数理解析研究所講究録
1535 巻 2007 年 128-134

Upper and lower bounds of numerical radius and an equality condition

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ABSTRACT

In this report, we give an inequality among operator norm and numerical radii of T and its Aluthge transform. It is a more precise estimation of the numerical radius than Kittaneh's result. Then we obtain an equivalent condition that the numerical radius is equal to the half of operator norm.

This is based on the following paper:
[Y] T. Yamazaki, *On upper and lower bounds of the numerical radius and an equality condition*, Studia Math., **178** (2007), 83–89.

1. Introduction

For a bounded linear operator T on a complex Hilbert space $\mathcal{H}$, we denote the operator norm and the numerical radius of T by $\|T\|$ and $w(T)$, respectively. It is well known that $w(T)$ is an equivalent norm of T as follows [5, Theorem 1.3-1]:

$$\frac{1}{2}\|T\| \leq w(T) \leq \|T\|. \tag{1.1}$$

On the second inequality, Kittaneh [8] has shown the following precise estimation of $w(T)$ by using several norm inequalities and ingenious techniques:

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}. \tag{1.2}$$

Obviously, (1.2) is sharper than the right inequality of (1.1). We remark that we can not compare $w(T)$ with $\|T^2\|^{\frac{1}{2}}$, generally. In fact, let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $0 = \|T^2\|^{\frac{1}{2}} < w(T) = \frac{1}{2}$. But let $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\frac{1}{\sqrt{2}} = w(T) < \|T^2\|^{\frac{1}{2}} = 1$.

We obtain a sufficient condition of $w(T) = \frac{1}{2}\|T\|$ by (1.1), (1.2) and [8] that is, if $T^2 = 0$, then $w(T) = \frac{1}{2}\|T\|$. But it is not to be a necessary condition. In fact, let $T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $w(T) = \frac{1}{2}\|T\| = 1$, but $T^2 \neq 0$. We remark that some conditions of $w(T) = \frac{1}{2}\|T\|$ are known in [5, Theorems 1.3-4 and 1.3-5], but any equivalent condition has not been known yet.

Let $T = U|T|$ be the polar decomposition of T. The Aluthge transform $\widetilde{T}$ of T is defined by $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ in [1]. It is well known the following properties of $\widetilde{T}$: (i) $\|\widetilde{T}\| \leq \|T\|$, (ii) $w(\widetilde{T}) \leq w(T)$ and (iii) $r(\widetilde{T}) = r(T)$. The first and last properties are easy by the definition of $\widetilde{T}$, and the second one is shown in [7], [9] and [11]. Moreover for a non-negative integer n, we denote n-th Aluthge transform by $\widetilde{T_n}$, i.e.,

$$\widetilde{T_n} = \widetilde{\left(\widetilde{T_{n-1}}\right)} \quad \text{and} \quad \widetilde{T_0} = T.$$

This was first considered by [7] and [10], independently.

In this paper, firstly, we shall obtain more precise estimation than (1.2). In the inequality, we use a bigger term $\|T\|$ and a smaller one $w(\widetilde{T})$ than $w(T)$. Moreover the proof is very simple and needs only generalized polarization identity. Next, we shall give an equivalent condition that $w(T) = \frac{1}{2}\|T\|$ holds.

2. Sharper inequality than Kittaneh's result

In this section, we shall show a sharper estimation of $w(T)$ than Kittaneh's one [8] as follows:

Theorem 2.1. *For any $T \in B(\mathcal{H})$, $w(T) \leq \dfrac{1}{2}\|T\| + \dfrac{1}{2}w(\widetilde{T})$.*

We remark that by the Heinz inequality [6] $\|A^rXB^r\| \leq \|AXB\|^r\|X\|^{1-r}$ for $A, B \geq 0$ and $r \in [0,1]$, we have

$$w(\widetilde{T}) \leq \|\widetilde{T}\| = \||T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}\| \leq \||T|U|T|\|^{\frac{1}{2}}\|U\|^{\frac{1}{2}} = \|T^2\|^{\frac{1}{2}}, \tag{2.1}$$

i.e., Theorem 2.1 is sharper than (1.2).

To prove Theorem 2.1, we use the following famous formula which is called the generalized polarization identity:

Theorem A (Generalized Polarization Identity). *For each $T \in B(\mathcal{H})$ and $x, y \in \mathcal{H}$,*

$$\begin{aligned} \langle Tx, y\rangle = &\frac{1}{4}\big(\langle T(x+y), x+y\rangle - \langle T(x-y), x-y\rangle\big) \\ &+ \frac{i}{4}\big(\langle T(x+iy), x+iy\rangle - \langle T(x-iy), x-iy\rangle\big). \end{aligned} \tag{2.2}$$

Proof of Theorem 2.1. First of all, we note that

$$w(T) = \sup_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)\| \tag{2.3}$$

holds, since

$$\sup_{\theta\in\mathbb{R}} \mathrm{Re}\{e^{i\theta}\langle Tx, x\rangle\} = |\langle Tx, x\rangle|$$

and

$$\sup_{\theta\in\mathbb{R}} \|\mathrm{Re}(e^{i\theta}T)\| = \sup_{\theta\in\mathbb{R}} w(\mathrm{Re}(e^{i\theta}T)) = w(T).$$

Let $T = U|T|$ be the polar decomposition. Then by (2.2), we have

$$\begin{aligned}\langle e^{i\theta}Tx, x\rangle &= \langle e^{i\theta}|T|x, U^*x\rangle \\ &= \frac{1}{4}\big(\langle |T|(e^{i\theta}+U^*)x, (e^{i\theta}+U^*)x\rangle - \langle |T|(e^{i\theta}-U^*)x, (e^{i\theta}-U^*)x\rangle\big) \\ &\quad + \frac{i}{4}\big(\langle |T|(e^{i\theta}+iU^*)x, (e^{i\theta}+iU^*)x\rangle - \langle |T|(e^{i\theta}-iU^*)x, (e^{i\theta}-iU^*)x\rangle\big).\end{aligned}$$

Noting that all inner products of the terminal side are all positive since $|T|$ is positive. Hence we have

$$\begin{aligned}\mathrm{Re}\langle e^{i\theta}Tx, x\rangle &= \frac{1}{4}\big(\langle (e^{-i\theta}+U)|T|(e^{i\theta}+U^*)x, x\rangle - \langle (e^{-i\theta}-U)|T|(e^{i\theta}-U^*)x, x\rangle\big) \\ &\leq \frac{1}{4}\langle (e^{-i\theta}+U)|T|(e^{i\theta}+U^*)x, x\rangle \\ &\leq \frac{1}{4}\|(e^{-i\theta}+U)|T|(e^{i\theta}+U^*)\| \\ &= \frac{1}{4}\||T|^{\frac{1}{2}}(e^{i\theta}+U^*)(e^{-i\theta}+U)|T|^{\frac{1}{2}}\| \quad (\text{by } \|X^*X\| = \|XX^*\|) \\ &= \frac{1}{4}\|2|T| + e^{i\theta}\widetilde{T} + e^{-i\theta}(\widetilde{T})^*\| \\ &= \frac{1}{2}\||T| + \mathrm{Re}(e^{i\theta}\widetilde{T})\| \\ &\leq \frac{1}{2}\|T\| + \frac{1}{2}\|\mathrm{Re}(e^{i\theta}\widetilde{T})\| \\ &\leq \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}) \quad (\text{by } (2.3)).\end{aligned}$$

Hence we have the desired inequality. □

Corollary 2.2. *If* $\widetilde{T} = 0$, *then* $w(T) = \frac{1}{2}\|T\|$.

Proof. The proof is easy by Theorem 2.1 and (1.1). □

Remark. (i) In Corollary 2.2, the conditions $\widetilde{T} = 0$ and $w(T) = \frac{1}{2}\|T\|$ are not equivalent. In fact, let $T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $w(T) = \frac{1}{2}\|T\| = 1$. But $\widetilde{T} = 1 \oplus 0 \neq 0$.

(ii) Conditions $\widetilde{T} = 0$ and $T^2 = 0$ are equivalent as follows: Let $T = U|T|$ be the polar decomposition of T. If $\widetilde{T} = 0$, then

$$T^2 = U|T|U|T| = U|T|^{\frac{1}{2}}\widetilde{T}|T|^{\frac{1}{2}} = 0.$$

Conversely, if $T^2 = 0$, then by (2.1) we have $\|\widetilde{T}\| \leq \|T^2\|^{\frac{1}{2}} = 0$.

Corollary 2.3. *For* $T \in B(\mathcal{H})$, $w(T) \leq \displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\|\widetilde{T_{n-1}}\|$.

Proof. By using Theorem 2.1 several times, we have

$$\begin{aligned} w(T) &\le \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}) \\ &\le \frac{1}{2}\|T\| + \frac{1}{2}\left\{\frac{1}{2}\|\widetilde{T}\| + \frac{1}{2}w(\widetilde{T_2})\right\} \\ &= \frac{1}{2}\|T\| + \frac{1}{4}\|\widetilde{T}\| + \frac{1}{4}w(\widetilde{T_2}) \\ &\le \frac{1}{2}\|T\| + \frac{1}{4}\|\widetilde{T}\| + \frac{1}{8}\|\widetilde{T_2}\| + \frac{1}{8}w(\widetilde{T_3}) \\ &\vdots \\ &\le \sum_{n=1}^{\infty} \frac{1}{2^n}\|\widetilde{T_{n-1}}\|. \end{aligned}$$

□

Let $s(T) = \sum_{n=1}^{\infty} \frac{1}{2^n}\|\widetilde{T_{n-1}}\|$. By (2.1), $\|\widetilde{A}\| \le \|A^2\|^{\frac{1}{2}} \le \|A\|$ hold for any $A \in B(\mathcal{H})$, and we obtain

$$r(T) \le w(T) \le \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}) \le s(T) \le \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}} \le \|T\|, \tag{2.4}$$

where $r(T)$ means the spectral radius of T.

It is well known that T is normaloid (i.e., $\|T\| = r(T)$) if and only if $\|T\| = w(T)$. Here we give more weaker conditions of normalodity of T than $\|T\| = w(T)$ as follows:

Corollary 2.4. *The following conditions are mutually equivalent:*

(i) *T is normaloid,*
(ii) $\|T\| = s(T)$,
(iii) $r(T) = \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T})$,
(iv) $s(T) = s(\widetilde{T})$.

Remark.

(i) In Corollary 2.4, the condition (ii) can not be replaced into more weaker condition $\|T\| = \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}$. For example, let $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then $\|T\| = \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}} = 1$ but $0 = r(T) < \|T\|$.

(ii) In Corollary 2.4, the condition (iii) can not be replaced into more weaker condition $r(T) = w(T)$, either. In fact let $T = 1 \oplus \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Then $1 = r(T) = w(T) < \|T\| = 2$. (We call the operator satisfying $r(T) = w(T)$ spectraloid.)

To prove Corollary 2.4, the following formula will be used.

Theorem B ([10]). *For any $T \in B(\mathcal{H})$, $\lim_{n\to\infty} \|\widetilde{T_n}\| = r(T)$.*

Proof. (i) $\Longrightarrow$ (ii), (iii) and (iv) are obvious by (2.4) and $r(T) = r(\widetilde{T}) \le s(\widetilde{T}) \le s(T) \le \|T\|$.

Proof of (ii) $\Longrightarrow$ (i). By the definition of $s(T)$,

$$s(T) = \frac{1}{2}\|T\| + \frac{1}{2}s(\widetilde{T}) \tag{2.5}$$

holds. Then by the assumption (ii), we have

$$s(T) = \frac{1}{2}\|T\| + \frac{1}{2}s(\widetilde{T}) = \|T\|$$

and $s(\widetilde{T}) = \|T\|$.

On the other hand, since the inequality $\|\widetilde{T}\| \le \|T\|$ always holds, then we have

$$s(\widetilde{T}) \le \|\widetilde{T}\| \le \|T\| = s(\widetilde{T}),$$

and we have $s(\widetilde{T}) = \|\widetilde{T}\| = \|T\|$. By using the same technique, we have $\|T\| = \|\widetilde{T_n}\|$ for all $n \in \mathbb{N}$. Hence by Theorem B, we have

$$\|T\| = \lim_{n\to\infty} \|\widetilde{T_n}\| = r(T),$$

that is, T is normaloid.

Proof of (iii) $\Longrightarrow$ (i). By (iii) and $r(\widetilde{T}) = r(T)$, we have

$$r(T) = \frac{1}{2}\|T\| + \frac{1}{2}w(\widetilde{T}) \ge \frac{1}{2}\|T\| + \frac{1}{2}r(\widetilde{T}) = \frac{1}{2}\|T\| + \frac{1}{2}r(T),$$

that is, $r(T) \ge \|T\|$ then $r(T) = \|T\|$.

Proof of (iv) $\Longrightarrow$ (ii). By (2.5) and the assumption (iv), i.e., $s(T) = s(\widetilde{T})$, we have (ii). □

In [2], Ando shows that $W(T) = W(\widetilde{T})$ is equivalent to $co\sigma(T) = W(T)$ (i.e., T is convexoid) for any matrix T, where $co\sigma(T)$ means the convex hull of the spectrum of T. The author thinks that this is a parallel result to the equivalence between (i) and (iv). So the author expects that $s(T)$ has some interesting properties.

3. Equivalent condition of $w(T) = \frac{1}{2}\|T\|$

In Corollary 2.2, we have obtained a sufficient condition that $w(T) = \frac{1}{2}\|T\|$ holds. Some conditions of $w(T) = \frac{1}{2}\|T\|$ are known in [5, Theorems 1.3-4 and 1.3-5]. But it has not been known any equivalent condition of $w(T) = \frac{1}{2}\|T\|$. In this section, we give an equivalent condition of $w(T) = \frac{1}{2}\|T\|$ holds as follows:

Theorem 3.1. *Let $T \in B(\mathcal{H})$. The following conditions are equivalent:*

(i) $w(T) = \frac{1}{2}\|T\|$,

(ii) $\|T\| = \|Re(e^{i\theta}T)\| + \|Im(e^{i\theta}T)\|$ *for all* $\theta \in \mathbb{R}$.

We remark that the condition (ii) should not be replaced into "$\|T\| = \|\mathrm{Re}(e^{i\theta}T)\| + \|\mathrm{Im}(e^{i\theta}T)\|$ for *some* $\theta \in \mathbb{R}$." Because if T is a non-zero self-adjoint operator, then $\|T\| = \|\mathrm{Re}T\| + \|\mathrm{Im}T\| = \|\mathrm{Re}T\|$, but $w(T) = \|T\| > \frac{1}{2}\|T\|$.

To prove Theorem 3.1, we need the following theorem:

Theorem C ([3]). *Let $A, B \in B(\mathcal{H})$ be non-zero. Then the equation $\|A+B\| = \|A\| + \|B\|$ holds if and only if $\|A\|\|B\| \in \overline{W(A^*B)}$.*

Proof of Theorem 3.1. Let $e^{i\theta}T = H_\theta + iK_\theta$ be the Cartesian decomposition of $e^{i\theta}T$. We remark that

$$K_\theta = H_{\theta-\frac{\pi}{2}}, \tag{3.1}$$

because $e^{i(\theta-\frac{\pi}{2})}T = -ie^{i\theta}T = K_\theta - iH_\theta$ holds.
Proof of (i) $\Longrightarrow$ (ii). Since $w(T) = \sup_{\theta\in\mathbb{R}} \|H_\theta\| = \sup_{\theta\in\mathbb{R}} \|K_\theta\|$ by (2.3) and (3.1), we have

$$\|T\| = \|e^{i\theta}T\| = \|H_\theta + iK_\theta\| \leq \|H_\theta\| + \|K_\theta\| \leq w(T) + w(T) = \|T\|.$$

Hence we have (ii).

Proof of (ii) $\Longrightarrow$ (i). For any $\theta \in \mathbb{R}$, (ii) ensures $\|H_\theta\|\|K_\theta\| \in \overline{W(H_\theta^*(iK_\theta))}$ by Theorem C, i.e., $-i\|H_\theta\|\|K_\theta\| \in \overline{W(H_\theta K_\theta)}$. Since $-i\|H_\theta\|\|K_\theta\|$ is a purely imaginary number and $\mathrm{Im}(H_\theta K_\theta) = \mathrm{Im}(H_0 K_0)$ holds for all $\theta \in \mathbb{R}$, we have

$$\|H_\theta\|\|K_\theta\| = w(H_\theta K_\theta) = \|\mathrm{Im}(H_\theta K_\theta)\| = \|\mathrm{Im}(H_0 K_0)\|.$$

Then for all $\theta \in \mathbb{R}$, we have the following conditions

$$\begin{cases} \|H_\theta\| + \|K_\theta\| = \|T\| \\ \|H_\theta\|\|K_\theta\| = \|\mathrm{Im}(H_0 K_0)\|, \end{cases}$$

that is,

$$\|H_\theta\| = \frac{\|T\| + \sqrt{\|T\|^2 - 4\|\mathrm{Im}(H_0K_0)\|}}{2} \quad \text{or} \quad \frac{\|T\| - \sqrt{\|T\|^2 - 4\|\mathrm{Im}(H_0K_0)\|}}{2},$$

and $\|K_\theta\|$ is another of the above. We remark that these values do not depend on $\theta \in \mathbb{R}$. So the function $\|H_\theta\|$ on $\theta \in \mathbb{R}$ takes only two values by (3.1). Here by the easy calculation, we have

$$H_\theta = H_0 \cos\theta - K_0 \sin\theta.$$

Hence by the continuity of operator norm, the function $\|H_\theta\|$ is continuous on $\theta \in \mathbb{R}$. Therefore the function $\|H_\theta\|$ must take only one value by intermediate value theorem, i.e.,

$$\|H_\theta\| = \|K_\theta\| = \frac{1}{2}\|T\|.$$

Hence we have (i). □

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