-1} & & & & 0 & \tilde{1}_{n'_{L-1},2n'_L} & & & & \\
2n'_L & & & & & 0 & -\tilde{1}_{2n'_L,n'_{L-1}} & & & \\
n'_{L-1} & & & & & & 0 & \ddots & & \\
 & & & & & & & \ddots & \ddots & \\
n'_2 & & & & & & & & 0 & -\tilde{1}_{n'_2 n'_1} \\
n'_1 & 0 & & & & & & & & 0
\end{array}$$

$\in \mathfrak{n}_{\Theta_\Pi}$ or $\mathfrak{n}_{\overline{\Theta}_\Pi}$. Then $\pi^\natural(X)^{2L-2} \neq 0$. Hence $\deg_x q_{\pi^\natural, M_\Theta(\lambda)}(x) \geq 2L-1$ for any $\lambda \in \mathfrak{a}_\Theta^*$. □

The remaining cases where the minimal polynomials are not completely determined by Theorem 2.6 are the case $\mathfrak{g} = \mathfrak{sp}_n$ with $\Theta = \overline{\Theta}_\Pi$ and the case $\mathfrak{g} = \mathfrak{o}_{2n}$ with $\Theta = \Theta_\Pi$. For the former case, we can prove the following theorem (the proof will be given in a subsequent paper).

Theorem 2.7. *Suppose $\mathfrak{g} = \mathfrak{sp}_n$. Then $q_{\pi^\natural, M_{\overline{\Theta}_\Pi}(\lambda)}(x) = q_{\overline{\Theta}_\Pi}(\mathfrak{sp}_n; x, \lambda)$ for any $\lambda \in \mathfrak{a}^*_{\overline{\Theta}_\Pi}$.*

For the latter case, Theorem 2.5 shows $q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)$ is not minimal for some $\lambda \in \mathfrak{a}^*_{\Theta_\Pi}$. The complete result is given as follows:

Theorem 2.8. *Suppose $\mathfrak{g} = \mathfrak{o}_{2n}$ and $\lambda = (\lambda_1, \ldots, \lambda_L) \in \mathfrak{a}^*_{\Theta_\Pi}$.*
(i) If there exists some $k = 1, \ldots, L$ for which both $n_k - n_{k-1} = 1$ and $\lambda_k = n_k - n$ hold, then $q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)$ has the double root $x = \frac{n-1}{2}$ and

$$q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x) = \frac{1}{x - \frac{n-1}{2}} q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda). \tag{2.7}$$

(ii) Conversely, if for each $k = 1, \ldots, L$ at least either $n_k - n_{k-1} > 1$ or $\lambda_k \neq n_k - n$ holds, then $q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x) = q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)$.

Here, we prove only (i) by using Theorem 2.31 of [OOs]. The proof of (ii) will be given in a subsequent paper.

Proof of (i). Suppose first $n = 1$. Then $\mathfrak{o}_2 = \mathfrak{a} = \mathbb{C}(E_{11} - E_{22})$, $L = 1$, $\Theta_\Pi = \emptyset$, and $M(\lambda)$ is one-dimensional space on which $\mathfrak{o}_2$ acts by λ. Furthermore $F_\pi = \begin{pmatrix} \frac{E_{11}-E_{22}}{2} & 0 \\ 0 & \frac{E_{22}-E_{11}}{2} \end{pmatrix} \in M(2; U(\mathfrak{o}_2)) \simeq U(\mathfrak{o}_2) \otimes \operatorname{End} V_2^*$. Since the assumption of (i) is equivalent to the condition $\lambda = 0$, the theorem (including (ii)) is immediate.

Secondly suppose $n \geq 2$ and the assumption of (i) is satisfied. Since $-\frac{\lambda_k}{2} + \frac{n_{k-1}}{2} = \frac{\lambda_k}{2} + \frac{2n-n_k-1}{2} = \frac{n-1}{2}$, $q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)$ has the double root $x = \frac{n-1}{2}$. The totality of weights of $(\pi^\natural, V_{2n})$ is $\{\pm e_1, \dots \pm e_n\}$:

$$\begin{array}{ccccccc} -e_1 \to -e_2 \to \cdots\cdots \to & -e_{n-1} & \to & -e_n & & & \\ & \downarrow & & \downarrow & & & \\ & e_n & \longrightarrow & e_{n-1} & \to \cdots\cdots \to e_2 \to e_1. \end{array}$$

(In the above weight diagram, an arrow is written if the difference of the weights at both ends equals some simple root.) Let $(\pi^\natural|_{\mathfrak{g}_{\Theta_\Pi}}, V_{2n}|_{\mathfrak{g}_{\Theta_\Pi}})$ be the restriction of the natural representation $(\pi^\natural, V_{2n})$ of $\mathfrak{g} = \mathfrak{o}_{2n}$ to the Lie subalgebra $\mathfrak{g}_{\Theta_\Pi}$. Then the set $\overline{\mathcal{W}}_{\Theta_\Pi}(\pi^\natural)$ of the lowest weights of $(\pi^\natural|_{\mathfrak{g}_{\Theta_\Pi}}, V_{2n}|_{\mathfrak{g}_{\Theta_\Pi}})$ is

$$\{-e_{n_0+1}, \dots, -e_{n_k+1}, \dots, -e_{n_{L-1}+1}, e_{n_L}, \dots, e_{n_k}, \dots, e_{n_1}\}$$

where a lowest weight vector is a weight vector for $\mathfrak{a}$ which is annihilated by $\mathfrak{g}_{\Theta_\Pi} \cap \bar{\mathfrak{n}}$. Let W_{D_n} be the Weyl group for $(\mathfrak{a}, \mathfrak{g})$. Since

$$\begin{aligned} \lambda_{\Theta_\Pi}|_{\mathfrak{a}} + \rho = (\lambda_1 + n - 1, \dots, \lambda_1 + n - n_1, \dots \\ \dots, \lambda_{k-1} + n - n_{k-1}, \overbrace{0}^{n_k}, \lambda_{k+1} + n - n_k - 1, \dots \\ \dots, \lambda_L + n - n_{L-1} - 1, \dots, \lambda_L + n - n_L), \end{aligned}$$

one can easily see that for a generic $(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L) \in \mathbb{C}^{L-1}$ each of $\{\lambda_{\Theta_\Pi}|_{\mathfrak{a}} + \rho - \varpi;\ \varpi \in \overline{\mathcal{W}}_{\Theta_\Pi}(\pi^\natural)\}$ is in a distinct W_{D_n}-orbit. Hence it follows from Theorem 2.31 of [OOs] that for such $(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L) \in \mathbb{C}^{L-1}$, all the roots of $q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x)$ are simple and in particular $q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x)$ divides $\frac{1}{x - \frac{n-1}{2}} q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)$. Let us show it is true for any $(\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L) \in \mathbb{C}^{L-1}$. Write $\left.\frac{1}{x - \frac{n-1}{2}} q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)\right|_{x \mapsto F_{\pi^\natural}} \in U(\mathfrak{g}) \otimes \operatorname{End} V_{2n}^*$ as

$$\sum_{i,j} Q_{ij} \otimes (v_i^* \otimes v_j) \text{ with } Q_{ij} \in U(\mathfrak{g}) \otimes \mathbb{C}[\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L].$$

Note that the canonical surjection $U(\mathfrak{g}) \to M_{\Theta_\Pi}(\lambda)$ induces an isomorphism $\nu : U(\bar{\mathfrak{n}}_{\Theta_\Pi}) \xrightarrow{\sim} M_{\Theta_\Pi}(\lambda)$. Thus for each Q_{ij} and any $u \in M_{\Theta_\Pi}(\lambda)$,

$$\nu^{-1}(Q_{ij} u) \in U(\bar{\mathfrak{n}}_{\Theta_\Pi}) \otimes \mathbb{C}[\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L].$$

Hence we have

$$\begin{aligned} & Q_{ij} \in \operatorname{Ann} M_{\Theta_\Pi}(\lambda) \text{ for a generic } (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L) \\ \Leftrightarrow\ & \nu^{-1}(Q_{ij} u) = 0 \text{ for any } u \in M_{\Theta_\Pi}(\lambda) \text{ and a generic } (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L) \\ \Leftrightarrow\ & \nu^{-1}(Q_{ij} u) = 0 \text{ for any } u \in M_{\Theta_\Pi}(\lambda) \text{ and any } (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L) \\ \Leftrightarrow\ & Q_{ij} \in \operatorname{Ann} M_{\Theta_\Pi}(\lambda) \text{ for any } (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_L). \end{aligned}$$

Thus $q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x)$ always divides $\frac{1}{x - \frac{n-1}{2}} q_{\Theta_\Pi}(\mathfrak{o}_{2n}; x, \lambda)$. But in view of Theorem 2.6 (iii) the two polynomials coincide. □

3 Degenerate affine Hecke algebras

Let $\mathfrak{g}$ be an arbitrary complex reductive Lie algebra.

Definition 3.1 (Harish-Chandra homomorphism). The *Harish-Chandra homomorphism* γ is the map of $U(\mathfrak{g})$ into $S(\mathfrak{a})$ defined as follows: If $D \in U(\mathfrak{g})$ there exists a unique $\tilde{D} \in U(\mathfrak{a})$ such that $D - \tilde{D} \in \bar{\mathfrak{n}}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}$; Consider $\tilde{D}$ as a polynomial function on $\mathfrak{a}^*$ by the identification $U(\mathfrak{a}) \simeq S(\mathfrak{a})$ and then put $\gamma(D)(\lambda) = \tilde{D}(\lambda - \rho)$ for $\lambda \in \mathfrak{a}^*$.

Let W be the Weyl group for $(\mathfrak{a}, \mathfrak{g})$, $Z(\mathfrak{g})$ the center of $U(\mathfrak{g})$, and $S(\mathfrak{a})^W$ the W-invariant subalgebra of $S(\mathfrak{a})$. It is well known that γ gives an algebra isomorphism $Z(\mathfrak{g}) \simeq S(\mathfrak{a})^W$ (the *Harish-Chandra isomorphism*). This isomorphism is generalized by [O] using the *degenerate affine Hecke algebra*. The generalized results applies to some calculations in this paper.

Definition 3.2 (degenerate affine Hecke algebra). There exists uniquely (up to equivalence) an algebra $\mathbf{H}$ over $\mathbb{C}$ with the following properties:

(i) $\mathbf{H} \simeq S(\mathfrak{a}) \otimes \mathbb{C}[W]$ as a $\mathbb{C}$-linear space.

(ii) The maps $S(\mathfrak{a}) \to \mathbf{H}, f \mapsto f \otimes 1$ and $\mathbb{C}[W] \to \mathbf{H}, w \mapsto 1 \otimes w$ are algebra homomorphisms.

(iii) $(f \otimes 1) \cdot (1 \otimes w) = f \otimes w$ for any $f \in S(\mathfrak{a})$ and $w \in W$.

(iv) $(1 \otimes s_\alpha) \cdot (\xi \otimes 1) = s_\alpha(\xi) \otimes s_\alpha + \alpha(\xi)$ for any $\alpha \in \Psi(\mathfrak{g})$ and $\xi \in \mathfrak{a}$. Here $s_\alpha \in W$ is the reflection corresponding to α.

We call $\mathbf{H}$ the degenerate affine Hecke algebra associated to the data $(\mathfrak{a}, \Psi(\mathfrak{g}))$.

Remark 3.3. (i) A usual 'degenerate affine Hecke algebra' has a deformation parameter, called the *multiplicity function*, while in our definition this parameter is fixed to a special one. Except for this point, the definition above is due to [Lu].
(ii) The center of $\mathbf{H}$ equals $S(\mathfrak{a})^W$ ([Lu, Theorem 6.5])
(iii) We identify $S(\mathfrak{a})$ and $\mathbb{C}[W]$ with subalgebras of $\mathbf{H}$. Then the relation in Definition 3.2 (iv) is simply written as

$$(3.1) \qquad s_\alpha \cdot \xi = s_\alpha(\xi) \cdot s_\alpha + \alpha(\xi) \qquad \forall \alpha \in \Psi(\mathfrak{g})\ \forall \xi \in \mathfrak{a}.$$

Define the left $\mathbf{H}$-module

$$S_{\mathbf{H}}(\mathfrak{a}) = \mathbf{H} \Big/ \sum_{w \in W \setminus \{1\}} \mathbf{H}(w-1).$$

Note that the inclusion map $S(\mathfrak{a}) \hookrightarrow \mathbf{H}$ induces the isomorphism $S(\mathfrak{a}) \simeq S_{\mathbf{H}}(\mathfrak{a})$ of left $S(\mathfrak{a})$-modules, through which we identify $S_{\mathbf{H}}(\mathfrak{a})$ with $S(\mathfrak{a})$. As a result, W acts on $S(\mathfrak{a})$ in two different ways. If $w \in W$ and $f \in S(\mathfrak{a})$, we use the notation wf for the usual action and let $\tilde{w}f$ denote the result of the left multiplication of $f \in S_{\mathbf{H}}(\mathfrak{a})$ by $w \in \mathbf{H}$.

Lemma 3.4. (i) *Suppose* $\alpha \in \Psi(\mathfrak{g})$ *and put* $\mathfrak{a}(\alpha) = \{H \in \mathfrak{a};\ \alpha(H) = 0\}$. *Then*

$$S(\mathfrak{a}) = S(\mathfrak{a}(\alpha)) \cdot \mathbb{C}[(\alpha^\vee)^2] \oplus S(\mathfrak{a}(\alpha)) \cdot \mathbb{C}[(\alpha^\vee)^2](\alpha^\vee - 1)$$

is the decomposition of $S(\mathfrak{a})$ *into the eigenspaces of* $\tilde{s}_\alpha$ *with eigenvalues* $1, -1$. *Here* $\alpha^\vee \in \mathfrak{a}$ *is the coroot corresponding to* α.
(ii) $S(\mathfrak{a})^{\widetilde{W}} = S(\mathfrak{a})^W$.
(iii) *For* $\alpha \in \Psi(\mathfrak{g})$ *and* $f \in S(\mathfrak{a})$,

$$\tilde{s}_\alpha f = s_\alpha f + \frac{f - s_\alpha f}{\alpha^\vee}. \tag{3.2}$$

Proof. From (3.1) we get the following relations in $\mathbf{H}$:

$$\begin{aligned} s_\alpha(\alpha^\vee - 1) &\equiv -(\alpha^\vee - 1) \mod \sum_{w \in W\setminus\{1\}} \mathbf{H}(w-1), \\ s_\alpha \cdot (\alpha^\vee)^2 &= (\alpha^\vee)^2 \cdot s_\alpha, \\ s_\alpha \cdot \xi &= \xi \cdot s_\alpha \quad \text{for} \quad \xi \in \mathfrak{a}(\alpha). \end{aligned}$$

From these (i) follows immediately. Next, (i) implies $S(\mathfrak{a})^{\tilde{s}_\alpha} = S(\mathfrak{a})^{s_\alpha}$ for any $\alpha \in \Psi(\mathfrak{g})$, proving (ii). Finally take any $\alpha \in \Psi(\mathfrak{g})$ and $f \in S(\mathfrak{a})$. Since

$$f = \left(\frac{f + s_\alpha f}{2} + \frac{f - s_\alpha f}{2\alpha^\vee}\right) + \frac{f - s_\alpha f}{2\alpha^\vee}(\alpha^\vee - 1)$$

corresponds to the decomposition of (i),

$$\tilde{s}_\alpha f = f - 2 \cdot \frac{f - s_\alpha f}{2\alpha^\vee}(\alpha^\vee - 1) = s_\alpha f + \frac{f - s_\alpha f}{\alpha^\vee}.$$

Thus we get (iii). □

Proposition 3.5. *Let* $\mathbf{V}$ *be a finite-dimensional* $\mathrm{ad}(\mathfrak{g})$*-subspace of* $U(\mathfrak{g})$ *and* $\mathbf{V}^\mathfrak{a}$ *the subspace of* $\mathbf{V}$ *consisting of the* 0*-weight vectors. Note that* W *naturally acts on* $\mathbf{V}^\mathfrak{a}$. *Put*

$$\Xi = \{\alpha \in \Psi(\mathfrak{g});\ 2\alpha \textit{ is not a weight of } \mathbf{V}\}.$$

Then

$$\gamma(s_\alpha D) = \tilde{s}_\alpha \gamma(D) \quad \textit{for any } D \in \mathbf{V}^\mathfrak{a} \textit{ and } \alpha \in \Xi.$$

Before proving the proposition we introduce some maps which should be considered as partial versions of the Harish-Chandra homomorphism.

Definition 3.6. Let $\Xi \subset \Psi(\mathfrak{g})$ and put

$$\Sigma(\mathfrak{g}_\Xi)^+ = \mathbb{Z}\Xi \cap \Sigma(\mathfrak{g})^+, \quad \rho_\Xi = \frac{1}{2}\sum_{\alpha \in \Sigma(\mathfrak{g}_\Xi)^+} \alpha, \quad \rho^\Xi = \rho - \rho_\Xi.$$

Define the map $\gamma^{\Xi} : U(\mathfrak{g}) \to U(\mathfrak{g}_{\Xi})$ by the projection

$$U(\mathfrak{g}) = (\bar{\mathfrak{n}}_{\Xi}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_{\Xi}) \oplus U(\mathfrak{g}_{\Xi}) \to U(\mathfrak{g}_{\Xi})$$

followed by the translation

$$U(\mathfrak{g}_{\Xi}) \simeq U(\mathfrak{m}_{\Xi}) \otimes S(\mathfrak{a}_{\Xi}) \ni \sum_i D_i \otimes f_i(\lambda) \longmapsto$$
$$\sum_i D_i \otimes f_i(\lambda - \rho^{\Xi}|_{\mathfrak{a}_{\Xi}}) \in U(\mathfrak{m}_{\Xi}) \otimes S(\mathfrak{a}_{\Xi}) \simeq U(\mathfrak{g}_{\Xi}).$$

Furthermore define the map $\gamma_{\Xi} : U(\mathfrak{g}_{\Xi}) \to S(\mathfrak{a})$ by the projection

$$U(\mathfrak{g}_{\Xi}) = ((\mathfrak{g}_{\Xi} \cap \bar{\mathfrak{n}})U(\mathfrak{g}_{\Xi}) + U(\mathfrak{g}_{\Xi})(\mathfrak{g}_{\Xi} \cap \mathfrak{n})) \oplus U(\mathfrak{a}) \to U(\mathfrak{a})$$

followed by the translation

$$U(\mathfrak{a}) \simeq S(\mathfrak{a}) \ni f(\lambda) \mapsto f(\lambda - \rho_{\Xi}) \in S(\mathfrak{a}).$$

Note that γ_{Ξ} is nothing but the ordinary Harish-Chandra homomorphism for $\mathfrak{g}_{\Xi}$.

One can easily observe γ^{Ξ} and γ_{Ξ} have the following properties:

Lemma 3.7. (i) $\gamma^{\emptyset} = \gamma$.
(ii) $\gamma_{\Xi} \circ \gamma^{\Xi} = \gamma$.
(iii) *Let* $U(\mathfrak{g})^{\mathfrak{a}_{\Xi}} = \{D \in U(\mathfrak{g}); [H, D] = 0 \ \forall H \in \mathfrak{a}_{\Xi}\}$. *For any* $D_1 \in U(\mathfrak{g})^{\mathfrak{a}_{\Xi}}$ *and* $D_2 \in U(\mathfrak{g})$ *we have* $\gamma^{\Xi}(D_1D_2) = \gamma^{\Xi}(D_1)\gamma^{\Xi}(D_2)$ *and* $\gamma^{\Xi}(D_2D_1) = \gamma^{\Xi}(D_2)\gamma^{\Xi}(D_1)$.
(iv) γ^{Ξ} *is an* $\mathrm{ad}(\mathfrak{g}_{\Xi})$*-homomorphism. In addition, if* $\alpha \in \Xi$ *and* $D \in U(\mathfrak{g})^{\mathfrak{a}}$ *then* $\gamma^{\Xi}(s_{\alpha}D) = s_{\alpha}\gamma^{\Xi}(D)$.

Proof of Proposition 3.5. Let Ξ be as in the proposition. We may assume $\Xi \neq \emptyset$ because the proposition is trivial if $\Xi = \emptyset$. Take an arbitrary $D \in \mathbf{V}^{\mathfrak{a}}$ and $\alpha \in \Xi$. Thanks to Lemma 3.7 (iv), $\gamma^{\Xi}(\mathbf{V})$ is an $\mathrm{ad}(\mathfrak{g}_{\Xi})$-subspace and $\gamma^{\Xi}(s_{\alpha}D) = s_{\alpha}\gamma^{\Xi}(D)$. Also, it follows from the definition of Ξ that if we consider $\gamma^{\Xi}(\mathbf{V})$ as an $\mathfrak{m}_{\Xi}$-module then each constituent of $\gamma^{\Xi}(\mathbf{V})$ is small in the sense of [Br]. Hence by Theorem 5.9 (iii) of [O] we have $\gamma_{\Xi}(s_{\alpha}\gamma^{\Xi}(D)) = \tilde{s}_{\alpha}\gamma_{\Xi}(\gamma^{\Xi}(D))$. Finally by Lemma 3.7 (ii), we conclude $\gamma(s_{\alpha}D) = \gamma_{\Xi}(\gamma^{\Xi}(s_{\alpha}D)) = \tilde{s}_{\alpha}\gamma_{\Xi}(\gamma^{\Xi}(D)) = \tilde{s}_{\alpha}\gamma(D)$. □

Hereafter in this section, we assume $\mathfrak{g} = \mathfrak{o}_{2n+1}, \mathfrak{sp}_n$, or $\mathfrak{o}_{2n}$ and $(\pi, V) = (\pi^{\natural}, V_N)$ ($N = \dim \pi^{\natural} = 2n+1$ or $2n$). Recall the notation in §2. For $i, j = 1, \ldots, N$ put $F_{ij} = p(E_{ij})$ and for $i = 1, \ldots, N$ put $H_i = F_{ii}$. Thus we have $F_{\pi^{\natural}} = \sum_{i,j} F_{ij} \otimes (v_i^* \otimes v_j)$ and $H_i = -H_{N+1-i}$. Note that $\{H_1, \ldots, H_n\}$ is a basis of $\mathfrak{a}$. For $i = 1, \ldots, n-1$ let $s_i \in W$ be the reflection corresponding to $e_i - e_{i+1} \in \Psi(\mathfrak{g})$.

Lemma 3.8. *Suppose* $\varphi(x) \in \mathbb{C}[x]$. *Then for* $i = 1, \ldots, n$,

$$\left(1 + \tilde{s}_1 + \tilde{s}_2\tilde{s}_1 + \cdots + \tilde{s}_{i-1}\tilde{s}_{i-2}\cdots\tilde{s}_1\right)\varphi(H_1) = \sum_{j=1}^{i} \varphi(H_j) \prod_{\substack{k \neq j \\ 1 \leq k \leq i}} \frac{H_j - H_k + \frac{1}{2}}{H_j - H_k}.$$

Here the right-hand side is the formula in the field of fractions of $S(\mathfrak{a})$.

Proof. Denote the field of fractions of $S(\mathfrak{a})$ by $K(\mathfrak{a})$. We assert the action of $\widetilde{W}$ on $S(\mathfrak{a})$ can be extended to $K(\mathfrak{a})$ by (3.2). In fact, if $s_\alpha s_{\alpha'} \cdots s_{\alpha''} = 1$ with $\alpha, \alpha' \dots, \alpha'' \in \Psi(\mathfrak{g})$, then

$$\tilde{s}_\alpha \tilde{s}_{\alpha'} \cdots \tilde{s}_{\alpha''} f = f \tag{3.3}$$

for any $f \in S(\mathfrak{a})$. But if we define the actions of $\tilde{s}_\alpha, \tilde{s}_{\alpha'}, \dots, \tilde{s}_{\alpha''}$ on $K(\mathfrak{a})$ by (3.2), then each $f \in K(\mathfrak{a})$ satisfies (3.3) since $f = f_1/f_2$ for some $f_1 \in S(\mathfrak{a})$ and $f_2 \in S(\mathfrak{a})^W$. It proves our assertion.

Now, clearly the lemma follows if we show for $i = 1, \dots, n$,

$$\begin{aligned} \tilde{s}_{i-1}\tilde{s}_{i-2}\cdots\tilde{s}_1\,\varphi(H_1) &= \varphi(H_i) \prod_{1\le k\le i-1} \frac{H_i - H_k + \frac{1}{2}}{H_i - H_k} \\ &\quad + \sum_{j=1}^{i-1} \frac{\varphi(H_j)}{2(H_j - H_i)} \prod_{\substack{k\neq j \\ 1\le k\le i-1}} \frac{H_j - H_k + \frac{1}{2}}{H_j - H_k}. \end{aligned} \tag{3.4}$$

Suppose (3.4) is valid for i ($< n$). Since the coroot corresponding to $e_i - e_{i+1}$ is $(e_i - e_{i+1})^\vee = 2(H_i - H_{i+1})$,

$$\tilde{s}_i f = \frac{1}{2(H_i - H_{i+1})} f + \frac{H_i - H_{i+1} - \frac{1}{2}}{H_i - H_{i+1}} s_i f \quad \text{for } f \in K(\mathfrak{a}).$$

Hence applying $\tilde{s}_i$ on both sides of (3.4), we have

$$\begin{aligned} &\tilde{s}_i\tilde{s}_{i-1}\cdots\tilde{s}_1\,\varphi(H_1) \\ &= \frac{\varphi(H_i)}{2(H_i - H_{i+1})} \prod_{1\le k\le i-1} \frac{H_i - H_k + \frac{1}{2}}{H_i - H_k} \\ &\quad + \varphi(H_{i+1}) \prod_{1\le k\le i} \frac{H_{i+1} - H_k + \frac{1}{2}}{H_{i+1} - H_k} \\ &\quad + \sum_{j=1}^{i-1} \frac{\varphi(H_j)}{2(H_i - H_{i+1})} \left\{ \frac{1}{2(H_j - H_i)} + \frac{H_i - H_{i+1} - \frac{1}{2}}{H_j - H_{i+1}} \right\} \prod_{\substack{k\neq j \\ 1\le k\le i-1}} \frac{H_j - H_k + \frac{1}{2}}{H_j - H_k} \\ &= \varphi(H_{i+1}) \prod_{1\le k\le i} \frac{H_{i+1} - H_k + \frac{1}{2}}{H_{i+1} - H_k} + \sum_{j=1}^{i} \frac{\varphi(H_j)}{2(H_j - H_{i+1})} \prod_{\substack{k\neq j \\ 1\le k\le i}} \frac{H_j - H_k + \frac{1}{2}}{H_j - H_k}. \end{aligned}$$

Since (3.4) is trivial for $i = 1$, we inductively get (3.4) for all i. □

The next theorem determines the image of each 'matrix coefficient' of $q(F_{\pi^\natural})$ under γ for any $q(x) \in \mathbb{C}[x]$.

Theorem 3.9. *Recall* $N = \dim \pi^\natural = 2n + 1$ *or* $2n$. *Suppose* $q(x) \in \mathbb{C}[x]$ *and define* $Q_{ij} \in U(\mathfrak{g})$ $(i, j = 1, \dots, N)$ *so that* $q(F_{\pi^\natural}) = \sum_{i,j} Q_{ij} \otimes (v_i^* \otimes v_j)$.

(i) If $i \neq j$ then $\gamma(Q_{ij}) = 0$.
(ii) Put $a(\mathfrak{g}) = \rho(H_1)$, that is,

$$a(\mathfrak{o}_{2n+1}) = \frac{n}{2} - \frac{1}{4}, \qquad a(\mathfrak{sp}_n) = \frac{n}{2}, \qquad a(\mathfrak{o}_{2n}) = \frac{n}{2} - \frac{1}{2}.$$

Then for $i = 1, \ldots, n$,

$$\gamma\Big(\sum_{j=1}^{i} Q_{N+1-j,N+1-j}\Big) = \sum_{j=1}^{i} q\big(a(\mathfrak{g}) - H_j\big) \prod_{\substack{k \neq j \\ 1 \leq k \leq i}} \frac{H_j - H_k + \frac{1}{2}}{H_j - H_k}.$$

(iii) Define $(F^t_{\pi\natural})_{ij} \in U(\mathfrak{g})$ ($t = 0, 1, \ldots$ and $i, j = 1, \ldots, N$) so that $F^t_{\pi\natural} = \sum_{i,j} (F^t_{\pi\natural})_{ij} \otimes (v_i^* \otimes v_j)$ and put $C^{(t)} = \sum_{i=1}^{N} (F^t_{\pi\natural})_{ii}$ ($t = 0, 1, \ldots$). Moreover put

$$\epsilon = \begin{cases} 1 & \text{if } \mathfrak{g} = \mathfrak{o}_{2n+1} \text{ or } \mathfrak{o}_{2n}, \\ -1 & \text{if } \mathfrak{g} = \mathfrak{sp}_n. \end{cases}$$

Then for $i = 1, \ldots, n$,

$$\gamma\Big(\sum_{j=1}^{i} Q_{jj}\Big) = \sum_{j=1}^{i} \Bigg\{ q\Big(H_j + a(\mathfrak{g}) + \frac{1}{2}\Big) - \frac{1}{2} \frac{q(y) - q\big(H_j + a(\mathfrak{g}) + \frac{1}{2}\big)}{y - H_j - a(\mathfrak{g}) - \frac{1}{2}}\Bigg|_{y^u \mapsto \gamma(C^{(u)}) - \epsilon(a(\mathfrak{g}) - H_j)^u} \Bigg\} \cdot \prod_{\substack{k \neq j \\ 1 \leq k \leq i}} \frac{H_j - H_k + \frac{1}{2}}{H_j - H_k}$$

where the substitutions for y^u in the right-hand side stand for the linear map $\mathbb{C}[y, H_j] \to S(\mathfrak{a})$ defined by $y^u H_j^{u'} \mapsto \big(\gamma(C^{(u)}) - \epsilon(a(\mathfrak{g}) - H_j)^u\big) H_j^{u'}$.

Proof. Clearly it suffices to prove the theorem only for the cases where $q(x) = x^t$ ($t = 0, 1, \ldots$). Let $(F^t_{\pi\natural})_{ij}$ and $C^{(t)}$ be as in (iii). Regard $U(\mathfrak{g}) \otimes \operatorname{End} V_N^*$ as a $\mathfrak{g}$-module by the adjoint action and denote the $\mathfrak{g}$-invariant subalgebra of $U(\mathfrak{g}) \otimes \operatorname{End} V_N^*$ by $(U(\mathfrak{g}) \otimes \operatorname{End} V_N^*)^{\mathfrak{g}}$. As stated in Remark 2.2 of [OOs], $F_{\pi\natural} \in (U(\mathfrak{g}) \otimes \operatorname{End} V_N^*)^{\mathfrak{g}}$. Hence for each t, $F^t_{\pi\natural} \in (U(\mathfrak{g}) \otimes \operatorname{End} V_N^*)^{\mathfrak{g}}$ and it holds that

(3.5) $\qquad [H, (F^t_{\pi\natural})_{ij}] = (e_i(H) - e_j(H))(F^t_{\pi\natural})_{ij} \quad \text{for } H \in \mathfrak{a} \text{ and } i, j = 1, \ldots, n.$

Since $\gamma : U(\mathfrak{g}) \to S(\mathfrak{a})$ is an ad($\mathfrak{a}$)-homomorphism, $\gamma\big((F^t_{\pi\natural})_{ij}\big) = 0$ if $i \neq j$. Thus we get (i).

Also, we see the linear map $p_t : \operatorname{End} V_N \to U(\mathfrak{g})$ defined by $p_t(E_{ij}) = (F^t_{\pi\natural})_{ij}$ is a $\mathfrak{g}$-homomorphism. Therefore $C^{(t)} = p_t(1_{V_N}) \in Z(\mathfrak{g})$ and for $t = 1, \ldots, n$,

$$\begin{aligned} \sum_{j=1}^{i} (F^t_{\pi\natural})_{N+1-j,N+1-j} &= \sum_{j=1}^{i} p_t(E_{N+1-j,N+1-j}) \\ &= p_t\big((1 + s_1 + s_2 s_1 + \cdots + s_{i-1} s_{i-2} \cdots s_1) E_{NN}\big) \\ &= (1 + s_1 + s_2 s_1 + \cdots + s_{i-1} s_{i-2} \cdots s_1)(F^t_{\pi\natural})_{NN}, \end{aligned}$$

$$\sum_{j=1}^{i}(F^t_{\pi\natural})_{jj} = \sum_{j=1}^{i} p_t(E_{jj}) = p_t\big((1+s_1+s_2s_1+\cdots+s_{i-1}s_{i-2}\cdots s_1)E_{11}\big)$$
$$= (1+s_1+s_2s_1+\cdots+s_{i-1}s_{i-2}\cdots s_1)(F^t_{\pi\natural})_{11}.$$

Now observe that $2(e_j - e_{j+1})$ is not a weight of the $\mathfrak{g}$-module $\operatorname{End} V_N = V_N \otimes V_N^*$ for $j = 1, \ldots, n-1$. Hence by use of Proposition 3.5 with $\mathbf{V} = p_t(\operatorname{End} V_N)$, we get

$$\gamma\left(\sum_{j=1}^{i}(F^t_{\pi\natural})_{N+1-j,N+1-j}\right) = (1+\tilde{s}_1+\tilde{s}_2\tilde{s}_1+\cdots+\tilde{s}_{i-1}\tilde{s}_{i-2}\cdots\tilde{s}_1)\gamma\big((F^t_{\pi\natural})_{NN}\big), \tag{3.6}$$

$$\gamma\left(\sum_{j=1}^{i}(F^t_{\pi\natural})_{jj}\right) = (1+\tilde{s}_1+\tilde{s}_2\tilde{s}_1+\cdots+\tilde{s}_{i-1}\tilde{s}_{i-2}\cdots\tilde{s}_1)\gamma\big((F^t_{\pi\natural})_{11}\big). \tag{3.7}$$

Let us show the following two equalities:

$$\gamma\big((F^t_{\pi\natural})_{NN}\big) = \big(a(\mathfrak{g}) - H_1\big)^t, \tag{3.8}$$

$$\gamma\big((F^t_{\pi\natural})_{11}\big) = \Big(H_1 + a(\mathfrak{g}) + \frac{1}{2}\Big)^t - \frac{1}{2}\frac{y^t - \big(H_1 + a(\mathfrak{g}) + \frac{1}{2}\big)^t}{y - H_1 - a(\mathfrak{g}) - \frac{1}{2}}\bigg|_{y^u \mapsto \gamma(C^{(u)}) - \epsilon(a(\mathfrak{g}) - H_1)^u}. \tag{3.9}$$

Then (3.6), (3.8) and Lemma 3.8 imply (ii). Also, since $\gamma(C^{(u)}) \in S(\mathfrak{a})^W$ $(u = 0, 1, \ldots)$, (3.7), (3.9), Remark 3.3 (ii) and Lemma 3.8 imply (iii). By virtue of (3.5) $F_{Nj} = (F^1_{\pi\natural})_{Nj} \in \bar{\mathfrak{n}}U(\mathfrak{g})$ $(j = 1, \ldots, N-1)$. Hence we get (3.8) by

$$(F^t_{\pi\natural})_{NN} = \sum_{j=1}^{N} F_{Nj}(F^{t-1}_{\pi\natural})_{jN} \equiv F_{NN}(F^{t-1}_{\pi\natural})_{NN} \pmod{\bar{\mathfrak{n}}U(\mathfrak{g})}$$
$$= (F^{t-1}_{\pi\natural})_{NN}(-H_1) \equiv \cdots \equiv (-H_1)^t \pmod{\bar{\mathfrak{n}}U(\mathfrak{g})}.$$

On the other hand, for $j = 1, \ldots, N$,

$$[E_{1j}, F_{j1}] = \left[E_{1j}, \frac{E_{j1} - \operatorname{sgn}\big(n + \frac{1}{2} - j\big)^{\frac{1-\epsilon}{2}} E_{N,N+1-j}}{2}\right]$$
$$= \frac{E_{11} - E_{jj}}{2} - \epsilon\delta_{jN}\frac{E_{11} - E_{NN}}{2}$$

and hence

$$[(F^t_{\pi\natural})_{1j}, F_{j1}] = [p_t(E_{1j}), F_{j1}] = p_t\Big([E_{1j}, F_{j1}]\Big)$$
$$= \frac{(F^t_{\pi\natural})_{11} - (F^t_{\pi\natural})_{jj}}{2} - \epsilon\delta_{jN}\frac{(F^t_{\pi\natural})_{11} - (F^t_{\pi\natural})_{NN}}{2}.$$

Therefore, since $F_{j1} \in \bar{\mathfrak{n}}U(\mathfrak{g})$ for $j = 2, \ldots, N$, we have for $t = 1, 2, \ldots$,

$$(F^t_{\pi\natural})_{11} = \sum_{j=1}^{N}(F^{t-1}_{\pi\natural})_{1j}F_{j1}$$

$$\equiv (F_{\pi\natural}^{t-1})_{11}F_{11} + \sum_{j=2}^{N}\big[(F_{\pi\natural}^{t-1})_{1j}, F_{j1}\big] \pmod{\bar{\mathfrak{n}}U(\mathfrak{g})}$$

$$= (F_{\pi\natural}^{t-1})_{11}\Big(H_1 + \frac{N-\epsilon}{2}\Big) - \frac{1}{2}\sum_{j=1}^{N}(F_{\pi\natural}^{t-1})_{jj} + \frac{\epsilon}{2}(F_{\pi\natural}^{t-1})_{NN}$$

$$\equiv (F_{\pi\natural}^{t-1})_{11}\Big(H_1 + \frac{N-\epsilon}{2}\Big) - \frac{1}{2}C^{(t-1)} + \frac{\epsilon}{2}(-H_1)^{t-1} \pmod{\bar{\mathfrak{n}}U(\mathfrak{g})}.$$

It inductively leads to

$$(F_{\pi\natural}^{t})_{11} \equiv \Big(H_1 + \frac{N-\epsilon}{2}\Big)^t - \frac{1}{2}\sum_{j=0}^{t-1}\big(C^{(j)} - \epsilon(-H_1)^j\big)\Big(H_1 + \frac{N-\epsilon}{2}\Big)^{t-1-j} \pmod{\bar{\mathfrak{n}}U(\mathfrak{g})}$$

for $t = 0, 1, \ldots$. Now (3.9) is immediate. □

Remark 3.10. In Theorem 3.9, each $\gamma(Q_{ij})$ is expressed in terms of H_k's and $\gamma(C^{(u)})$'s. But each $\gamma(C^{(u)})$ can be also expressed in terms of H_k's by Gould's result [Go1]. In §4 the explicit form of $\gamma(C^{(u)})$ will be studied in detail.

4 A variant of Gould's result

Let (π, V) be an irreducible, faithful, finite-dimensional representation of a complex reductive Lie algebra $\mathfrak{g}$. Then the bilinear form $\langle X, Y\rangle := \mathrm{Trace}\,(XY)$ on $\mathrm{End}\,V \times \mathrm{End}\,V$ is non-degenerate on $\pi(\mathfrak{g}) \times \pi(\mathfrak{g})$ and we can use the setting and notation in the preceding sections. Let $(U(\mathfrak{g}) \otimes \mathrm{End}\,V^*)^{\mathfrak{g}}$ denote the subalgebra of $\mathfrak{g}$-fixed elements in $U(\mathfrak{g}) \otimes \mathrm{End}\,V^*$ with respect to the adjoint action of $\mathfrak{g}$. For any $q(x) \in \mathbb{C}[x]$ one has $q(F_\pi) \in (U(\mathfrak{g}) \otimes \mathrm{End}\,V^*)^{\mathfrak{g}}$ and hence $q(F_\pi)$ is identified with an element $p_{q(x)} \in \mathrm{Hom}_{\mathfrak{g}}(\mathrm{End}\,V, U(\mathfrak{g}))$ (cf. the proof of Theorem 3.9). Hence

$$p_{q(x)}(\mathbf{1}_V) = \big(\mathbf{1}_{U(\mathfrak{g})} \otimes \mathrm{Trace}\big)(q(F_\pi)) \in U(\mathfrak{g}) \tag{4.1}$$

is a central element. This type of central elements are introduced by [Ge] and Gould calculates the image of (4.1) under the Harish-Chandra homomorphism γ in [Go1]. Suppose $\Theta \subset \Psi(\mathfrak{g})$. We consider $\mathfrak{a}_\Theta^* \hookrightarrow \mathfrak{a}^*$ by $\lambda \mapsto \lambda_\Theta|_{\mathfrak{a}}$. (Hereafter $\lambda_\Theta|_{\mathfrak{a}}$ will be briefly denoted by λ_Θ.) Then $\mathfrak{a}_\Theta^* + \rho$ is an affine subspace of $\mathfrak{a}^*$. Gould's formula for $\gamma\big(p_{q(x)}(\mathbf{1}_V)\big)$ is simplified if we restrict it to $\mathfrak{a}_\Theta^* + \rho$. Denote the symmetric bilinear form on $\mathfrak{a}^* \times \mathfrak{a}^*$ induced from the $\langle \pi(\cdot), \pi(\cdot)\rangle$ also by $\langle \cdot, \cdot \rangle$.

Theorem 4.1. *Let $(\pi|_{\mathfrak{g}_\Theta}, V|_{\mathfrak{g}_\Theta})$ be the restriction of the representation (π, V) of $\mathfrak{g}$ to $\mathfrak{g}_\Theta$. Let $\{\varpi_1, \ldots, \varpi_d\}$ be the totality of the lowest weights of $(\pi|_{\mathfrak{g}_\Theta}, V|_{\mathfrak{g}_\Theta})$ counting their* *multiplicities. Here a lowest weight vector is a weight vector for* $\mathfrak{a}$ *which is annihilated by* $\mathfrak{g}_\Theta \cap \bar{\mathfrak{n}}$. *Then for* $q(x) \in \mathbb{C}[x]$ *and* $\lambda \in \mathfrak{a}_\Theta^*$

$$\gamma\big(p_{q(x)}(\mathbf{1}_V)\big)(\lambda_\Theta + \rho)$$

$$= \sum_{k=1}^{d} q\left(\langle \varpi_k, \lambda_\Theta + \rho\rangle + \frac{\langle \bar{\pi}, \bar{\pi} - 2\rho\rangle - \langle \varpi_k, \varpi_k\rangle}{2}\right)\frac{\tilde{d}(\lambda_\Theta - \varpi_k)}{\tilde{d}(\lambda_\Theta)} \tag{4.2}$$

$$\text{with } \tilde{d}(\mu) = \prod_{\alpha \in \Sigma(\mathfrak{g})^+} \langle \mu + \rho, \alpha\rangle \text{ for } \mu \in \mathfrak{a}^*.$$

Here $\bar{\pi}$ *is the lowest weight of* (π, V).

Before proving the theorem we first recall a key fact used in [OOs] which is originally due to [Go1]. Let $\{X_i\}$ and $\{X_i^*\}$ be two bases of $\mathfrak{g}$ such that $\langle \pi(X_i), \pi(X_j^*)\rangle = \delta_{ij}$ and define the *Casimir element* $\Delta_\pi = \sum_i X_i X_i^* \in U(\mathfrak{g})$.

Lemma 4.2. *Suppose* M *is a highest weight module of* $\mathfrak{g}$ *with highest weight* $\mu \in \mathfrak{a}^*$. *Then we have*

(i) Δ_π *acts on* M *by the scalar* $\langle \mu, \mu + 2\rho\rangle$.

(ii) *The natural action of* $F_\pi \in U(\mathfrak{g}) \otimes \operatorname{End} V^*$ *on* $M \otimes V^*$ *coincides with the action of*

$$\frac{1}{2}\left(\langle \mu, \mu + 2\rho\rangle + \langle \bar{\pi}, \bar{\pi} - 2\rho\rangle - \Delta_\pi\right) \in U(\mathfrak{g})$$

on a $\mathfrak{g}$*-module* $M \otimes V^*$. *Here* $\bar{\pi}$ *denotes the lowest weight of* (π, V).

Proof. By [OOs, Lemma 2.19 ii)] and [ibid., Lemma 2.26]. □

Proof of Theorem 4.1. The proof is just a simple modification of Gould's one. For a dominant and algebraically integral weight $\mu \in \mathfrak{a}^*$, let (π_μ, V_μ) be the irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight μ. Suppose $\lambda \in \mathfrak{a}_\Theta^*$ satisfies (i) λ_Θ is dominant and algebraically integral, and also (ii) for each $k = 1, \ldots, d$, $\lambda_\Theta - \varpi_k$ is dominant and algebraically integral. (Such λ's constitute a Zariski dense subset of $\mathfrak{a}_\Theta^*$.) Then Proposition 2.27 of [OOs] gives the irreducible decomposition

$$(\pi_{\lambda_\Theta} \otimes \pi^*, V_{\lambda_\Theta} \otimes V^*) = \bigoplus_{k=1}^{d} (\pi_{\lambda_\Theta - \varpi_k}, V_{\lambda_\Theta - \varpi_k}).$$

It, combined with Lemma 4.2, leads to

$$(\pi_{\lambda_\Theta} \otimes \mathbf{1}_{\operatorname{End} V^*})(F_\pi)$$

$$= \frac{1}{2}\Big(\big(\langle \lambda_\Theta, \lambda_\Theta + 2\rho\rangle + \langle \bar{\pi}, \bar{\pi} - 2\rho\rangle\big)\mathbf{1}_{V_{\lambda_\Theta} \otimes V^*} - (\pi_{\lambda_\Theta} \otimes \pi^*)(\Delta_\pi)\Big)$$

$$= \bigoplus_{k=1}^{d}\left(\langle \varpi_k, \lambda_\Theta + \rho\rangle + \frac{\langle \bar{\pi}, \bar{\pi} - 2\rho\rangle - \langle \varpi_k, \varpi_k\rangle}{2}\right)\mathbf{1}_{V_{\lambda_\Theta - \varpi_k}}.$$

Now

$$\begin{aligned}\gamma(p_{q(x)}(\mathbf{1}_V))(\lambda_\Theta+\rho) &= \frac{1}{\dim V_{\lambda_\Theta}} \operatorname{Trace}\circ\pi_{\lambda_\Theta}(p_{q(x)}(\mathbf{1}_V)) \\ &= \frac{1}{\dim V_{\lambda_\Theta}} \operatorname{Trace}\circ(\pi_{\lambda_\Theta}\otimes \mathbf{1}_{\operatorname{End} V^*})(q(F_\pi)) \\ &= \frac{1}{\dim V_{\lambda_\Theta}} \operatorname{Trace} q((\pi_{\lambda_\Theta}\otimes \mathbf{1}_{\operatorname{End} V^*})(F_\pi)) \\ &= \sum_{k=1}^{d} \frac{\dim V_{\lambda_\Theta-\varpi_k}}{\dim V_{\lambda_\Theta}} q\left(\langle \varpi_k, \lambda_\Theta+\rho\rangle + \frac{\langle\bar\pi,\bar\pi-2\rho\rangle - \langle\varpi_k,\varpi_k\rangle}{2}\right).\end{aligned}$$

Hence by Weyl's dimension formula, (4.2) holds. Since the left-hand side of (4.2) is a polynomial function on $\mathfrak{a}_\Theta^*$, so is the right-hand side and the equality holds for any $\lambda \in \mathfrak{a}_\Theta^*$. □

In the rest of this section, we assume $\mathfrak{g} = \mathfrak{o}_{2n+1}$, $\mathfrak{sp}_n$, or $\mathfrak{o}_{2n}$ and $(\pi, V) = (\pi^\natural, V_N)$ ($N = \dim \pi^\natural = 2n+1$ or $2n$) and $\Theta = \Theta_\Pi$ defined by (2.4). Here Π is as in (2.3). In these cases $p_{x^t}(\mathbf{1}_{V_N})$ ($t = 0, 1, \ldots$) equals $C^{(t)}$ in Theorem 3.9 (iii). We shall now calculate the explicit forms of (4.2), which will be used later. Note that

$$\langle e_i, e_j\rangle = \frac{1}{2}\delta_{ij} \text{ for } i,j = 1,\ldots,n.$$

Lemma 4.3. *Suppose $\varphi(x) \in \mathbb{C}[x]$ and $\mu_1,\ldots,\mu_K,\chi$ are $K+1$ indeterminates. Then*

$$\sum_{k=1}^{K} \varphi(\mu_k) \prod_{\substack{1\le \ell\le K \\ \ell\ne k}} \frac{\mu_k+\mu_\ell-\chi}{\mu_k-\mu_\ell} \in \mathbb{C}[\mu_1,\ldots,\mu_K,\chi].$$

Proof. Using the difference product

$$\mathfrak{D} = \prod_{1\le k<\ell\le K} (\mu_k - \mu_\ell)$$

in $\mu_1,\ldots,\mu_K$, we can express

$$\sum_{k=1}^{K} \varphi(\mu_k) \prod_{\substack{1\le \ell\le K \\ \ell\ne k}} \frac{\mu_k+\mu_\ell-\chi}{\mu_k-\mu_\ell} = \frac{(-1)^{K-1}}{\mathfrak{D}} \sum_{k=1}^{K} \varphi(\mu_k)(\mathfrak{D}|_{\mu_k\mapsto\chi-\mu_k}).$$

Since $\sum_{k=1}^{K} \varphi(\mu_k)(\mathfrak{D}|_{\mu_k\mapsto\chi-\mu_k})$ is alternating with respect to $\mu_1,\ldots,\mu_K$, it is divisible by $\mathfrak{D}$. □

Example 4.4 (B_n). Suppose $\mathfrak{g} = \mathfrak{o}_{2n+1}$. Then $\rho = (n-\frac{1}{2}, n-\frac{3}{2},\ldots,\frac{1}{2})$ and the totality of weights of $(\pi^\natural, V_{2n+1})$ is $\{0, \pm e_1,\ldots,\pm e_n\}$:

$$\bar\pi^\natural = -e_1 \to -e_2 \to \cdots\cdots - e_n \to 0 \to e_n \to \cdots\cdots e_2 \to e_1.$$

(In the above weight diagram, an arrow is written if the difference of the weights at both ends equals some simple root.) The set $\overline{\mathcal{W}}_{\Theta_\Pi}(\pi^\natural)$ of the lowest weights of $(\pi^\natural|_{\mathfrak{g}_{\Theta_\Pi}}, V_{2n+1}|_{\mathfrak{g}_{\Theta_\Pi}})$ is given by

$$\overline{\mathcal{W}}_{\Theta_\Pi}(\pi^\natural) = \{-e_{n_0+1}, \ldots, -e_{n_{L-1}+1}, 0, e_{n_L}, \ldots, e_{n_1}\}.$$

Suppose $\lambda = (\lambda_1, \ldots, \lambda_L) \in \mathfrak{a}^*_{\Theta_\Pi}$ and put

$$\mu_{-k} = -\frac{\lambda_k}{2} + \frac{n_{k-1}}{2}, \quad \mu_k = \frac{\lambda_k}{2} + \frac{2n - n_k}{2} \tag{4.3}$$

for $k = 1, \ldots, L$. Then we have for $q(x) \in \mathbb{C}[x]$

(4.4)

$$\begin{aligned}
&\gamma\big(p_{q(x)}(\mathbf{1}_{V_{2n+1}})\big)(\lambda_{\Theta_\Pi} + \rho) \\
&= \sum_{k=1}^{L} q\big(-\tfrac{1}{2}(\lambda_k + n - n_{k-1} - \tfrac{1}{2}) + \tfrac{1}{2}(n - \tfrac{1}{2})\big) \\
&\quad \cdot \frac{\langle \rho + e_{n_{k-1}+1}, e_{n_{k-1}+1} - e_{n_{k-1}+2}\rangle \cdots \langle \rho + e_{n_{k-1}+1}, e_{n_{k-1}+1} - e_{n_k}\rangle}{\langle \rho, e_{n_{k-1}+1} - e_{n_{k-1}+2}\rangle \quad \cdots \quad \langle \rho, e_{n_{k-1}+1} - e_{n_k}\rangle} \\
&\quad \cdot \prod_{\substack{1 \le \ell \le L \\ \ell \ne k}} \frac{\langle \lambda_{\Theta_\Pi} + \rho + e_{n_{k-1}+1}, e_{n_{k-1}+1} - e_{n_{\ell-1}+1}\rangle \cdots \langle \lambda_{\Theta_\Pi} + \rho + e_{n_{k-1}+1}, e_{n_{k-1}+1} - e_{n_\ell}\rangle}{\langle \lambda_{\Theta_\Pi} + \rho, e_{n_{k-1}+1} - e_{n_{\ell-1}+1}\rangle \quad \cdots \quad \langle \lambda_{\Theta_\Pi} + \rho, e_{n_{k-1}+1} - e_{n_\ell}\rangle} \\
&\quad \cdot \prod_{\ell=1}^{L} \frac{\langle \lambda_{\Theta_\Pi} + \rho + e_{n_{k-1}+1}, e_{n_{k-1}+1} + e_{n_{\ell-1}+1}\rangle \cdots \langle \lambda_{\Theta_\Pi} + \rho + e_{n_{k-1}+1}, e_{n_{k-1}+1} + e_{n_\ell}\rangle}{\langle \lambda_{\Theta_\Pi} + \rho, e_{n_{k-1}+1} + e_{n_{\ell-1}+1}\rangle \quad \cdots \quad \langle \lambda_{\Theta_\Pi} + \rho, e_{n_{k-1}+1} + e_{n_\ell}\rangle} \\
&\quad + q\big(0 + \tfrac{n}{2}\big) \\
&\quad + \sum_{k=1}^{L} q\big(\tfrac{1}{2}(\lambda_k + n - n_k + \tfrac{1}{2}) + \tfrac{1}{2}(n - \tfrac{1}{2})\big) \\
&\qquad \cdot \frac{\langle \rho - e_{n_k}, e_{n_{k-1}+1} - e_{n_k}\rangle \cdots \langle \rho - e_{n_k}, e_{n_k - 1} - e_{n_k}\rangle}{\langle \rho, e_{n_{k-1}+1} - e_{n_k}\rangle \quad \cdots \quad \langle \rho, e_{n_k-1} - e_{n_k}\rangle} \\
&\qquad \cdot \prod_{\substack{1 \le \ell \le L \\ \ell \ne k}} \frac{\langle \lambda_{\Theta_\Pi} + \rho - e_{n_k}, e_{n_k} - e_{n_{\ell-1}+1}\rangle \cdots \langle \lambda_{\Theta_\Pi} + \rho - e_{n_k}, e_{n_k} - e_{n_\ell}\rangle}{\langle \lambda_{\Theta_\Pi} + \rho, e_{n_k} - e_{n_{\ell-1}+1}\rangle \quad \cdots \quad \langle \lambda_{\Theta_\Pi} + \rho, e_{n_k} - e_{n_\ell}\rangle} \\
&\qquad \cdot \prod_{\ell=1}^{L} \frac{\langle \lambda_{\Theta_\Pi} + \rho - e_{n_k}, e_{n_k} + e_{n_{\ell-1}+1}\rangle \cdots \langle \lambda_{\Theta_\Pi} + \rho - e_{n_k}, e_{n_k} + e_{n_\ell}\rangle}{\langle \lambda_{\Theta_\Pi} + \rho, e_{n_k} + e_{n_{\ell-1}+1}\rangle \quad \cdots \quad \langle \lambda_{\Theta_\Pi} + \rho, e_{n_k} + e_{n_\ell}\rangle} \\
&= q\big(\tfrac{n}{2}\big) - 2 \sum_{k \in \{\pm 1, \ldots, \pm L\}} q(\mu_k)\big(2\mu_k - n - \tfrac{1}{2}\big) \prod_{\substack{\ell \in \{\pm 1, \ldots, \pm L\} \\ \ell \ne k}} \frac{\mu_k + \mu_\ell - n}{\mu_k - \mu_\ell}.
\end{aligned}$$

Using the relation

$$n = \frac{1}{L - \frac{1}{2}} \sum_{k=1}^{L} (\mu_{-k} + \mu_k), \tag{4.5}$$

we can eliminate n from the final form of (4.4) and the result is a symmetric polynomial in the $2L$ variables μ_k by Lemma 4.3. Since $\mathfrak{a}^*_{\overline{\Theta}_\Pi} + \rho \subset \mathfrak{a}^*_{\Theta_\Pi} + \rho$, (4.4) applies to $\lambda_{\overline{\Theta}_\Pi} + \rho \in \mathfrak{a}^*_{\overline{\Theta}_\Pi} + \rho$ by letting $\lambda_L = 0$ in (4.3).

Example 4.5 (C_n, D_n). Suppose $\mathfrak{g} = \mathfrak{sp}_n$ or $\mathfrak{o}_{2n}$ and put

$$\epsilon = \begin{cases} 1 & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}, \\ -1 & \text{if } \mathfrak{g} = \mathfrak{sp}_n. \end{cases}$$

Suppose $\lambda = (\lambda_1, \ldots, \lambda_L) \in \mathfrak{a}^*_{\Theta_\Pi}$ and put

$$\mu_{-k} = -\frac{\lambda_k}{2} + \frac{n_{k-1}}{2}, \quad \mu_k = \frac{\lambda_k}{2} + \frac{2n - n_k - \epsilon}{2} \tag{4.6}$$

for $k = 1, \ldots, L$. Then for $q(x) \in \mathbb{C}[x]$ a similar calculation to (4.4) implies

$$\begin{aligned} &\gamma\big(p_{q(x)}(\mathbf{1}_{V_{2n}})\big)(\lambda_{\Theta_\Pi} + \rho) \\ &= -2 \sum_{k \in \{\pm 1, \ldots, \pm L\}} q(\mu_k)(2\mu_k - n + \epsilon) \prod_{\substack{\ell \in \{\pm 1, \ldots, \pm L\} \\ \ell \neq k}} \frac{\mu_k + \mu_\ell - n + \frac{\epsilon}{2}}{\mu_k - \mu_\ell}. \end{aligned} \tag{4.7}$$

Since

$$n = \frac{1}{L - \frac{1}{2}} \sum_{k=1}^{L} (\mu_{-k} + \mu_k) + \frac{L\epsilon}{2L - 1}, \tag{4.8}$$

we can eliminate n from the right-hand side of (4.7) and the result is a symmetric polynomial in the $2L$ variables μ_k by Lemma 4.3. Since $\mathfrak{a}^*_{\overline{\Theta}_\Pi} + \rho \subset \mathfrak{a}^*_{\Theta_\Pi} + \rho$, (4.7) applies to $\lambda_{\overline{\Theta}_\Pi} + \rho \in \mathfrak{a}^*_{\overline{\Theta}_\Pi} + \rho$ by letting $\lambda_L = 0$ in (4.6).

Remark 4.6. If $\mathfrak{g} = \mathfrak{o}_2$, then $(\pi^\natural, V_2)$ is reducible and we cannot apply Theorem 4.1 to deduce Example 4.5. Nevertheless, it can be directly checked that Example 4.5 is also valid for $\mathfrak{g} = \mathfrak{o}_2$.

5 Two-sided ideals

Retain the settings for the classical cases. Namely, $\mathfrak{g} = \mathfrak{o}_{2n+1}, \mathfrak{sp}_n$, or $\mathfrak{o}_{2n}$, $(\pi, V) = (\pi^\natural, V_N)$, $\Theta = \Theta_\Pi$ or $\overline{\Theta}_\Pi$, and so forth. For $\mu \in \mathfrak{a}^*$ put

$$I(\mu) = \sum_{\Delta \in Z(\mathfrak{g})} U(\mathfrak{g})(\Delta - \gamma(\Delta)(\mu + \rho)).$$

Then one has $\operatorname{Ann} M(\mu) = I(\mu)$ and

$$\gamma(I(\mu)) = \sum_{f \in S(\mathfrak{a})^W} S(\mathfrak{a})(f - f(\mu + \rho)). \tag{5.1}$$

In addition, for $q(x) \in \mathbb{C}[x]$ and $\lambda \in \mathfrak{a}^*_\Theta$ define $Q_{ij} \in U(\mathfrak{g})$ $(i, j = 1, \ldots, N)$ so that $q(F_{\pi^\natural}) = \sum_{i,j} Q_{ij} \otimes (v_i^* \otimes v_j)$ and put

$$I_{q(x)}(\lambda) = \sum_{i,j} U(\mathfrak{g}) Q_{ij} + I(\lambda_\Theta).$$

From Lemma 2.1 of [OOs] one has $I_{q(x)}(\lambda)$ is a two-sided ideal in $U(\mathfrak{g})$ and

(5.2) $$\gamma\big(I_{q(x)}(\lambda)\big) = \sum_{i,j} S(\mathfrak{a})\gamma(Q_{ij}) + \gamma\big(I(\lambda)\big).$$

If $q(x)$ is a multiple of the minimal polynomial $q_{\pi^{\natural},M_\Theta(\lambda)}(x)$ then clearly

(5.3) $$\operatorname{Ann} M(\lambda) \subset I_{q(x)}(\lambda) \subset I_{q_{\pi^{\natural},M_\Theta(\lambda)}(x)}(\lambda) \subset \operatorname{Ann} M_\Theta(\lambda).$$

Let $J_\Theta(\lambda)$ and $J(\lambda_\Theta)$ be as in §2. In [Os] and [OOs] it is shown that the equalities

(5.4) $$I_{q_{\pi^{\natural},M_\Theta(\lambda)}(x)}(\lambda) = \operatorname{Ann} M_\Theta(\lambda)$$

(5.5) $$J_\Theta(\lambda) = I_{q_{\pi^{\natural},M_\Theta(\lambda)}(x)}(\lambda) + J(\lambda_\Theta)$$

hold for a generic $\lambda \in \mathfrak{a}_\Theta^*$. In this section we give the key result to determine exactly for which $\lambda \in \mathfrak{a}_\Theta^*$ (5.5) holds.

Lemma 5.1. *The equality* (5.5) *holds if and only if* $\lambda_\Theta + \rho - \alpha$ *is not a common zero of* $\gamma\big(I_{q_{\pi^{\natural},M_\Theta(\lambda)}(x)}(\lambda)\big)$ *for each* $\alpha \in \Theta$.

Proof. By Lemma 3.4 of [OOs]. □

Lemma 5.2. $\lambda_\Theta + \rho$ *is a common zero of* $\gamma\big(\operatorname{Ann} M_\Theta(\lambda)\big)$.

Proof. For any $D \in U(\mathfrak{g})$,

$$D \equiv \gamma(D)(\lambda+\rho) \mod \bar{\mathfrak{n}}U(\mathfrak{g}) + J(\lambda_\Theta).$$

On the other hand,

$$\begin{aligned} J_\Theta(\lambda) &= U(\mathfrak{g})\mathfrak{m}_\Theta + J(\lambda_\Theta) = U(\bar{\mathfrak{n}}_\Theta)U(\mathfrak{m}_\Theta)U(\mathfrak{a}_\Theta + \mathfrak{n}_\Theta)\mathfrak{m}_\Theta + J(\lambda_\Theta) \\ &= U(\bar{\mathfrak{n}}_\Theta)U(\mathfrak{m}_\Theta)\mathfrak{m}_\Theta + J(\lambda_\Theta) \subset \bar{\mathfrak{n}}U(\mathfrak{g}) + J(\lambda_\Theta). \end{aligned}$$

Since $\operatorname{Ann} M_\Theta(\lambda) \subset J_\Theta(\lambda)$, we have the lemma. □

Proposition 5.3. *Define* $Q_{ij} \in U(\mathfrak{g})$ $(i,j = 1,\ldots,N)$ *so that* $q_{\pi^{\natural},M_\Theta(\lambda)}(F_{\pi^{\natural}}) = \sum_{i,j} Q_{ij} \otimes (v_i^* \otimes v_j)$. *Let* $\kappa = (-1)^{\dim q_{\pi^{\natural},M_\Theta(\lambda)}(x)}$. *Then for* $i = 1,\ldots,n$,

(5.6) $$\gamma(Q_{ii}) \equiv \kappa\gamma(Q_{N+1-i,N+1-i}) \mod \gamma\big(I(\lambda_\Theta)\big).$$

Proof. We first assert it suffices to show (5.6) for $i = 1$. In fact, owing to (5.1) and Lemma 3.4 (ii), $\gamma\big(I(\lambda_\Theta)\big)$ is $\widetilde{W}$-stable. On the other hand, it follows from Proposition 3.5 that for $i = 1,\ldots,n$,

$$\begin{aligned} &\sum_{j=1}^{i} \gamma(Q_{jj}) - \kappa\sum_{j=1}^{i}\gamma(Q_{N+1-j,N+1-j}) \\ &\qquad = (1 + \tilde{s}_1 + \tilde{s}_2\tilde{s}_1 + \cdots + \tilde{s}_{i-1}\tilde{s}_{i-2}\cdots\tilde{s}_1)\big(\gamma(Q_{11}) - \kappa\gamma(Q_{NN})\big) \end{aligned}$$

(cf. the proof of Theorem 3.9). Hence if (5.6) is valid for $i = 1$ then we inductively have

$$\sum_{j=1}^{i} \gamma(Q_{jj}) - \kappa \sum_{j=1}^{i} \gamma(Q_{N+1-j,N+1-j}) \in \gamma\big(I(\lambda_\Theta)\big)$$

for each $i = 1, \ldots, n$, which proves our assertion. In what follows we give two separate arguments according to whether $\mathfrak{g} = \mathfrak{o}_{2n+1}$ or not.

Case $\mathfrak{g} = \mathfrak{o}_{2n+1}$. Define the linear map

$$T_L : \mathbb{C}[x, \mu_{-L}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_L] \to \mathbb{C}[x, \mu_{-L}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_L]$$

by

$$q(x, \boldsymbol{\mu}) \longmapsto q(\tilde{n} - x, \boldsymbol{\mu}) - \frac{1}{2} \frac{q(y, \boldsymbol{\mu}) - q(\tilde{n} - x, \boldsymbol{\mu})}{y - \tilde{n} + x} \Bigg|_{y^u \mapsto \widetilde{C}^{(u)} - x^u} \tag{5.7}$$

with

$$\boldsymbol{\mu} = (\mu_{-L}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_L), \tag{5.8}$$

$$\tilde{n} = \frac{1}{L - \frac{1}{2}} \sum_{k \in \{\pm 1, \ldots, \pm L\}} \mu_k, \tag{5.9}$$

$$\widetilde{C}^{(u)} = \left(\tfrac{\tilde{n}}{2}\right)^u - 2 \sum_{k \in \{\pm 1, \ldots, \pm L\}} (\mu_k)^u (2\mu_k - \tilde{n} - \tfrac{1}{2}) \prod_{\substack{\ell \in \{\pm 1, \ldots, \pm L\} \\ \ell \neq k}} \frac{\mu_k + \mu_\ell - \tilde{n}}{\mu_k - \mu_\ell} \qquad (u = 0, 1, \ldots). \tag{5.10}$$

Here the substitutions for y^u in the right-hand side of (5.7) stand for the linear map $\mathbb{C}[y, x, \boldsymbol{\mu}] \to \mathbb{C}[x, \boldsymbol{\mu}]$ defined by $y^u x^{u'} \boldsymbol{\mu}^{u''} \mapsto \widetilde{C}^{(u)} x^{u'} \boldsymbol{\mu}^{u''} - x^{u+u'} \boldsymbol{\mu}^{u''}$. Since $\widetilde{C}^{(u)} \in \mathbb{C}[\boldsymbol{\mu}]$ by Lemma 4.3, the map T_L is well-defined. Also, clearly T_L maps a symmetric function in $\boldsymbol{\mu}$ to a function of the same kind. Put

$$q_0(x, \boldsymbol{\mu}) = \left(x - \frac{\tilde{n}}{2}\right) \prod_{k \in \{\pm 1, \ldots, \pm L\}} (x - \mu_k),$$

$$\bar{q}_0(x, \boldsymbol{\mu}) = \prod_{k \in \{-L, \ldots, -1, 1, \ldots, L-1\}} (x - \mu_k)$$

and let us prove

$$(T_L q_0)(x, \boldsymbol{\mu}) = -q_0(x, \boldsymbol{\mu}), \tag{5.11}$$

$$(T_L \bar{q}_0)(x, \boldsymbol{\mu}) \equiv -\bar{q}_0(x, \boldsymbol{\mu}) \mod \mathbb{C}[x, \boldsymbol{\mu}] \left(2(L-1)\mu_L - \sum_{k \in \{-L, \ldots, -1, 1, \ldots, L-1\}} \mu_k \right). \tag{5.12}$$

For this purpose define

$$S_L = \left\{ (\hat{\lambda}_1, \ldots, \hat{\lambda}_L, \hat{n}_1, \ldots, \hat{n}_L) \in \mathbb{C}^L \times \mathbb{Z}^L;\ \hat{n}_0 := 0 < \hat{n}_1 < \cdots < \hat{n}_L \right\},$$

$$
\begin{aligned}
&\mathbf{m}_L : S_L \longrightarrow \mathbb{C}^{2L};\\
&\quad (\hat{\lambda}_1,\dots,\hat{\lambda}_L,\hat{n}_1,\dots,\hat{n}_L) \longmapsto \hat{\boldsymbol{\mu}} = (\hat{\mu}_{-L},\dots,\hat{\mu}_{-1},\hat{\mu}_1,\dots,\hat{\mu}_L)\\
&\quad \text{with } \hat{\mu}_{-k} = -\frac{\hat{\lambda}_k}{2} + \frac{\hat{n}_{k-1}}{2} \text{ and } \hat{\mu}_k = \frac{\hat{\lambda}_k}{2} + \frac{2\hat{n}_L - \hat{n}_k}{2} \ (\hat{n}_0 := 0),\\
&\overline{S}_L = \left\{ (\hat{\lambda}_1,\dots,\hat{\lambda}_{L-1},\hat{n}_1,\dots,\hat{n}_L) \in \mathbb{C}^{L-1} \times \mathbb{Z}^L;\ \hat{n}_0 := 0 < \hat{n}_1 < \cdots < \hat{n}_L \right\},\\
&\overline{P}_L = \left\{ \boldsymbol{\mu} \in \mathbb{C}^{2L};\ 2(L-1)\mu_L = \sum_{k\in\{-L,\dots,-1,1,\dots,L-1\}} \mu_k \right\},\\
&\overline{\mathbf{m}}_L : \overline{S}_L \longrightarrow \overline{P}_L;\\
&\quad (\hat{\lambda}_1,\dots,\hat{\lambda}_{L-1},\hat{n}_1,\dots,\hat{n}_L) \longmapsto \hat{\boldsymbol{\mu}} = (\hat{\mu}_{-L},\dots,\hat{\mu}_{-1},\hat{\mu}_1,\dots,\hat{\mu}_L)\\
&\quad \text{with } \hat{\mu}_{-k} = -\frac{\hat{\lambda}_k}{2} + \frac{\hat{n}_{k-1}}{2} \text{ and } \hat{\mu}_k = \frac{\hat{\lambda}_k}{2} + \frac{2\hat{n}_L - \hat{n}_k}{2} \ (\hat{\lambda}_L := 0, \hat{n}_0 := 0).
\end{aligned}
$$

Suppose $(\hat{\lambda}_1,\dots,\hat{\lambda}_L,\hat{n}_1,\dots,\hat{n}_L) \in S_L$ and $\hat{\boldsymbol{\mu}} = \mathbf{m}_L(\hat{\lambda}_1,\dots,\hat{\lambda}_L,\hat{n}_1,\dots,\hat{n}_L)$. Then $q_0(x,\hat{\boldsymbol{\mu}})$ equals the minimal polynomial "$q_{\pi^\natural, M_{\Theta_{\widehat{\Pi}}}(\hat{\lambda})}(x)$" for the natural representation "$\pi^\natural$" of $\mathfrak{o}_{2\hat{n}_L+1}$ and the generalized Verma module "$M_{\Theta_{\widehat{\Pi}}}(\hat{\lambda})$" of $\mathfrak{o}_{2\hat{n}_L+1}$ corresponding to $\widehat{\Pi} : \hat{n}_0 < \hat{n}_1 < \cdots < \hat{n}_L$ and $\hat{\lambda} = (\hat{\lambda}_1,\dots,\hat{\lambda}_L)$. Let $\widehat{Q}_{ij}$ and $\hat{\rho}$ be "Q_{ij}" and "ρ" for this setting. In view of the definition of T_L, Theorem 3.9 (iii), (4.4) and Lemma 5.2, we have

$$\gamma(\widehat{Q}_{11})(\lambda_{\Theta_{\widehat{\Pi}}} + \hat{\rho}) = (T_L q_0)(\hat{\mu}_{-1},\hat{\boldsymbol{\mu}}) = 0.$$

But since $\mathbf{m}_L\big(S_L\big)$ is a Zariski dense subset of $\mathbb{C}^{2L}$,

$$(T_L q_0)(\mu_{-1},\boldsymbol{\mu}) = 0 \text{ for any } \boldsymbol{\mu} \in \mathbb{C}^{2L}.$$

It shows $(T_L q_0)(x,\boldsymbol{\mu})$ is divisible by $x - \mu_{-1}$. Moreover, since $(T_L q_0)(x,\boldsymbol{\mu})$ is symmetric in $\boldsymbol{\mu}$, $(T_L q_0)(x,\boldsymbol{\mu})$ is divisible by each $x - \mu_k$ $(k = -L,\dots,-1,1,\dots,L)$. On the other hand,

$$
\begin{aligned}
(T_L q_0)\left(\tfrac{\tilde{n}}{2},\boldsymbol{\mu}\right) &= q_0\left(\tfrac{\tilde{n}}{2},\boldsymbol{\mu}\right) - \frac{1}{2}\frac{q_0(y,\boldsymbol{\mu}) - q_0\left(\frac{\tilde{n}}{2},\boldsymbol{\mu}\right)}{y - \frac{\tilde{n}}{2}}\Bigg|_{y^u \mapsto \widetilde{C}^{(u)} - \left(\frac{\tilde{n}}{2}\right)^u}\\
&= -\tfrac{1}{2} \prod_{k\in\{\pm1,\dots,\pm L\}} (y - \mu_k)\Big|_{y^u \mapsto \widetilde{C}^{(u)} - \left(\frac{\tilde{n}}{2}\right)^u}\\
&= -\tfrac{1}{2} \prod_{k\in\{\pm1,\dots,\pm L\}} (y - \mu_k)\Big|_{y^u \mapsto \widetilde{C}^{(u)}} + \tfrac{1}{2} \prod_{k\in\{\pm1,\dots,\pm L\}} \left(\tfrac{\tilde{n}}{2} - \mu_k\right)\\
&= -\tfrac{1}{2} \prod_{k\in\{\pm1,\dots,\pm L\}} \left(\tfrac{\tilde{n}}{2} - \mu_k\right) + \tfrac{1}{2} \prod_{k\in\{\pm1,\dots,\pm L\}} \left(\tfrac{\tilde{n}}{2} - \mu_k\right) \quad \text{(by (5.10))}\\
&= 0.
\end{aligned}
$$

Thus $(T_L q_0)(x, \boldsymbol{\mu})$ is divisible also by $x - \frac{\tilde{n}}{2}$. Since it is immediate from (5.7) that $\deg_x(T_L q_0)(x, \boldsymbol{\mu}) = 2L+1$ and the coefficient of x^{2L+1} of $(T_L q_0)(x, \boldsymbol{\mu})$ is -1, we get (5.11).

Likewise, suppose $(\hat{\lambda}_1, \ldots, \hat{\lambda}_{L-1}, \hat{n}_1, \ldots, \hat{n}_L) \in \overline{S}_L$ and $\hat{\boldsymbol{\mu}} = \overline{\mathbf{m}}_L(\hat{\lambda}_1, \ldots, \hat{\lambda}_{L-1}, \hat{n}_1, \ldots, \hat{n}_L)$. Then $\bar{q}_0(x, \hat{\boldsymbol{\mu}})$ equals the minimal polynomial "$q_{\pi^\natural, M_{\overline{\Theta}_{\hat{\Pi}}}(\hat{\lambda})}(x)$" for the natural representation "$\pi^\natural$" of $\mathfrak{o}_{2\hat{n}_L+1}$ and the generalized Verma module "$M_{\overline{\Theta}_{\hat{\Pi}}}(\hat{\lambda})$" of $\mathfrak{o}_{2\hat{n}_L+1}$ corresponding to $\hat{\Pi} : \hat{n}_0 < \hat{n}_1 < \cdots < \hat{n}_L$ and $\hat{\lambda} = (\hat{\lambda}_1, \ldots, \hat{\lambda}_{L-1})$. Hence by the same reason as above we have $(T_L \bar{q}_0)(\hat{\mu}_{-1}, \hat{\boldsymbol{\mu}}) = 0$. Since $\overline{\mathbf{m}}_L\big(\overline{S}_L\big)$ is a Zariski dense subset of $\overline{P}_L$, $(T_L \bar{q}_0)(\mu_{-1}, \boldsymbol{\mu}) = 0$ for any $\boldsymbol{\mu} \in \overline{P}_L$. It implies

$$(5.13) \qquad (T_L \bar{q}_0)(\mu_k, \boldsymbol{\mu}) = 0 \text{ for any } \boldsymbol{\mu} \in \overline{P}_L \text{ and } k \in \{-L, \ldots, -1, 1, \ldots, L-1\}$$

because $(T_L \bar{q}_0)(x, \boldsymbol{\mu})$ and the definition of $\overline{P}_L$ are symmetric in $\mu_{-L}, \ldots, \mu_{-1}$, $\mu_1, \ldots, \mu_{L-1}$. Since $\deg_x(T_L \bar{q}_0)(x, \boldsymbol{\mu}) = 2L-1$ and the coefficient of x^{2L-1} of $(T_L \bar{q}_0)(x, \boldsymbol{\mu})$ is -1, (5.13) proves (5.12).

Now consider the case where $\Theta = \Theta_\Pi$ and $\lambda \in \mathfrak{a}^*_{\Theta_\Pi}$. Let $\dot{\boldsymbol{\mu}} = \mathbf{m}_L(\lambda_1, \ldots, \lambda_L, n_1, \ldots, n_L)$. Then $q_0(x, \dot{\boldsymbol{\mu}}) = q_{\pi^\natural, M_{\Theta_\Pi}(\lambda)}(x)$ and it follows from Theorem 3.9 (ii) that

$$(5.14) \qquad \gamma(Q_{NN}) = q_0\Big(\frac{n}{2} - \frac{1}{4} - H_1, \dot{\boldsymbol{\mu}}\Big).$$

Also, it follows from the definition of T_L, Theorem 3.9 (iii) and (4.4) that

$$(5.15) \qquad \gamma(Q_{11}) \equiv (T_L q_0)\Big(\frac{n}{2} - \frac{1}{4} - H_1, \dot{\boldsymbol{\mu}}\Big) \mod \gamma\big(I(\lambda_{\Theta_\Pi})\big).$$

Thanks to (5.11), (5.14), and (5.15), we get (5.6) for $i = 1$.

Next, consider the case where $\Theta = \overline{\Theta}_\Pi$ and $\lambda \in \mathfrak{a}^*_{\overline{\Theta}_\Pi}$. Let $\ddot{\boldsymbol{\mu}} = \overline{\mathbf{m}}_L(\lambda_1, \ldots, \lambda_{L-1}, n_1, \ldots, n_L)$. Then $\bar{q}_0(x, \ddot{\boldsymbol{\mu}}) = q_{\pi^\natural, M_{\overline{\Theta}_\Pi}(\lambda)}(x)$ and as in the previous paragraph,

$$(5.16) \qquad \gamma(Q_{NN}) = \bar{q}_0\Big(\frac{n}{2} - \frac{1}{4} - H_1, \ddot{\boldsymbol{\mu}}\Big),$$

$$(5.17) \qquad \gamma(Q_{11}) \equiv (T_L \bar{q}_0)\Big(\frac{n}{2} - \frac{1}{4} - H_1, \ddot{\boldsymbol{\mu}}\Big) \mod \gamma\big(I(\lambda_{\overline{\Theta}_\Pi})\big).$$

In this case (5.12), (5.16), and (5.17) lead to (5.6) for $i = 1$.

Case $\mathfrak{g} = \mathfrak{sp}_n$, or $\mathfrak{o}_{2n}$. Because the outline is the same as in the case $\mathfrak{g} = \mathfrak{o}_{2n+1}$, we shall omit the detailed explanation. Put

$$\epsilon = \begin{cases} 1 & \text{if } \mathfrak{g} = \mathfrak{o}_{2n}, \\ -1 & \text{if } \mathfrak{g} = \mathfrak{sp}_n \end{cases}$$

and define the linear map

$$T_L : \mathbb{C}[x, \mu_{-L}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_L] \to \mathbb{C}[x, \mu_{-L}, \ldots, \mu_{-1}, \mu_1, \ldots, \mu_L]$$

by

$$q(x,\boldsymbol{\mu}) \longmapsto q\big(\tilde n - x - \tfrac{\epsilon}{2},\boldsymbol{\mu}\big) - \frac{1}{2}\frac{q(y,\boldsymbol{\mu}) - q\big(\tilde n - x - \frac{\epsilon}{2},\boldsymbol{\mu}\big)}{y - \tilde n + x + \frac{\epsilon}{2}}\Bigg|_{y^u \mapsto \widetilde C^{(u)} - \epsilon x^u} \tag{5.18}$$

with

$$\boldsymbol{\mu} = (\mu_{-L},\dots,\mu_{-1},\mu_1,\dots,\mu_L), \tag{5.19}$$

$$\tilde n = \frac{1}{L-\frac{1}{2}} \sum_{k\in\{\pm1,\dots,\pm L\}} \mu_k + \frac{L\epsilon}{2L-1}, \tag{5.20}$$

$$\widetilde C^{(u)} = -2 \sum_{k\in\{\pm1,\dots,\pm L\}} (\mu_k)^u (2\mu_k - \tilde n + \epsilon) \prod_{\substack{\ell\in\{\pm1,\dots,\pm L\}\\ \ell\neq k}} \frac{\mu_k + \mu_\ell - \tilde n + \frac{\epsilon}{2}}{\mu_k - \mu_\ell} \qquad (u = 0,1,\dots). \tag{5.21}$$

Moreover, put

$$q_0(x,\boldsymbol{\mu}) = \prod_{k\in\{\pm1,\dots,\pm L\}} (x-\mu_k),$$

$$\bar q_0(x,\boldsymbol{\mu}) = \prod_{k\in\{-L,\dots,-1,1,\dots,L-1\}} (x-\mu_k),$$

$$S_L = \Big\{ (\hat\lambda_1,\dots,\hat\lambda_L,\hat n_1,\dots,\hat n_L) \in \mathbb{C}^L \times \mathbb{Z}^L;\ \hat n_0 := 0 < \hat n_1 < \dots < \hat n_L \Big\},$$

$$\mathbf{m}_L : S_L \longrightarrow \mathbb{C}^{2L};$$

$$(\hat\lambda_1,\dots,\hat\lambda_L,\hat n_1,\dots,\hat n_L) \longmapsto \hat{\boldsymbol{\mu}} = (\hat\mu_{-L},\dots,\hat\mu_{-1},\hat\mu_1,\dots,\hat\mu_L)$$

$$\text{with } \hat\mu_{-k} = -\frac{\hat\lambda_k}{2} + \frac{\hat n_{k-1}}{2} \text{ and } \hat\mu_k = \frac{\hat\lambda_k}{2} + \frac{2\hat n_L - \hat n_k - \epsilon}{2} \ (\hat n_0 := 0),$$

$$\overline{S}_L = \left\{ (\hat\lambda_1,\dots,\hat\lambda_{L-1},\hat n_1,\dots,\hat n_L) \in \mathbb{C}^{L-1}\times\mathbb{Z}^L;\ \begin{array}{r} \hat n_0 := 0 < \hat n_1 < \dots < \hat n_{L-1}, \\ \hat n_{L-1} + \frac{1+\epsilon}{2} < n_L \end{array} \right\},$$

$$\overline{P}_L = \left\{ \boldsymbol{\mu} \in \mathbb{C}^{2L};\ (L-1)\big(2\mu_L + \tfrac{\epsilon}{2}\big) = \sum_{k\in\{-L,\dots,-1,1,\dots,L-1\}} \mu_k \right\},$$

$$\overline{\mathbf{m}}_L : \overline{S}_L \longrightarrow \overline{P}_L;$$

$$(\hat\lambda_1,\dots,\hat\lambda_{L-1},\hat n_1,\dots,\hat n_L) \longmapsto \hat{\boldsymbol{\mu}} = (\hat\mu_{-L},\dots,\hat\mu_{-1},\hat\mu_1,\dots,\hat\mu_L)$$

$$\text{with } \hat\mu_{-k} = -\frac{\hat\lambda_k}{2} + \frac{\hat n_{k-1}}{2} \text{ and } \hat\mu_k = \frac{\hat\lambda_k}{2} + \frac{2\hat n_L - \hat n_k - \epsilon}{2} \ (\hat\lambda_L := 0, \hat n_0 := 0)$$

and let us prove

$$(T_L q_0)(x,\boldsymbol{\mu}) = q_0(x,\boldsymbol{\mu}), \tag{5.22}$$

$$(T_L \bar q_0)(x,\boldsymbol{\mu}) \equiv -\bar q_0(x,\boldsymbol{\mu}) \mod \mathbb{C}[x,\boldsymbol{\mu}]\left((L-1)\big(2\mu_L + \tfrac{\epsilon}{2}\big) - \sum_{k\in\{-L,\dots,-1,1,\dots,L-1\}} \mu_k \right). \tag{5.23}$$

Suppose $(\hat{\lambda}_1,\ldots,\hat{\lambda}_L,\hat{n}_1,\ldots,\hat{n}_L) \in S_L$ and $\hat{\boldsymbol{\mu}} = \mathbf{m}_L(\hat{\lambda}_1,\ldots,\hat{\lambda}_L,\hat{n}_1,\ldots,\hat{n}_L)$. Then $q_0(x,\hat{\boldsymbol{\mu}})$ equals the polynomial $q_{\Theta_{\widehat{\Pi}}}(\hat{\mathfrak{g}};x,\hat{\lambda})$ in Definition 2.2 corresponding to $\hat{\mathfrak{g}} = \mathfrak{o}_{2\hat{n}_L}$ or $\mathfrak{sp}_{\hat{n}_L}$, $\widehat{\Pi} : \hat{n}_0 < \hat{n}_1 < \cdots < \hat{n}_L$, and $\hat{\lambda} = (\hat{\lambda}_1,\ldots,\hat{\lambda}_L)$. In view of the definition of T_L, Theorem 3.9 (iii), (4.7) and Lemma 5.2, we have

$$(T_L q_0)(\hat{\mu}_{-1},\hat{\boldsymbol{\mu}}) = 0.$$

Since $\mathbf{m}_L\Big(S_L\Big)$ is a Zariski dense subset of $\mathbb{C}^{2L}$,

$$(T_L q_0)(\mu_{-1},\boldsymbol{\mu}) = 0 \text{ for any } \boldsymbol{\mu} \in \mathbb{C}^{2L}.$$

It shows $(T_L q_0)(x,\boldsymbol{\mu})$ is divisible by $x-\mu_{-1}$. Because of the symmetry of $(T_L q_0)(x,\boldsymbol{\mu})$ in $\boldsymbol{\mu}$, $(T_L q_0)(x,\boldsymbol{\mu})$ is divisible by each $x-\mu_k$ $(k=\pm1,\ldots,\pm L)$. Thus we get (5.22).

Likewise, suppose $(\hat{\lambda}_1,\ldots,\hat{\lambda}_{L-1},\hat{n}_1,\ldots,\hat{n}_L) \in \overline{S}_L$ and $\hat{\boldsymbol{\mu}} = \overline{\mathbf{m}}_L(\hat{\lambda}_1,\ldots,\hat{\lambda}_{L-1},\hat{n}_1,\ldots,\hat{n}_L)$. Then $\bar{q}_0(x,\hat{\boldsymbol{\mu}})$ equals the polynomial $q_{\overline{\Theta}_{\widehat{\Pi}}}(\hat{\mathfrak{g}};x,\hat{\lambda})$ in Definition 2.2 corresponding to $\hat{\mathfrak{g}} = \mathfrak{o}_{2\hat{n}_L}$ or $\mathfrak{sp}_{\hat{n}_L}$, $\widehat{\Pi} : \hat{n}_0 < \hat{n}_1 < \cdots < \hat{n}_L$, and $\hat{\lambda} = (\hat{\lambda}_1,\ldots,\hat{\lambda}_{L-1})$. By the same reason as above, $(T_L\bar{q}_0)(\hat{\mu}_{-1},\hat{\boldsymbol{\mu}}) = 0$. Since $\overline{\mathbf{m}}_L\Big(\overline{S}_L\Big)$ is a Zariski dense subset of $\overline{P}_L$, $(T_L\bar{q}_0)(\mu_{-1},\boldsymbol{\mu}) = 0$ for any $\boldsymbol{\mu} \in \overline{P}_L$. It implies

$$(T_L\bar{q}_0)(\mu_k,\boldsymbol{\mu}) = 0 \text{ for any } \boldsymbol{\mu} \in \overline{P}_L \text{ and } k \in \{-L,\ldots,-1,1,\ldots,L-1\}$$

because $(T_L\bar{q}_0)(x,\boldsymbol{\mu})$ and the definition of $\overline{P}_L$ are symmetric in $\mu_{-L},\ldots,\mu_{-1}$, $\mu_1,\ldots,\mu_{L-1}$. Thus we get (5.23).

If $\epsilon = 1$ (namely $\mathfrak{g} = \mathfrak{o}_{2n}$) then for $i = 1,\ldots,L$ we also put

$$S_L^{(i)} = \left\{ \begin{array}{r} (\hat{\lambda}_1,\ldots,\hat{\lambda}_{i-1},\hat{\lambda}_{i+1},\ldots\hat{\lambda}_L,\hat{n}_1,\ldots,\hat{n}_{i-1},\hat{n}_{i+1},\ldots,\hat{n}_L) \in \mathbb{C}^{L-1}\times\mathbb{Z}^{L-1}; \\ \hat{n}_0 := 0 < \hat{n}_1 < \cdots < \hat{n}_{i-1}, \\ \hat{n}_i := \hat{n}_{i-1}+1 < \hat{n}_{i+1} < \cdots < \hat{n}_L \end{array} \right\},$$

$$P_L^{(i)} = \left\{ \boldsymbol{\mu} \in \mathbb{C}^{2L};\ \Big(L-\frac{3}{2}\Big)(\mu_{-i}+\mu_i) + \frac{L-1}{2} = \sum_{\substack{1\le k\le L\\ k\ne i}} (\mu_{-k}+\mu_k),\ \mu_{-i} = \mu_i \right\},$$

$$\begin{aligned} \mathbf{m}_L^{(i)} : S_L^{(i)} &\longrightarrow P_L^{(i)}; \\ (\hat{\lambda}_1,\ldots,\hat{\lambda}_{i-1},\hat{\lambda}_{i+1},\ldots\hat{\lambda}_L,&\hat{n}_1,\ldots,\hat{n}_{i-1},\hat{n}_{i+1},\ldots,\hat{n}_L) \\ &\longmapsto \hat{\boldsymbol{\mu}} = (\hat{\mu}_{-L},\ldots,\hat{\mu}_{-1},\hat{\mu}_1,\ldots,\hat{\mu}_L) \\ \text{with } \hat{\mu}_{-k} = -\frac{\hat{\lambda}_k}{2} + \frac{\hat{n}_{k-1}}{2} &\text{ and } \hat{\mu}_k = \frac{\hat{\lambda}_k}{2} + \frac{2\hat{n}_L-\hat{n}_k-1}{2} \\ &(\hat{n}_0 := 0, \hat{n}_i := \hat{n}_{i-1}+1, \hat{\lambda}_i := \hat{n}_{i-1}-\hat{n}_L+1), \end{aligned}$$

$$I_L^{(i)} = \left\{ f(x,\boldsymbol{\mu}) \in \mathbb{C}[x,\boldsymbol{\mu}];\ f(x,\hat{\boldsymbol{\mu}}) = 0 \text{ for any } \hat{\boldsymbol{\mu}} \in P_L^{(i)} \right\},$$

$$q_0^{(i)}(x,\boldsymbol{\mu}) = \Big(x-\frac{\tilde{n}-1}{2}\Big) \prod_{\substack{1\le k\le L\\ k\ne i}} (x-\mu_{-k})(x-\mu_k) \qquad (\tilde{n} \text{ is defined by (5.20)})$$

and try to prove

$$(5.24)\qquad \big(T_L q_0^{(i)}\big)(x,\boldsymbol{\mu}) \equiv -q_0^{(i)}(x,\boldsymbol{\mu}) \mod I_L^{(i)}.$$

Suppose

$$i = 1,\dots,L,$$
$$(\hat{\lambda}_1,\dots,\hat{\lambda}_{i-1},\hat{\lambda}_{i+1},\dots\hat{\lambda}_L,\hat{n}_1,\dots,\hat{n}_{i-1},\hat{n}_{i+1},\dots,\hat{n}_L) \in S_L^{(i)},$$
$$\hat{\boldsymbol{\mu}} = \mathbf{m}_L^{(i)}(\hat{\lambda}_1,\dots,\hat{\lambda}_{i-1},\hat{\lambda}_{i+1},\dots\hat{\lambda}_L,\hat{n}_1,\dots,\hat{n}_{i-1},\hat{n}_{i+1},\dots,\hat{n}_L).$$

Then it follows from Theorem 2.8 that $q_0^{(i)}(x,\hat{\boldsymbol{\mu}})$ coincides with the minimal polynomial "$q_{\pi^\natural,M_{\Theta_{\hat{\Pi}}}(\hat{\lambda})}(x)$" for the natural representation "$\pi^\natural$" of $\mathfrak{o}_{2\hat{n}_L}$ and the generalized Verma module "$M_{\Theta_{\hat{\Pi}}}(\hat{\lambda})$" of $\mathfrak{o}_{2\hat{n}_L}$ corresponding to $\hat{\Pi} : \hat{n}_0 < \hat{n}_1 < \cdots < \hat{n}_L$ and $\hat{\lambda} = (\hat{\lambda}_1,\dots,\hat{\lambda}_L)$ $(\hat{n}_0 := 0, \hat{n}_i := \hat{n}_{i-1}+1, \hat{\lambda}_i := \hat{n}_{i-1} - \hat{n}_L + 1)$. By the same reason as above , $\big(T_L q_0^{(i)}\big)(\hat{\mu}_{-1},\hat{\boldsymbol{\mu}}) = 0$. Since $\mathbf{m}_L^{(i)}\big(S_L^{(i)}\big)$ is a Zariski dense subset of $P_L^{(i)}$,

$$(5.25)\qquad \big(T_L q_0^{(i)}\big)(\mu_{-1},\boldsymbol{\mu}) \in I_L^{(i)}.$$

Observe that $\mu_{-1} - \frac{\hat{n}-1}{2} \in I_L^{(1)}$. Hence (5.25) for $i=1$ implies

$$(5.26)\qquad \big(T_L q_0^{(1)}\big)\big(\tfrac{\hat{n}-1}{2},\boldsymbol{\mu}\big) \in I_L^{(1)}.$$

Thus, if $L = 1$ then (5.26) shows (5.24). Suppose $L > 1$. Applying the permutation of the variable $\mu_{-1} \leftrightarrow \mu_{-2}, \mu_1 \leftrightarrow \mu_2$ to (5.25) for $i = 2$, we get

$$(5.27)\qquad \big(T_L q_0^{(1)}\big)(\mu_{-2},\boldsymbol{\mu}) \in I_L^{(1)}.$$

But since $\big(T_L q_0^{(1)}\big)(x,\boldsymbol{\mu})$ is symmetric in $\mu_{-L},\dots,\mu_{-2},\mu_2,\dots,\mu_L$,

$$(5.28)\qquad \big(T_L q_0^{(1)}\big)(\mu_k,\boldsymbol{\mu}) \in I_L^{(1)} \text{ for any } k = \pm 2,\dots,\pm L.$$

Thus (5.26) and (5.28) imply (5.24) for $i = 1$. Finally (5.24) for an arbitrary i is obtained by applying the permutation of the variable $\mu_{-1} \leftrightarrow \mu_{-i}, \mu_1 \leftrightarrow \mu_i$ to (5.24) for $i = 1$.

Now consider the case where $\Theta = \Theta_\Pi$ and $\lambda \in \mathfrak{a}^*_{\Theta_\Pi}$. Let $\dot{\boldsymbol{\mu}} = \mathbf{m}_L(\lambda_1,\dots,\lambda_L, n_1,\dots,n_L)$. It follows from Theorem 2.6 and Theorem 2.8 that

$$q_{\pi^\natural,M_{\Theta_\Pi}(\lambda)}(x) = \begin{cases} q_0(x,\dot{\boldsymbol{\mu}}) & \text{if } \epsilon = -1 \text{ or } \dot{\boldsymbol{\mu}} \notin \bigcup_{i=1}^L P_L^{(i)}, \\ q_0^{(i)}(x,\dot{\boldsymbol{\mu}}) & \text{if } \epsilon = 1 \text{ and } \dot{\boldsymbol{\mu}} \in P_L^{(i)}. \end{cases}$$

Therefore by Theorem 3.9 (ii)

$$(5.29)\qquad \gamma(Q_{NN}) = \begin{cases} q_0\big(\frac{n}{2} - \frac{1}{4} - \frac{\epsilon}{4} - H_1,\dot{\boldsymbol{\mu}}\big) & \text{if } \epsilon = -1 \text{ or } \dot{\boldsymbol{\mu}} \notin \bigcup_{i=1}^L P_L^{(i)}, \\ q_0^{(i)}\big(\frac{n}{2} - \frac{1}{4} - \frac{\epsilon}{4} - H_1,\dot{\boldsymbol{\mu}}\big) & \text{if } \epsilon = 1 \text{ and } \dot{\boldsymbol{\mu}} \in P_L^{(i)}. \end{cases}$$

Also, it follows from the definition of T_L, Theorem 3.9 (iii) and (4.7) that

$$\gamma(Q_{11}) \equiv \begin{cases} (T_L q_0)\left(\frac{n}{2} - \frac{1}{4} - \frac{\epsilon}{4} - H_1, \dot{\boldsymbol{\mu}}\right) & \text{if } \epsilon = -1 \text{ or } \dot{\boldsymbol{\mu}} \notin \bigcup_{i=1}^{L} P_L^{(i)}, \\ (T_L q_0^{(i)})\left(\frac{n}{2} - \frac{1}{4} - \frac{\epsilon}{4} - H_1, \dot{\boldsymbol{\mu}}\right) & \text{if } \epsilon = 1 \text{ and } \dot{\boldsymbol{\mu}} \in P_L^{(i)}. \end{cases} \quad (5.30)$$
$$\bmod \gamma\big(I(\lambda_{\Theta_\Pi})\big)$$

Thanks to (5.22), (5.24), (5.29), and (5.30), we get (5.6) for $i = 1$.

Next, consider the case where $\Theta = \overline{\Theta}_\Pi$ and $\lambda \in \mathfrak{a}^*_{\overline{\Theta}_\Pi}$. Let $\ddot{\boldsymbol{\mu}} = \overline{\mathbf{m}}_L(\lambda_1, \ldots, \lambda_{L-1}, n_1, \ldots, n_L)$. Then $\bar{q}_0(x, \ddot{\boldsymbol{\mu}}) = q_{\pi^\natural, M_{\overline{\Theta}_\Pi}(\lambda)}(x)$ and as in the previous paragraph,

$$\gamma(Q_{NN}) = \bar{q}_0\left(\frac{n}{2} - \frac{1}{4} - \frac{\epsilon}{4} - H_1, \ddot{\boldsymbol{\mu}}\right), \quad (5.31)$$

$$\gamma(Q_{11}) \equiv (T_L \bar{q}_0)\left(\frac{n}{2} - \frac{1}{4} - \frac{\epsilon}{4} - H_1, \ddot{\boldsymbol{\mu}}\right) \bmod \gamma\big(I(\lambda_{\overline{\Theta}_\Pi})\big). \quad (5.32)$$

In this case (5.23), (5.31), and (5.32) lead to (5.6) for $i = 1$. □

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