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附録：
ON THE PROOF OF THE CAPELLI IDENTITIES

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以下に収録するのは，外積代数を用いて直接に古典的な Capelli 恒等式を導くことを実行した論文である．Noumi-Umeda-Wakayama の Duke 論文 (1994) において量子群版の Capelli 恒等式を導く計算がなされたが，そのアイディアを古典的な場合に適用することで，量子群や R 行列を知らなくても読める形にして置くことが有益であろうと考え，1997 年に exposition のようなつもりで書いた．基本的な内容だけに，自分も含め，いろいろな論文で引用する機会は多く生じたが，雑誌に発表することなく時間が過ぎてしまった．今回の研究集会のテーマに関しても，その手法の原型ともなっているので，ここに附録として収録しておけば，幾分なりとも利用価値があるかと考えた．

内容的には，古典的な Capelli 恒等式 (GL_n case) だけでなく，Howe-Umeda (1991) の附録にある直交 Lie 環の普遍包絡環の行列式型中心元に関して，Howe-Umeda とは違う扱いも述べているので，他の手法 (たとえば Itoh-Umeda(2001)) との比較としても若干の興味があると思う．更に述べれば，同じ結果でありながら証明法で，その明らかさが異なるという例も与える点にも注意しておきたい．つまり或る元が中心に属することを示すのに，群で不変ということを見るか，Lie 環で消されることを見るか，という証明方針の違いは，GL_n では少しの差しか生じないが，直交 Lie 環では，非可換性 (交換関係の複雑さ) に由来して，よりはっきり違いが出るということである．

このような違いは「現象」として見る以上には出ず，根本的な理由を説明する段階には至っていない．その意味で，最初の「総説的論説」の具体的な補いとして読んでいただければ幸いである．

ON THE PROOF OF THE CAPELLI IDENTITIES

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ABSTRACT. Using exterior calculus, we present detailed proofs of the classical Capelli identities in a purely computational manner. The proof of the fact that the Capelli elements are central is also given in a similar way. In the course of proofs of these two facts, one can easily see the mechanism of the multiplication formula of determinant with non-commutative entries. Simple treatments of the related facts from the Appendix of [HU] are also given.

Introduction: The celebrated Capelli identities [Ca1–3] played important roles in classical invariant theory (see e.g. [My], [Wy]), and have been generalized in several directions. This makes it desirable to have simple proofs of them available. In this article, we give an exposition of the complete set of proofs for the classical Capelli identities utilizing exterior calculus. This approach is based on ideas in [NUW], which proves a quantum group version of the Capelli identity. (Similar treatments are also found in the papers [Kz], [Na].) However, a separate treatment, as it does not require any knowledge of R-matrix for this case, provides more direct access to the proof, and, indeed, yields a short, simple, computaional proof. In the same spirit, we also present a reason why the Capelli elements are central elements of the enveloping algebra of $\mathfrak{gl}_n$. In the proof of this fact as well as in the proof of Capelli identity, one can recognize the multiplication formula for determinant. As our framework gives a clear view point for the proofs given in the Appendix of [HU], we discuss centrality also.

In this paper, we work over $\mathbb{C}$ for simplicity, though this restriction is not needed in any crucial points.

1. Capelli identity as the multiplication formula for determinant: Consider the space $\mathrm{Mat}(m,n)$ of $m \times n$ matrices, on which the two general linear groups GL_m and GL_n acts respectively from the left and the right. Then the space $\mathcal{P}(\mathrm{Mat}(m,n))$ of polynomial functions on $\mathrm{Mat}(m,n)$ is naturally endowed with the structures of a right GL_m-module and a left GL_n-module. We denote these two actions by λ and ρ, and use

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the same notation for the infinitesimal actions of the Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ and for the extensions of these actions to the universal enveloping algebras $U(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_n)$. For the matrix space $\mathrm{Mat}(m,n)$ or the Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$, the matrix units, which form the standard basis, are denoted by E_{ij}. The coordinate functions of $\mathrm{Mat}(m,n)$ with respect to E_{ij} is denoted by t_{ij} and the partial differential operator $\frac{\partial}{\partial t_{ij}}$ is often abbreviated as ∂_{ij}. It is easy to check that for $E_{ij} \in \mathfrak{gl}_n$ or $\mathfrak{gl}_m$, its image under λ and ρ are respectively expressed as

$$\rho(E_{ij}) = \sum_{\alpha=1}^{m} t_{\alpha i}\partial_{\alpha j}, \qquad \lambda(E_{ij}) = \sum_{\beta=1}^{n} t_{j\beta}\partial_{i\beta}. \tag{1.1}$$

Introducing the four matrices

$$T = (t_{ij})_{1\le i\le m, 1\le j\le n}, \qquad D = (\partial_{ij})_{1\le i\le m, 1\le j\le n},$$
$$\Pi = (\rho(E_{ij}))_{1\le i,j\le n}, \qquad \Pi^\circ = (\lambda(E_{ij}))_{1\le i,j\le m},$$

we can write the relations above in matrix form

$$\Pi = {}^tTD, \qquad {}^t\Pi^\circ = T\,{}^tD. \tag{1.2}$$

Here ^{t} stands for the transpose of a matrix. Roughly speaking, the Capelli identities are the multiplication formulas (or more generally the Binet-Cauchy theorem in the rectangular case) of the determinant of these matrices.

In this paper we understand the determinant $\det(a)$ of an $n\times n$ matrix $a = (a_{ij})_{i,j=1}^n$, whose entries are in a (possibly non-commutative) algebra $\mathcal{A}$, as the alternating sum

$$\det(a) = \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}(\sigma)\, a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n}.$$

The exterior calculus is useful for manipulating the determinant of this type even with non-commutative entries. The exterior algebra Λ_n is an associative algebra generated by the n elements $e_1, e_2, \cdots, e_n$ subject to the relations $e_ie_j + e_je_i = 0$. We work in an extended algebra $\Lambda_n \otimes \mathcal{A}$, in which the two subalgebras Λ_n and $\mathcal{A}$ commute. The determinant $\det(a)$ then comes in the following way: forming the elements η_i from the columns of a by $\eta_i = \sum_{\alpha=1}^n e_\alpha a_{\alpha i}$ and multiplying them, we find that $e_1e_2\cdots e_n\det(a) = \eta_1\eta_2\cdots\eta_n$, as in the commutative case. In fact,

$$\begin{aligned}
\eta_1\eta_2\cdots\eta_n &= \sum_{1\le\alpha_1,\alpha_2,\cdots,\alpha_n\le n} e_{\alpha_1}e_{\alpha_2}\cdots e_{\alpha_n}a_{\alpha_1 1}a_{\alpha_2 2}\cdots a_{\alpha_n n}\\
&= \sum_{\sigma\in\mathfrak{S}_n} e_{\sigma(1)}e_{\sigma(2)}\cdots e_{\sigma(n)}a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n}\\
&= \sum_{\sigma\in\mathfrak{S}_n} e_1e_2\cdots e_n\,\mathrm{sign}(\sigma)a_{\sigma(1)1}a_{\sigma(2)2}\cdots a_{\sigma(n)n}\\
&= e_1e_2\cdots e_n\det(a).
\end{aligned}$$

For our Capelli identities, we use this framework with $\mathcal{A} = \mathrm{End}(\mathcal{P}(\mathrm{Mat}(m,n)))$. According to the multiplication of matrices $\Pi = {}^tTD$, we form the following two sets of elements in $\Lambda_n \otimes \mathcal{A}$:

$$(1.3) \qquad \xi_i = \sum_{\alpha=1}^{n} e_\alpha t_{i\alpha} \quad (1 \le i \le m), \qquad \zeta_j = \sum_{i=1}^{m} \xi_i \partial_{ij} \quad (1 \le j \le n).$$

Then from the relation (1.1), we see

$$(1.4) \qquad \zeta_j = \sum_{\alpha=1}^{n} e_\alpha \rho(E_{\alpha j}).$$

Noting that the variables $\{t_{ij}\}$ commutate with each other, we see easily that the commutation relations among ξ_i are $\xi_i\xi_j + \xi_j\xi_i = 0$. For the computation of a product like $\zeta_1\zeta_2\cdots\zeta_n$, we need to know the commutation relations between ξ_i and ∂_{pq}.

Lemma 1. *The following commutation relations hold:*

$$(1) \qquad [\partial_{pq}, \xi_i] = e_q\delta_{pi},$$
$$(2) \qquad \zeta_q\xi_i + \xi_i\zeta_q = \xi_i e_q.$$

Proof: The assertion (1) is easy to see by a direct computation.

$$[\partial_{pq}, \xi_i] = \sum_\alpha e_\alpha[\partial_{pq}, t_{i\alpha}] = \sum_\alpha e_\alpha\delta_{pi}\delta_{q\alpha} = e_q\delta_{pi}.$$

For (2), multiply ξ_p from the left on both sides of (1) and sum up with respect to p. Then noting that ξ_p and ξ_i anti-commute, we come to the assertion. ◇

Suggested by the formula (2) in Lemma 1, we introduce the elements $\zeta_j(u)$ with parameter u as

$$(1.5) \qquad \zeta_j(u) = \zeta_j + ue_j = \sum_{\alpha=1}^{n} e_\alpha\rho(E_{\alpha j} + u\delta_{\alpha j}).$$

Then we see from Lemma 1 (2)

$$(1.6) \qquad \zeta_q(u+1)\xi_i + \xi_i\zeta_q(u) = 0,$$

because ξ_i and e_q anti-commute.

To describe submatrices made from a given matrix, we introduce some notation here. For subsets $I \subset M = \{1, \cdots, m\}$ and $J \subset N = \{1, \cdots, n\}$, making their arrangements under the natural order, we denote by a_{IJ} the submatrices of a matrix $a = (a_{ij})$ formed from those a_{ij} such that $i \in I$ and $j \in J$. Also as we will soon see below, we need some shift in the diagonal of Π for the Capelli identity. For this shift, we introduce the following convention: for any matrix a of size k, we put $a^\natural = a + \mathrm{diag}(k-1, k-2, \cdots, 0)$.

The following is the Capelli identity for rectangular matrices, which can be regarded as a non-commutative version of the Binet-Cauchy theorem.

Theorem 2. (*Capelli identity for rectangular matrices*) : *The following equality holds:*

$$\det(\Pi^{\natural}_{NN}) = \sum_{\sharp I=n} \det({}^tT_{NI}) \det(D_{IN}).$$

When $m < n$, we understand that the empty sum in the right-hand side represents 0.

Corollary 3. (*Capelli identity for square matrices*) : *For the case $m = n$, we have the following equality:*

$$\det(\Pi^{\natural}) = \det({}^tT) \det(D).$$

Proof of Theorem 2 : We compute the product $\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0)$ in two ways. Using the definition (1.5), we see on the one hand

$$\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0) = e_1e_2\cdots e_n \det(\rho(E_{ij} + (n-i)\delta_{ij}).$$

On the other hand, by the relation (1.6) we have

$$\begin{aligned}
\zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_n(0) &= \sum_{1\le k_n\le m} \zeta_1(n-1)\zeta_2(n-2)\cdots\zeta_{n-1}(1)\xi_{k_n}\partial_{k_n n} \\
&= (-)^{n-1} \sum_{1\le k_n\le m} \xi_{k_n}\zeta_1(n-2)\zeta_2(n-3)\cdots\zeta_{n-1}(0)\partial_{k_n n} \\
&= (-)^{n-1} \sum_{1\le k_{n-1},k_n\le m} \xi_{k_n}\zeta_1(n-2)\zeta_2(n-3)\cdots\zeta_{n-2}(1)\xi_{k_{n-1}}\partial_{k_{n-1} n-1}\partial_{k_n n} \\
&= (-)^{2(n-1)} \sum_{1\le k_{n-1},k_n\le m} \xi_{k_{n-1}}\xi_{k_n}\zeta_1(n-3)\zeta_2(n-4)\cdots\zeta_{n-2}(0)\partial_{k_{n-1} n-1}\partial_{k_n n} \\
&= \quad \cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots \\
&= (-)^{n(n-1)} \sum_{1\le k_1,k_2,\cdots,k_n\le m} \xi_{k_1}\xi_{k_2}\cdots\xi_{k_n}\partial_{k_1 1}\partial_{k_2 2}\cdots\partial_{k_n n} \\
&= \sum_{\substack{I=\{i_1<\cdots<i_n\} \\ \sigma\in\mathfrak{S}(I)}} \xi_{i_{\sigma(1)}}\xi_{i_{\sigma(2)}}\cdots\xi_{i_{\sigma(n)}}\partial_{i_{\sigma(1)}1}\partial_{i_{\sigma(2)}2}\cdots\partial_{i_{\sigma(n)}n} \\
&= \sum_{\substack{I=\{i_1<\cdots<i_n\} \\ \sigma\in\mathfrak{S}(I)}} \operatorname{sign}(\sigma)\xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}\partial_{i_{\sigma(1)}1}\partial_{i_{\sigma(2)}2}\cdots\partial_{i_{\sigma(n)}n} \\
&= \sum_{I=\{i_1<\cdots<i_n\}} \xi_{i_1}\xi_{i_2}\cdots\xi_{i_n}\det(D_{IN}) \\
&= e_1e_2\cdots e_n \sum_{I=\{i_1<\cdots<i_n\}} \det({}^tT_{NI})\det(D_{IN}).
\end{aligned}$$

Thus by comparing these two calculations, we obtain our formula. ◇

We can extend Theorem 2 slightly, as follows. Starting from the $n \times n$ square matrix space, we choose k columns according to a subset $J = \{j_1, j_2, \cdots, j_k\} \subset N$. Then we have the relations $\Pi_{JJ} = {}^tT_{JN}D_{NJ}$ as a subset of the relations $\Pi = {}^tTD$. Applying Theorem 2 to this new $n \times k$ rectangular matrix with the replacement $(m,n) \mapsto (n,k)$, we see

$$\det(\Pi^{\natural}_{JJ}) = \sum_{\sharp I=k} \det({}^tT_{JI}) \det(D_{IJ}). \tag{1.7}$$

We will use this form of Theorem 2 for the proof of the lower order Capelli identities in the next section.

Remark: We have similar formula for the left action λ as follows.

Theorem 2′. (*Capelli identity for rectangular matrices*) *We have the following equality:*

$$\det(\Pi^{\circ\natural}_{MM}) = \sum_{\sharp J=m} \det(T_{MJ}) \det({}^tD_{JM}).$$

When $n < m$, we understand that the empty sum in the right-hand side represents 0.

2. The lower order Capelli identities: In the above, we have proved the Capelli identity as a multiplication formula of non-commutaive determinant. The formula itself seems very nice. However, from the representation-theoretic point of view, the significance of the Capelli identity lies in the fact that it presents an equality between the two invariant differential operators on the space $\mathrm{Mat}(m,n)$ (see [H], [HU], [U1]). In this sense, what we proved under the name of "the Capelli identity for rectangular matrices" is *not* the *real* Capelli identity except for $m = n$. Let us explain what should be done to get the real Capelli identities. First, if $m = n$, it is easy to see that the right-hand side of Corollary 3 is invariant under the actions of GL_n both from the right and the left. Admitting that the representations ρ and λ of $U(\mathfrak{gl}_n)$ are faithful in this case, we see that the element $C = \det(E_{ij} + (n-i)\delta_{ij})$ is in the center of $U(\mathfrak{gl}_n)$. Then noting the operator $(\det T)^z \rho(C)(\det T)^{-z} = \det(\rho(E_{ij} + (n-i-z)\delta_{ij}))$ is also invariant, we can conclude that the element $C(z) = \det(E_{ij} + (n-i-z)\delta_{ij})$ is central for any z. As this $C(z)$ is polynomial in z of degree n, we will get n central elements from $C(z)$ as the coefficients once we develope it in z in some way. To be more specific, we expand $C(z)$ in a form

$$C(z) = \sum_{k=0}^{n} (-)^k z^{(k)} C_{n-k}$$

with $z^{(k)} = z(z-1)\cdots(z-k+1)$, and call C_k the kth Capelli element. The explicit form of these C_k in terms of minor determinants will be given in Proposition 4 below. According to the transition from the ring of differential operators on $\mathcal{P}(\mathrm{Mat}(m,n))$ to $U(\mathfrak{gl}_n)$, we introduce an $n \times n$ matrix $\boldsymbol{E} = (E_{ij})_{1 \le i,j \le n}$, so that we have $C(z) = \det(\boldsymbol{E}^{\natural} - z)$ by definition. As the preimage in $\Lambda_n \otimes U(\mathfrak{gl}_n)$ of ζ_j and $\zeta_j(u)$ defined by (1.4) and (1.5), we put

$$\omega_j = \sum_{\alpha=1}^{n} e_\alpha E_{\alpha j}; \qquad \omega_j(u) = \omega_j + ue_j = \sum_{\alpha=1}^{n} e_\alpha (E_{\alpha j} + u\delta_{\alpha j}). \tag{2.1}$$

Then we see clearly $\omega_1(n-1-z)\omega_2(n-2-z)\cdots\omega_n(-z) = e_1 e_2 \cdots e_n C(z)$.

Proposition 4. *The kth Capelli element is expressed as the sum of $k \times k$ minor determinants:*

$$C_k = \sum_{\sharp J=k} \det(\boldsymbol{E}^{\natural}_{JJ}).$$

Proof: Let us denote by Δ the difference operator defined by $\Delta\varphi(z) = \varphi(z+1) - \varphi(z)$. Note that $\Delta z^{(k)} = kz^{(k-1)}$. Then r applications of Δ to the expansion for $C(z)$ yields

$$\Delta^r C(z) = \sum_{k=0}^{n} (-)^k \binom{k}{r} r!\, z^{(k-r)} C_{n-k},$$

so that

$$\Delta^r C(z)\big|_{z=0} = (-)^r r!\, C_{n-r}.$$

Our task is thus now to compute the difference of the determinant $C(z) = \det(\boldsymbol{E}^{\natural} - z)$. For this we recall the formula $\Delta(\varphi(z)\psi(z)) = \Delta\varphi(z) \cdot \psi(z) + \varphi(z+1) \cdot \Delta\psi(z)$ for the difference of a product of functions. Note that this is valid even for the case where $\varphi(z)$ and $\psi(z)$ do not necessarily commute. We apply this formula k times in succession on the left-hand side of $\omega_1(n-1-z)\omega_2(n-2-z)\cdots\omega_n(-z) = e_1e_2\cdots e_nC(z)$. Then we get

$$\Delta^r C(z) = (-)^r r! \sum_{\sharp I=r} \det(\boldsymbol{E}^{\natural}_{I^cI^c} - z)$$

Here I^c stands for the complement of the r-set $I = \{i_1, i_2, \cdots, i_r\}$ in $N = \{1, \cdots, n\}$. The factor $r!$ for the term $\det(\boldsymbol{E}^{\natural}_{I^cI^c} - z)$ can be counted as follows. Under an operation of the difference operator Δ on a product made from $\omega_i(\bullet - z)$'s with $\bullet$ a suitable constant, it will get replacements $\omega_i(\bullet - z) \mapsto -e_i$ from the product formula above. Thus the term $\det(\boldsymbol{E}^{\natural}_{I^cI^c} - z)$ will be produced from $\det(\boldsymbol{E}^{\natural} - z)$ under the successive r operations of Δ as many times as the number of the processes that I appears in the end, each of which process can be identified with the numbering of I by $\{1, 2, \cdots, r\}$. The number $r!$ of permutations of the r-set I hence comes as the factor. Putting $z = 0$ in this formula and comparing it with the other formula above, we see our assertion with $k = n - r$. ◊

Remark: If we introduce a $U(\mathfrak{gl}_n)$-valued polynomial $C_k(w)$ by the formula

$$C(z+w) = \sum_{k=0}^{n} (-)^k z^{(k)} C_{n-k}(w),$$

then we have similarly its expression as

$$C_k(w) = \sum_{\sharp I=k} \det(\boldsymbol{E}^{\natural}_{II} - w).$$

It is also easy to see that $\Delta C_k(w) = -(n-k+1)C_{k-1}(w)$.

With Proposition 4 in hand, we can now show the kth Capelli identity.

Theorem 5. *(The kth Capelli identity) : The image of the kth Capelli element under ρ is expressed as:*

$$\rho(C_k) = \sum_{\sharp I=k,\, \sharp J=k} \det({}^tT_{JI}) \det(D_{IJ}).$$

Proof: For each k-set $J \subset N$, we have the relation (1.7) deduced from Theorem 2:

$$\det(\Pi^{\natural}_{JJ}) = \sum_{\sharp I=k} \det({}^tT_{JI}) \det(D_{IJ}).$$

Summing up over k-set J, we get the proof of our assertion, because the resulting left-hand side is just $\rho(C_k)$ by Propositon 4. $\diamondsuit$

3. Centrality of the Capelli elements: In the previous section, we stressed the fact that $C(z)$ is central, and gave an explanation for it from the representation-theoretic point of view, especially by the Capelli identity itself. It should be natural, however, that this can be proved independently of the Capelli identity, and we shall do so here.

We observe the following basic fact.

Lemma 6. *The commutation relations for $\omega_i(u)$ are given by*

$$\omega_i(u+1)\omega_j(u) + \omega_j(u+1)\omega_i(u) = 0.$$

In particular, we have

$$\omega_i(u+1)\omega_i(u) = 0.$$

Proof. This can be shown by an easy calculation:

$$\begin{aligned}
\omega_i(u)\omega_j(v) + \omega_j(v)\omega_i(u) &= \sum_{\alpha,\beta=1}^{n} e_\alpha e_\beta [E_{\alpha i} + u\delta_{\alpha i}, E_{\beta j} + v\delta_{\beta j}] \\
&= \sum_{\alpha,\beta=1}^{n} e_\alpha e_\beta (E_{\alpha j}\delta_{i\beta} - E_{\beta i}\delta_{j\alpha}) \\
&= \sum_{\alpha=1}^{n} e_\alpha e_i E_{\alpha j} - \sum_{\beta=1}^{n} e_j e_\beta E_{\beta i} \\
&= -e_i\omega_j - e_j\omega_i.
\end{aligned}$$

From this, we see that $\omega_i(u+1)\omega_j(v)+\omega_j(v+1)\omega_i(u) = e_i\omega_j(v)+e_j\omega_i(u)-e_i\omega_j-e_j\omega_i = (v-1)e_ie_j + (u-1)e_je_i$. Put here $u = v$. Then the last term vanishes and we come to the conclusion. $\diamondsuit$

Recall that GL_n acts on $\mathfrak{gl}_n$ by the adjoint action, so that it acts on the enveloping algebra $U(\mathfrak{gl}_n)$ as an automorphism group. We will first show that the element $C(z)$ is invariant under this action. This statement can be paraphrased as follows. Take any

$g = (g_{ij}) \in GL_n$, and write its inverse as $g^{-1} = h = (h_{ij})$. From $\boldsymbol{E} = (E_{ij})$, we form $g\boldsymbol{E}g^{-1} = \boldsymbol{E}^* = (E^*_{ij})$, so that

$$E^*_{ij} = \sum_{k,\ell=1}^{n} g_{ik} E_{k\ell} h_{\ell j}.$$

Under the algebra automorphism of $U(\mathfrak{gl}_n)$ extending this $\boldsymbol{E} \mapsto \boldsymbol{E}^*$, the polynomial $C(z) = \det(\boldsymbol{E}^\natural - z)$ is transformed to $C^*(z) = \det(\boldsymbol{E}^{*\natural} - z)$. The invariancy of $C(z)$ then amounts to the equality $C(z) = C^*(z)$, which is what we will prove below.

Theorem 7. *The Capelli elements are invariant under the adjoint action of* GL_n, *i.e.,* $C(z) \in U(\mathfrak{gl}_n)^{GL_n}$. *In particular,* $C(z)$ *is central in* $U(\mathfrak{gl}_n)$.

Proof: As remarked above, it suffices to prove the equality $C(z) = C^*(z)$. For this calculation, let us introduce some more suitably transformed elements corresponding to $\omega_j(u)$ and e_i:

$$\omega^*_j(u) = \sum_{\alpha=1}^{n} e_\alpha (E^*_{\alpha j} + u\delta_{\alpha j}), \quad e'_i = \sum_{\alpha=1}^{n} e_\alpha g_{\alpha i}, \quad \omega'_j(u) = \sum_{\alpha=1}^{n} e'_\alpha (E_{\alpha j} + u\delta_{\alpha j}).$$

Then we have

$$\begin{aligned}
\omega^*_j(u) &= \sum_{\alpha} e_\alpha (E^*_{\alpha j} + u\delta_{\alpha j}) = \sum_{\alpha,k,\ell} e_\alpha (g_{\alpha k} E_{k\ell} h_{\ell j} + u g_{\alpha k} \delta_{k\ell} h_{\ell j}) \\
&= \sum_{\alpha,k,\ell} e_\alpha g_{\alpha k} (E_{k\ell} + u\delta_{k\ell}) h_{\ell j} \\
&= \sum_{k,\ell} e'_k (E_{k\ell} + u\delta_{k\ell}) h_{\ell j} = \sum_{\ell} \omega'_\ell(u) h_{\ell j}
\end{aligned}$$

Note that Lemma 6 above is also valid for $\omega'_j(u)$, because e'_i is just transformed linearly from e_i with scalar coefficients. Put $u_i = n - i - z$ for brevity. Then we see from the relation above and Lemma 6

$$\begin{aligned}
e_1 e_2 \cdots e_n C^*(z) &= \omega^*_1(u_1)\omega^*_2(u_2) \cdots \omega^*_n(u_n) \\
&= \sum_{1 \le \ell_1, \ell_2, \cdots, \ell_n \le n} \omega'_{\ell_1}(u_1)\omega'_{\ell_2}(u_2) \cdots \omega'_{\ell_n}(u_n) h_{\ell_1 1} h_{\ell_2 2} \cdots h_{\ell_n n} \\
&= \sum_{\sigma \in \mathfrak{S}_n} \omega'_{\sigma(1)}(u_1)\omega'_{\sigma(2)}(u_2) \cdots \omega'_{\sigma(n)}(u_n) h_{\sigma(1)1} h_{\sigma(2)2} \cdots h_{\sigma(n)n} \\
&= \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sign}(\sigma) \omega'_1(u_1)\omega'_2(u_2) \cdots \omega'_n(u_n) h_{\sigma(1)1} h_{\sigma(2)2} \cdots h_{\sigma(n)n} \\
&= \omega'_1(u_1)\omega'_2(u_2) \cdots \omega'_n(u_n) \det(h) \\
&= e'_1 e'_2 \cdots e'_n \det(\boldsymbol{E}^\natural - z) \det(g)^{-1} \\
&= e_1 e_2 \cdots e_n \det(g) C(z) \det(g)^{-1} = e_1 e_2 \cdots e_n C(z)
\end{aligned}$$

Thus verified our assertion. ◇

In the mechanism of the proof above, we can easily recognize the (two) multiplication formulas of determinant as in the classical counterpart $\det(gag^{-1}) = \det a$. We can of course prove that the Capelli elements are central in $U(\mathfrak{gl}_n)$ within the framework of the Lie algebra, not using the group GL_n. For this we first prepare a lemma, which is more fundamental than Lemma 6. Let us write $E_{pq}(u) = E_{pq} + u\delta_{pq}$ for short.

Lemma 8. *The commutation relation for E_{pq} and $\omega_i(u)$ is given by*

$$[E_{pq}, \omega_j(u)] = e_q E_{pj}(u) - \delta_{pj}\omega_q(u).$$

Proof: This is shown also by an easy calculation:

$$\begin{aligned}[E_{pq}, \omega_j(u)] &= \sum_\alpha e_\alpha [E_{pq}, E_{\alpha j} + u\delta_{\alpha j}] = \sum_\alpha e_\alpha (E_{pj}\delta_{q\alpha} - E_{\alpha q}\delta_{pj}) \\ &= e_q E_{pj} - \delta_{pj}\omega_q = e_q E_{pj}(u) - \delta_{pj}\omega_q(u). \qquad \diamondsuit\end{aligned}$$

Remark: As an easy variant on the proof of Lemma 8, we can obtaine a more general formula as follows: for any u, v, w, we have

$$[E_{pq}(u), \omega_j(v)] = e_q E_{pj}(w) - \delta_{pj}\omega_q(w).$$

Multiplying e_p from the left on both sides of this with $u = v = w$, we will get Lemma 6 again.

The following is essentially the same as the proof in the Appendix A of [HU].

Theorem 9. *The Capelli elements are invariant under the adjoint action of $\mathfrak{gl}_n$. In other words, $C(z)$ is in the center of $U(\mathfrak{gl}_n)$.*

Proof: We will show that $[E_{pq}, C(z)] = 0$ for any $1 \le p, q \le n$. From the expression $e_1 e_2 \cdots e_n C(z) = \omega_1(u_1)\omega_2(u_2)\cdots\omega_n(u_n)$ with $u_j = n - j + z$, using Lemma 8, we have

$$\begin{aligned} &e_1 e_2 \cdots e_n [E_{pq}, C(z)] \\ &\quad = \sum_{j=1}^n \omega_1(u_1)\cdots\omega_{j-1}(u_{j-1})\underbrace{(e_q E_{pj}(u_j) - \delta_{pj}\omega_q(u_j))}_{j}\omega_{j+1}(u_{j+1})\cdots\omega_n(u_n) \\ &\quad = \sum_{j=1}^n \omega_1(u_1)\cdots\omega_{j-1}(u_{j-1})\underbrace{e_q E_{pj}(u_j)}_{j}\omega_{j+1}(u_{j+1})\cdots\omega_n(u_n) \\ &\qquad\qquad - \omega_1(u_1)\cdots\omega_{p-1}(u_{p-1})\underbrace{\omega_q(u_p)}_{p}\omega_{p+1}(u_{p+1})\cdots\omega_n(u_n). \end{aligned}$$

Here the second term can be calculated by Lemma 6 as

$$\omega_1(u_1)\cdots\omega_{p-1}(u_{p-1})\underbrace{\omega_q(u_p)}_{p}\omega_{p+1}(u_{p+1})\cdots\omega_n(u_n) = \begin{cases} 0 & (\text{for } p \neq q), \\ e_1 e_2 \cdots e_n C(z) & (\text{for } p = q). \end{cases}$$

To calculate the first term, we introduce $\omega_j^{(q)}(u) = \omega_j(u) - e_q E_{qj}(u)$ by omitting the qth component of $\omega_j(u)$. Then we have

$$
\begin{aligned}
(\omega_1^{(q)}(u_1) + e_q E_{p1}(u_1))(\omega_2^{(q)}(u_2) + e_q E_{p2}(u_2)) \cdots (\omega_n^{(q)}(u_n) + e_q E_{pn}(u_n)) \\
= \sum_{j=1}^{n} \omega_1^{(q)}(u_1) \cdots \omega_{j-1}^{(q)}(u_{j-1}) \underbrace{e_q E_{pj}(u_j)}_{j} \omega_{j+1}^{(q)}(u_{j+1}) \cdots \omega_n^{(q)}(u_n) \\
= \sum_{j=1}^{n} \omega_1(u_1) \cdots \omega_{j-1}(u_{j-1}) \underbrace{e_q E_{pj}(u_j)}_{j} \omega_{j+1}(u_{j+1}) \cdots \omega_n(u_n),
\end{aligned}
$$

because $e_q^2 = 0$ and $\omega_1^{(q)}(u_1) \cdots \omega_n^{(q)}(u_n) = 0$. Thus the first term is expressed as the product of the elements $\omega_j^{(q)}(u_j) + e_q E_{pj}(u_j)$. Note that $\omega_j^{(q)}(u_j) + e_q E_{pj}(u_j)$ is gotten by the replacement $E_{qj}(u_j) \mapsto E_{pj}(u_j)$ in $\omega_j(u_j)$. Then the product of these elements vanishes if $p \neq q$, because it gives a determinant with two identical rows. For $p = q$, since $\omega_j^{(q)}(u_j) + e_q E_{pj}(u_j) = \omega_j(u_j)$, their product is nothing but $e_1 e_2 \cdots e_n C(z)$. In any case, the first term is cancelled by the second, so that $[E_{pq}, C(z)] = 0$ as desired.◇

Remark: Comparing the proof here and that given in [HU], although they are essentially the same, we will get different impressions from them, especially of the length of the treatments of the 'first' and 'second' terms. The introduction of the elements $\omega_j^{(q)}(u) = \omega_j(u) - e_q E_{qj}(u)$ to treat 'first' term corresponds to the simple trick (A.1.3) in [HU]. Also we should compare the proofs of Theorems 7 and 9. We notice that the 'first' and the 'second' terms above correspond respectively to the determinants $\det(g)$ from the right and $\det(g)^{-1}$ from the left in the proof of Theorem 7.

Appendix

Some central elements in $U(\mathfrak{o}_n)$: Here, in analogy with Theorem 9, we will translate the proof of A.2 in [HU]. First we take a realization of the orthogonal Lie algebra $\mathfrak{o}_n$ as

$$\mathfrak{o}_n = \{X \in \mathfrak{gl}_n \,;\, X + {}^t X = 0\},$$

and accordingly consider the standard elements $A_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}_n$. Parallel to the preceeding sections, we introduce

$$
\begin{aligned}
\boldsymbol{A} &= (A_{ij})_{1 \le i,j \le n}, & A_{ij}(u) &= A_{ij} + u\delta_{ij}, \\
\psi_j &= \sum_{\alpha=1}^{n} e_\alpha A_{\alpha j}, & \psi_j(u) &= \sum_{\alpha=1}^{n} e_\alpha A_{\alpha j}(u) = \psi_j + u e_j
\end{aligned}
$$

Our objective is the determinant $\det(\boldsymbol{A}^\natural - z)$, which is to be shown central in the universal enveloping algebra $U(\mathfrak{o}_n)$. We know $e_1 e_2 \cdots e_n \det(\boldsymbol{A}^\natural - z) = \psi_1(u_1)\psi_2(u_2) \cdots \psi_n(u_n)$ with $u_j = n - j - z$. Let us compute the basic commutation relations.

Lemma A1. *We have the following:*

(1) $$[A_{ij}, A_{k\ell}] = A_{i\ell}\delta_{jk} - A_{ik}\delta_{j\ell} - A_{j\ell}\delta_{ik} + A_{jk}\delta_{i\ell},$$

(2) $$[A_{pq}(u), \psi_j(v)] = e_q A_{pj}(x) - \delta_{pj}\psi_q(x) + \delta_{qj}\psi_p(y) - e_p A_{qj}(y),$$

(3) $$\psi_i(u+1)\psi_j(u) + \psi_j(u+1)\psi_i(u) = -\delta_{ij}\Omega,$$

where $\Omega = \sum_{1\le\alpha,\beta\le n} e_\alpha e_\beta A_{\alpha\beta} = -\sum_\alpha e_\alpha \psi_\alpha = -\sum_\alpha e_\alpha\psi_\alpha(u)$. *Note that in (2), the parameters* u, v, x, y *are independent.*

Proof: The assertion (1) is immediate from the definition. For (2), we compute:

$$\begin{aligned}[A_{pq}(u), \psi_j(v)] &= \sum_\alpha e_\alpha [A_{pq}(u), A_{\alpha j}(v)] = \sum_\alpha e_\alpha [A_{pq}, A_{\alpha j}] \\ &= \sum_\alpha e_\alpha (A_{pj}\delta_{q\alpha} - A_{p\alpha}\delta_{qj} - A_{qj}\delta_{p\alpha} + A_{q\alpha}\delta_{pj}) \\ &= e_q A_{pj} - \delta_{qj}\sum_\alpha e_\alpha A_{p\alpha} - e_p A_{qj} + \delta_{pj}\sum_\alpha e_\alpha A_{q\alpha} \\ &= e_q A_{pj} + \delta_{qj}\psi_p - e_p A_{qj} - \delta_{pj}\psi_q \\ &= e_q A_{pj}(x) + \delta_{qj}\psi_p(y) - e_p A_{qj}(y) - \delta_{pj}\psi_q(x).\end{aligned}$$

For (3), multiply e_p from the left on both sides of (2) and sum up over p. Then we have

$$\psi_q(u)\psi_j(v) + \psi_j(v)\psi_q(u) = -e_q\psi_j(x) - e_j\psi_q(x) + \delta_{qj}\sum_p e_p\psi_p(y),$$

so that for $u = v = x$

$$\psi_q(u+1)\psi_j(u) + \psi_j(u+1)\psi_q(u) = -\delta_{qj}\Omega,$$

the conclusion. ◇

Theorem A. *The determinant* $\det(\boldsymbol{A}^\natural - z)$ *is central in* $U(\mathfrak{o}_n)$.

Proof: Using the relation $e_1 e_2 \cdots e_n \det(\boldsymbol{A}^\natural - z) = \psi_1(u_1)\psi_2(u_2)\cdots\psi_n(u_n)$ with $u_j = n - j - z$, we show $[A_{pq}, \det(\boldsymbol{A}^\natural - z)] = 0$ for $p < q$. By the formula (2) in Lemma A1 with $u = 0, v = x = y = u_j$, we see

$$\begin{aligned}&e_1 e_2 \cdots e_n [E_{pq}, \det(\boldsymbol{A}^\natural - z)] \\ &\quad= \sum_{j=1}^n \psi_1(u_1)\cdots\psi_{j-1}(u_{j-1})\underbrace{[A_{pq}, \psi_j(u_j)]}_{j}\psi_{j+1}(u_{j+1})\cdots\psi_n(u_n) \\ &\quad= \sum_{j=1}^n \psi_1(u_1)\cdots\psi_{j-1}(u_{j-1}) e_q \underbrace{A_{pj}(u_j)}_{j}\psi_{j+1}(u_{j+1})\cdots\psi_n(u_n) \\ &\quad- \sum_{j=1}^n \psi_1(u_1)\cdots\psi_{j-1}(u_{j-1}) e_p \underbrace{A_{qj}(u_j)}_{j}\psi_{j+1}(u_{j+1})\cdots\psi_n(u_n) \\ &\quad- \psi_1(u_1)\cdots\psi_{p-1}(u_{p-1})\underbrace{\psi_q(u_p)}_{p}\psi_{p+1}(u_{p+1})\cdots\psi_n(u_n) \\ &\quad+ \psi_1(u_1)\cdots\psi_{q-1}(u_{q-1})\underbrace{\psi_p(u_q)}_{q}\psi_{q+1}(u_{q+1})\cdots\psi_n(u_n).\end{aligned}$$

Then the first and the second terms both vanish by the same reasoning as in the proof of Theorem 9, because each of those sums gives a determinant with two identical rows. The products in the third and the fourth terms actually amount to be the same, so that they are cancelled with the opposite signatures. In fact, both of them are calculated by the formula (3) in Lemma A1 as

$$-\psi_1(u_1)\cdots\psi_{p-1}(u_{p-1})\overset{p}{\check{\cdot}}\psi_{p+1}(u_p)\cdots\psi_{q-1}(u_{q-2})\overset{q}{\check{\cdot}}\psi_{q+1}(u_{q-1})\cdots\psi_n(u_{n-2})\Omega/2,$$

because $\psi_p(u+1)\psi_p(u) = \psi_q(u+1)\psi_q(u) = -\Omega/2$. Thus proved our assertion. $\diamondsuit$

REFERENCES

[Ca1] A. Capelli, *Über die Zurückführung der Cayley'schen Operation Ω auf gewöhnliche Polar-Operationen*, Math. Ann. **29** (1887), 331–338.

[Ca2] ———, *Ricerca delle operazioni invariantive fra piu serie di variabili permutabili con ogni altra operazione invariantiva fra le stesse serie*, Atti delle Scinze Fis. e Mat. di Napoli **(2) I** (1888), 1–17.

[Ca3] ———, *Sur les opérations dans la théorie des formes algébriques*, Math. Ann. **37** (1890), 1–37.

[H] R. Howe, *Remarks on classical invariant theory*, Trans. Amer. Math. Soc. **313** (1989), 539–570*Erratum*, Trans. Amer. Math. Soc. **318** (1990), 823.

[HU] R. Howe and T. Umeda, *The Capelli identity, the double commutant theorem, and multiplicity-free actions*, Math. Ann. **290** (1991), 565–619.

[Kz] J-L. Koszul, *Les algèbre de Lie graduée de type* $\mathfrak{sl}(n,1)$ *et l'opérateur de A. Capelli*, C.R. Acad. Sc. Paris **292** (1981), 139–141.

[My] F. Meyer, *Bericht über den gegenwärtingen Stand der Invariantentheorie*, Jber. d. Dt. Math.-Verein **1** (1892), 79–292.

[MNO] A. Molev, M. Nazarov and G. Olshanskii, *Yangians and classical Lie algebras*, Russian Math. Surveys **51** (1996), 205–282.

[Na] M. Nazarov, *Quantum Berezinian and the classical Capelli identity*, Lett. Math. Phys. **21** (1991), 123–131.

[NUW] M. Noumi, T. Umeda and M. Wakayama, *A quantum analogue of the Capelli identity and an elementary differential calculas on* $GL_q(n)$, Duke Math. J. **76** (1994), 567–594.

[U1] T. Umeda, *The Capelli identities, a century after*, Sugaku **46** (1994), 206–227; (in Japanese); English transl. in "Selected Papers on Harmonic Analysis, Groups, and Invariants", AMS Translations, Series 2, vol. 183 (1998), pp. 51–78, ed. by K. Nomizu

[U2] ———, *Newton's formula for* $\mathfrak{gl}_n$, Proc. Amer. Math. Soc. **126** (1998), 3169–3175.

[Wy] H. Weyl, *The Classical Groups, their Invariants and Representations*, Princeton Univ. Press, 1946.

[Z] D.P. Želobenko, *Compact Lie Groups and their Representations*, Transl. Math. Monographs **40** (Amer. Math. Soc.), 1973.