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The Smith Problem and a Counterexample to Laitinen's Conjecture

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1. Introduction

In this article, we describe current study of the Smith problem. Let G be a finite group. P. A. Smith [24] asked

> whether the tangential G-representations U and V at two fixed points of an arbitrary smooth G-action on a sphere with exactly two fixed points are isomorphic to each other.

So it has been referred to as the *Smith problem.* We say that U and V above are *S-equivalent.* Important breakthroughs on the problem came in

- Atiyah-Bott-Milnor [1], [8]: Affirmative in semi-free actions,
- Cappell-Shaneson [2]–[4]: Negative for $G = C_{4m}$, $m \geq 2$, and
- T. Petrie [15]–[19]: Negative for $G = C_{pqr} \times C_{pqr}$ where p, q, r are distinct odd primes.

Surveys of relevant study so far are given in [22], [21], [7] and [13]. In case where G is an Oliver group, E. Laitinen [7] lighted the problem again with an conjecture which would guarantee existence of nonisomorphic S-equivalent G-representations U and V. Laitinen-Pawałowski [7] and Pawałowski-Solomon [13] showed that Laitinen's conjecture is true for most Oliver groups G:

(1) perfect groups G
(2) Oliver groups G of odd order
(3) Oliver groups G with $N \triangleleft G$ such that $G/N \cong C_{pq}$, where p and q are distinct odd primes,
(4) nonsolvable gap groups $G \ncong \mathrm{P\Sigma L}(2, 27)$.

A finite group G is called a *gap group* if there exists a (finite dimensional) real G-representation V such that

(1) $\dim V^P > 2\dim V^H$ for any subgroup $P \subset G$ of prime power order and any subgroup $H \subset G$ with $P \subsetneq H$, and

(2) $V^N = 0$ for any normal subgroup $N \subset G$ such that $|G/N|$ is a prime power.

Although various affirmative evidences have been found, it turns out that there is an counterexample [9] to Laitinen's conjecture. That is, the conjecture is invalid in the case $G = \mathrm{Aut}(A_6)$. We remark that $\mathrm{Aut}(A_6)$ is not a 'gap group'. The counterexample indicates that the notion 'gap group' may be crucial to Laitinen's conjecture and hence relevant to the Smith problem. As for gap groups, readers can refer to [6], [11], [10], and [25].

2. S-, D-equivalences and Laitinen's conjecture

Let G be a finite group. For a smooth G-manifold X and a G-fixed point x in X, the tangent space $T_x(X)$ of X at x is regarded as a real G-module, namely a real G-representation space. Thus $T_x(X)$ is referred to as the *tangential G-representation of X at x*. Let U and V be (finite dimensional) real G-modules. We say that U and V are *S-equivalent*, in symbol $U \sim_{\mathrm{S}} V$, if there exists a homotopy sphere Σ with smooth G-action such that $\Sigma^G = \{a, b\}$, $T_a(\Sigma) \cong U$ and $T_b(\Sigma) \cong V$. Such a homotopy sphere with smooth G-action is called a *2FP sphere* (two-fixed-point sphere) for U and V. It is easy to show that $\mathrm{res}^G_P U \cong \mathrm{res}^G_P V$ holds if $U \sim_{\mathrm{S}} V$ and $|P| = 2, 4$. Sanchez further showed that $\mathrm{res}^G_P U \cong \mathrm{res}^G_P V$ holds if $U \sim_{\mathrm{S}} V$ and P is a p-subgroup of G for an odd prime p.

Let $\mathcal{P}(G)$ denote the set of all subgroups of G of prime power order, where the trivial subgroup $\{e\}$ lies in $\mathcal{P}(G)$. Define $\mathcal{P}(G)^*$ by

$$\mathcal{P}(G)^* = \mathcal{P}(G) \smallsetminus \{P \leq G \mid 8 \text{ divides } |P|\}.$$

Let $\mathrm{RO}(G)$ denote the real representation ring. Define the *Smith set* $\mathrm{Sm}(G)$ to be

$$\{[U] - [V] \in \mathrm{RO}(G) \mid U \sim_{\mathrm{S}} V\}.$$

Let $\mathrm{Irr}(G)^*$ denote the set of all irreducible real G-modules of dimension ≥ 2. Then $\mathrm{Sm}(G)$ is a subset of the submodule $\mathrm{RO}(G)^*$ of $\mathrm{RO}(G)$ generated by $\mathrm{Irr}(G)^*$.

Define the homomorphism $d_G : \mathrm{RO}(G)^* \to \mathbb{Z}$ by

$$d_G([U] - [V]) = \dim U^G - \dim V^G,$$

and the *Petrie kernel* $\mathcal{P}^*\mathrm{RO}(G)^*$ to be

$$\bigcap_{P \in \mathcal{P}(G)^*} \mathrm{Ker}[\mathrm{res}^G_P : \mathrm{Ker}(d_G) \to \mathrm{RO}(P)].$$

Then we have

$$\mathrm{Sm}(G) \subset \mathcal{P}^*\mathrm{RO}(G)^*.$$

There exists a disk Δ with smooth G-action such that Δ^G consists of exactly two points if and only if G is an Oliver group. We remark that a finite group G is an Oliver group if and only if G never admits a normal series

$$P \triangleleft H \triangleleft G$$

such that $|P|$ and $|G/H|$ are prime powers and H/P is a cyclic group. We say that U and V are *D-equivalent*, in symbol $U \sim_{\mathrm{D}} V$, if there exists a disk Δ with smooth G-action such that $\Delta^G = \{a, b\}$, $T_a(\Delta) \cong U$ and $T_b(\Delta) \cong V$. Such a disk with smooth G-action is called an *2FP disk* for U and V. Define the *Oliver set* $\mathrm{Oli}(G)$ by

$$\mathrm{Oli}(G) = \{[U] - [V] \in \mathrm{RO}(G) \mid U \sim_{\mathrm{D}} V\},$$

and the kernel $\mathcal{P}\mathrm{RO}(G)^*$ by

$$\mathcal{P}\mathrm{RO}(G)^* = \bigcap_{P \in \mathcal{P}(G)} \mathrm{Ker}[\mathrm{res}^G_P : \mathrm{Ker}(d_G) \to \mathrm{RO}(P)].$$

B. Oliver [14] proved the equality

$$\mathrm{Oli}(G) = \mathcal{P}\mathrm{RO}(G)^*.$$

Thus the Oliver set is an additive subgroup of $\mathrm{RO}(G)^*$.

For an element $g \in G$, let (g) denote the conjugacy class of g in the group G and let $(g)^{\pm}$ denote the real conjugacy class of g in G, namely $(g)^{\pm} = (g) \cup (g^{-1})$. Let a_G

denote the number of real conjugacy classes $(g)^{\pm}$, $g \in G$, such that the order of g is not a power of a prime. Laitinen-Pawalowski [7] showed that if $a_G \neq 0$ then

$$a_G - 1 = \text{rank } \mathcal{P}\mathrm{RO}(G)^*$$

and hence

Theorem 2.1. *If* $\mathrm{Oli}(G) \neq 0$ *then* $a_G \geq 2$.

The next conjecture has been referred to as Laitinen's conjecture.

Conjecture. Let G be an Oliver group. If $a_G \geq 2$ then $\mathrm{Sm}(G) \cap \mathrm{Oli}(G) \neq 0$.

3. New observations on 2FP spheres

Recall a classical lemma.

Lemma 3.1. *Let M be a connected, closed smooth C_2-manifold of dimension ≥ 1. If the C_2-fixed point set F of M is nonempty then $|F| \geq 2$.*

A simple proof of the lemma is given in [9].

Let G be a finite group. The lemma above implies the next proposition.

Proposition 3.2. *Let Σ be a 2FP sphere for U and V. If K is a subgroup of G with index 2 then* $\dim U^K = \dim V^K$.

For each $[W] \in \mathrm{Irr}(G)^*$, we have an associated homomorphism $f_W : \mathrm{RO}(G)^* \to \mathbb{Z}$; if $x \in \mathrm{RO}(G)^*$ then $f_W(x)$ is the multiplicity of $[W]$ in x. Thus, for $x \in \mathrm{RO}(G)^*$, we have

$$x = \sum_{[W]\in \mathrm{Irr}(G)^*} f_{[W]}(x)\ [W].$$

Define the subset $\mathrm{RO}(G)^*_0$ of $\mathrm{RO}(G)^*$ by

$$\mathrm{RO}(G)^*_0 = \bigcap_{[W]\in \mathrm{Irr}(G)^*,\ P\in \mathcal{P}(G)} ((\dim W^P) f_{[W]} \mathrm{res}^G_P)^{-1}(0),$$

where $(\dim W^P) f_{[W]} \mathrm{res}^G_P$ is a map $\mathrm{RO}(G)^* \to \mathrm{RO}(P)$;

$$(\dim W^P) f_{[W]} \mathrm{res}^G_P(x) = (\dim W^P) f_{[W]}(x) \mathrm{res}^G_P(x)$$

for $x \in \mathrm{RO}(G)^*$. Further define the subset $\mathrm{SRO}(G)$ of $\mathrm{RO}(G)^*$ by

$$\mathrm{SRO}(G) = \mathcal{P}^*\mathrm{RO}(G)^* \cap \mathrm{RO}(G)_0^*.$$

With this notation, we have the following results.

Theorem 3.3. *The implication* $\mathrm{Sm}(G) \subseteq \mathrm{SRO}(G)$ *holds for any finite group* G.

Theorem 3.4. *If* $G = \mathrm{Aut}(A_6)$ *then* $\mathrm{SRO}(G) = 0$ *and hence* $\mathrm{Sm}(G) = 0$.

Note that if $G = \mathrm{Aut}(A_6)$ then $a_G = 2$ and $\mathrm{Sm}(G) = 0$, which disagrees with Laitinen's conjecture.

4. New Conjectures

At present, we have the following conjectures.

Conjecture 4.1. Let G be an Oliver gap group. If $a_G \geq 2$ then $\mathrm{Sm}(G) \cap \mathrm{Oli}(G) \neq 0$.

Conjecture 4.2. Let G be an Oliver group. If $\mathrm{SRO}(G) \neq 0$ then $\mathrm{Sm}(G) \neq 0$.

The author wishes that readers are interested in the conjectures above.

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