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Julia sets of quartic polynomials and a topology of the symbol space

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Abstract

For a certain quartic polynomial, there exists a homeomorphism between the set of all components of the filled-in Julia set with the Hausdorff metric and some subset of the corresponding symbol space with the ordinary metric known well. But these sets are not compact with respect to each metric. We introduce new topologies with respect to which these sets are compact.

1 Introduction and the main results

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere and let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function of degree $d \geq 2$. In the theory of the complex dynamics, there are two important sets called the Fatou set $F(f)$ and the Julia set $J(f)$. The Fatou set $F(f)$ is the set of normality in the sense of Montel for the family $\{f^n\}_{n=0}^{\infty}$, where $f^n = f \circ \cdots \circ f$ is n iterates of f. The Julia set $J(f)$ is the complement $\hat{\mathbb{C}} \setminus F(f)$. $J(f)$ is either connected or else has uncountably many connected components. In the case that f is a polynomial, we define the filled-in Julia set $K(f)$ as

$$K(f) = \{z \in \mathbb{C} : \{f^n(z)\}_{n=0}^{\infty} \text{ is bounded}\}.$$

$J(f)$ is the topological boundary of $K(f)$. We call $A(f) = \hat{\mathbb{C}} \setminus K(f)$ the attracting basin of the point at infinity.

We often consider another model in order to simplify dynamics of f. The model is the symbol space and the shift map defines a dynamical system on the symbol space. Let X^{ω} be the countable product of a set X.

Definition 1.1. The *symbol space* of q-symbols is the countable product $\Sigma_q = \{1, 2, \dots, q\}^\omega$. For $s = (s_n)$ and $t = (t_n) \in \Sigma_q$, a metric ρ on Σ_q is defined as

$$\rho(s,t) = \sum_{n=0}^{\infty} \frac{\delta(s_n, t_n)}{2^n}, \quad \text{where} \quad \delta(k,l) = \begin{cases} 1 & \text{if } k \neq l, \\ 0 & \text{if } k = l. \end{cases}$$

Then (Σ_q, ρ) is a compact metric space. The *shift map* $\sigma : \Sigma_q \to \Sigma_q$ is defined as

$$\sigma((s_0, s_1, s_2, \dots)) = (s_1, s_2, \dots).$$

The shift map σ is continuous with respect to the metric ρ.

The connectivity of the Julia set of a polynomial of degree two or more is affected by the behavior of finite critical points.

Theorem 1.2. *Let f be a polynomial of degree $d \geq 2$. If all finite critical points of f are in $A(f)$, then $J(f)$ is totally disconnected. Furthermore $f|_{J(f)}$ is topologically conjugate to the shift map $\sigma : \Sigma_d \to \Sigma_d$. On the other hand, $J(f)$ is connected if and only if all finite critical points of f are in $K(f)$.*

If some critical orbits of a polynomial converge to the point at infinity but all critical orbits do not converge to it, then the Julia set is disconnected and not generally totally disconnected. It is a problem whether dynamics of a polynomial on the Julia set can be simplified as dynamics of the shift map on some symbol space when the Julia set is disconnected and not totally disconnected. But it is difficult to make points of non-trivial connected components of the Julia set correspond to points of some symbol space. Therefore we consider the set of all components of the Julia set and make it correspond to some symbol space. On account of the following arguments, we consider not the set of all components of the Julia set but the set of all components of the filled-in Julia set. But nothing essentially changes.

Definition 1.3. Let f be a polynomial of degree $d \geq 2$. The *Green's function* associated with $K(f)$ is defined as

$$G(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ x = \max\{\log x, 0\}$. $G(z)$ is zero for $z \in K(f)$ and $G(z)$ is positive for $z \in \mathbb{C} \setminus K(f)$. Note that G satisfies the identity $G(f(z)) = d \cdot G(z)$.

Definition 1.4. The triple (f, U, V) is a *polynomial-like map* of degree d if U and V are topological disks with $\overline{U} \subset V$ and $f : U \to V$ is a holomorphic proper map of degree d. We define the *filled-in Julia set* $K(f)$ of a polynomial-like map (f, U, V) as

$$K(f) = \{z \in U : \{f^n(z)\}_{n=0}^{\infty} \subset U\}.$$

Definition 1.5. Let X be a metric space. For a compact subset $A \subset X$ and $\delta > 0$, let $A[\delta]$ be a δ-neighborhood of A. For compact subsets A and $B \subset X$, we define the *Hausdorff metric* d_H as

$$d_H(A, B) = \inf\{\delta : A \subset B[\delta] \text{ and } B \subset A[\delta]\}.$$

Let f be a quartic polynomial and let c_1, c_2 and c_3 be finite critical points of f. Suppose that c_1 and c_2 are in $K(f)$ and c_3 is in $A(f)$. Let U be a bounded component of $\mathbb{C} \setminus G^{-1}(G(f(c_3)))$ and let U_A and U_B be bounded components of $\mathbb{C} \setminus G^{-1}(G(c_3))$. In other words, $U = \{z \in \mathbb{C} : G(z) < G(f(c_3))\}$ and $U_A \cup U_B = \{z \in \mathbb{C} : G(z) < G(c_3)\}$, where G is the Green's function associated with $K(f)$. Suppose that c_1 is in U_A and c_2 is in U_B. Then $(f|_{U_A}, U_A, U)$ and $(f|_{U_B}, U_B, U)$ are polynomial-like maps of degree 2. Suppose that filled-in Julia sets $K_A = K(f|_{U_A})$ and $K_B = K(f|_{U_B})$ are connected.

Let $K(f)^*$ be the set of all components of $K(f)$. Since c_3 is in $A(f)$, $K(f)^*$ is uncountable. $K(f)^*$ becomes a metric space with the Hausdorff metric d_H. We define a map $F : (K(f)^*, d_H) \to (K(f)^*, d_H)$ as $F(K) = f(K)$ for $K \in K(f)^*$. Then F is continuous.

Let $\Sigma_6 = \{1, 2, 3, 4, A, B\}^\omega$ be the symbol space. We define a subset Σ of Σ_6 as follows: $s = (s_n) \in \Sigma$ if and only if

(S1) if $s_n = A$, then $s_{n+1} = A$,

(S2) if $s_n = B$, then $s_{n+1} = B$,

(S3) if $s_n = A$ and $s_{n-1} \neq A$, then $s_{n-1} = 3$ or 4,

(S4) if $s_n = B$ and $s_{n-1} \neq B$, then $s_{n-1} = 1$ or 2,

(S5) if $s \in \Sigma_4 = \{1, 2, 3, 4\}^\omega$, then there exist subsequences $(s_{n(k)})_{k=1}^{\infty}$ and $(s'_{n(l)})_{l=1}^{\infty}$ such that $s_{n(k)} = 1$ or 2 for all $k \geq 1$ and $s'_{n(l)} = 3$ or 4 for all $l \geq 1$.

The author proved the following theorem in [Ka].

Theorem 1.6. *Suppose that a quartic polynomial f, its finite critical points c_1, c_2, c_3 and domains U, U_A, U_B are as above. And suppose that filled-in Julia sets K_A and K_B are connected. Then there exists a homeomorphism $\Lambda : (K(f)^*, d_H) \to (\Sigma, \rho)$ such that $\Lambda \circ F = \sigma \circ \Lambda$.*

Theorem 1.6 means that componentwise dynamics of f on $K(f)$ (of course, also on $J(f)$) be simplified as dynamics of the shift map on Σ. But (Σ, ρ) is not compact. For example, a sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, B, B, \dots) \right\}_{n=0}^{\infty}$$

in Σ converges to $s = (1, 1, 1, \dots)$ but s is not in Σ. Can we define a topology of Σ with respect to which the sequence $\{s^{(n)}\}_{n=0}^{\infty}$ converges to a "appropriate" point in Σ, and furthermore Σ is compact? The meaning of "appropriate" is explained in another section. In this paper we answer the question:

Theorem 1.7. *Let Σ be as above. Then there exists a topology $\mathcal{O}$ of Σ such that $(\Sigma, \mathcal{O})$ is compact, metrizable, perfect and totally disconnected. Moreover the shift map $\sigma : (\Sigma, \mathcal{O}) \to (\Sigma, \mathcal{O})$ is continuous.*

By Theorem 1.6, there exists a homeomorphism $\Lambda : (K(f)^*, d_H) \to (\Sigma, \rho)$ such that $\Lambda \circ F = \sigma \circ \Lambda$. Especially $\Lambda^{-1} : \Sigma \to K(f)^*$ is bijective. Let $\mathcal{G}$ be the quotient topology of $K(f)^*$ relative to Λ^{-1} and the topology $\mathcal{O}$ of Σ in Theorem 1.7, that is,

$$\mathcal{G} = \{G \subset K(f)^* : \Lambda(G) \in \mathcal{O}\}.$$

Then $\Lambda : (K(f)^*, \mathcal{G}) \to (\Sigma, \mathcal{O})$ is a homeomorphism such that $\Lambda \circ F = \sigma \circ \Lambda$.

Corollary 1.8. *$(K(f)^*, \mathcal{G})$ is compact, metrizable, perfect and totally disconnected. Moreover $F : (K(f)^*, \mathcal{G}) \to (K(f)^*, \mathcal{G})$ is continuous.*

2 Definition of a new topology of Σ

We define a topology of Σ. If $s = (A, A, A, \dots) \in \Sigma$, we define subsets $N_s^{(k)}$ of Σ as

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 1 \text{ or } 2 \text{ for } n \leq k\}.$$

Similarly, if $s = (B, B, B, \dots) \in \Sigma$,

$$N_s^{(k)} = \{s\} \cup \{t = (t_n) \in \Sigma : t_n = 3 \text{ or } 4 \text{ for } n \leq k\}.$$

If $s = (s_0, \dots, s_l, A, A, A, \dots) \in \Sigma$ with $s_l \neq A$,

$$N_s^{(k)} = \{s\} \cup \left\{ t = (t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \leq l, \\ 1 \text{ or } 2 & \text{if } l+1 \leq n \end{cases} \text{ for } n \leq k \right\}.$$

Similarly, if $s = (s_0, \dots, s_l, B, B, B, \dots) \in \Sigma$ with $s_l \neq B$,

$$N_s^{(k)} = \{s\} \cup \left\{ t = (t_n) \in \Sigma : t_n = \begin{cases} s_n & \text{if } n \leq l, \\ 3 \text{ or } 4 & \text{if } l+1 \leq n \end{cases} \text{ for } n \leq k \right\}.$$

Finally, if $s = (s_n) \in \Sigma \cap \Sigma_4$,

$$N_s^{(k)} = \{t = (t_n) \in \Sigma : t_n = s_n \text{ for } n \leq k\}.$$

Note that $N_s^{(k+1)} \subset N_s^{(k)}$ for all $s \in \Sigma$ and $k \geq 0$. Let $\mathcal{N}(s) = \{N_s^{(k)}\}_{k=0}^{\infty}$ and $\mathcal{N} = \{\mathcal{N}(s) : s \in \Sigma\}$. Then $\mathcal{N}$ is a neighborhood system of Σ and hence $(\Sigma, \mathcal{O})$ is a topological space, where

$$\mathcal{O} = \{O \subset \Sigma : \text{if } s \in O, \text{ then there exists } N \in \mathcal{N}(s) \text{ such that } N \subset O\}.$$

The topology $\mathcal{O}$ satisfies Theorem 1.7.

3 Appropriateness of the convergence with respect to $\mathcal{O}$

We can formulate $\Lambda : K(f)^* \to \Sigma$ concretely. Refer to [Ka] for the detailed proof. Let U, U_A, U_B, K_A and K_B be the same as the section 1. There exist forward invariant rays R_{A1} and R_{B2} under f such that R_{A1} lands at a point on ∂K_A and R_{B1} lands at a point on ∂K_B. These landing points are repelling or parabolic fixed points of f. Let R_{A2} and R_{B2} be components of $f^{-1}(R_{A1})$ and $f^{-1}(R_{B1})$ which satisfy $R_{A2} \cap U_A \neq \emptyset$ and $R_{B2} \cap U_B \neq \emptyset$ and differ from R_{A1} and R_{B1} respectively. We set $V_A = U \setminus (K_A \cup R_{A1})$ and $V_B = U \setminus (K_B \cup R_{B1})$. Let I_1, I_2, I_3 and I_4 be branches of f^{-1} such that

$$I_1 : V_A \to U_1, \;\; I_2 : V_A \to U_2,$$
$$I_3 : V_B \to U_3, \;\; I_4 : V_B \to U_4,$$

where U_1 and U_2 are components of $U_A \setminus K_A \cup R_{A1} \cup R_{A2}$ respectively. Similarly, U_3 and U_4 are components of $U_B \setminus K_B \cup R_{B1} \cup R_{B2}$ respectively. We define $\Lambda : K(f)^* \to \Sigma$ as follows: for $K \in K(f)^*$,

$$[\Lambda(K)]_n = \begin{cases} i & \text{if } f^n(K) \subset U_i, \\ A & \text{if } f^n(K) = K_A, \\ B & \text{if } f^n(K) = K_B, \end{cases}$$

where $n \geq 0$ and $i = 1, 2, 3, 4$. We can also formulate $\Lambda^{-1} : \Sigma \to K(f)^*$ as follows: if $s_n = A$ and $s_{n-1} \neq A$,

$$\Lambda^{-1}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_A).$$

If $s_n = B$ and $s_{n-1} \neq B$,

$$\Lambda^{-1}(s) = I_{s_0} \circ \cdots \circ I_{s_{n-1}}(K_B).$$

If $s \in \Sigma_4$, there exists a subsequence $(s_{n(l)})_{l=1}^{\infty}$ such that $s_{n(l)} = 1$ or 2 and $s_{n(l)-1} = 3$ or 4. We set $K_s^{(l)} = I_{s_0} \circ \cdots \circ I_{s_{n(l)-1}}(\overline{U_A})$. Then $K_s^{(l+1)} \subset K_s^{(l)}$ and

$$\Lambda^{-1}(s) = \bigcap_{l=1}^{\infty} K_s^{(l)}.$$

Note that $\bigcap_{l=1}^{\infty} K_s^{(l)}$ is a one-point set since each I_k decreases the Poincaré distance on V_A or V_B.

We reconsider the sequence

$$\left\{ s^{(n)} = (\underbrace{1, 1, \dots, 1}_{n \text{ times}}, B, B, \dots) \right\}_{n=0}^{\infty}$$

in Σ. It converges to $s = (1, 1, 1, \dots) \notin \Sigma$ with respect to ρ. However, it converges to $s = (A, A, A, \dots) \in \Sigma$ with respect to $\mathcal{O}$. We check that the convergence with respect to $\mathcal{O}$ is "appropriate". By definition of Λ^{-1},

$$\Lambda^{-1}(s^{(n)}) = \underbrace{I_1 \circ \cdots \circ I_1}_{n \text{ times}}(K_B).$$

Let $K^{(n)} = \Lambda^{-1}(s^{(n)})$. Since I_1 decreases the Poincaré distance on V_A, the sequence $\{K^{(n)}\}_{n=0}^{\infty} \subset K(f)^*$ converges to not $K_A \in K(f)^*$ but a one-point set $K = \{\zeta\}$ with respect to the Hausdorff metric d_H. The point ζ is actually in ∂K_A, and therefore $K \notin K(f)^*$. We expect that $\{K^{(n)}\}_{n=0}^{\infty}$ converges to K_A with respect to $\mathcal{G}$. In fact,

$$\lim_{n\to\infty} K^{(n)} = \lim_{n\to\infty} \Lambda^{-1}(s^{(n)}) = \Lambda^{-1}\left(\lim_{n\to\infty} s^{(n)}\right) = \Lambda^{-1}(s) = K_A$$

since $\Lambda^{-1} : (\Sigma, \mathcal{O}) \to (K(f)^*, \mathcal{G})$ is continuous. Therefore we express that the convergence of $\{s^{(n)}\}_{n=0}^{\infty}$ with respect to $\mathcal{O}$ is "appropriate" in the sense that $\{K^{(n)}\}_{n=0}^{\infty}$ converges to K_A with respect to $\mathcal{G}$.

4 Applications

For a rational function of degree at least two, the backward orbit of a point in the Julia set and the set of all repelling periodic points are dense in the Julia set:

Theorem 4.1. *Let g be a rational function of degree at least two. If $z \in J(g)$, then*

$$J(g) = \overline{\bigcup_{k=1}^{\infty} g^{-k}(z)}.$$

Theorem 4.2. *Let g be a rational function of degree at least two. Then*

$$J(g) = \overline{\{\text{repelling periodic point of } g\}}.$$

We obtain analogies of Theorem 4.1 and 4.2.

Theorem 4.3. *Let $(\Sigma, \mathcal{O})$ be as in Theorem 1.7 and let $s \in \Sigma$. Then*

$$\Sigma = \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}(s)},$$

where the closure is taken in $(\Sigma, \mathcal{O})$.

Remark 4.4. The closure of the backward orbit of $s \in \Sigma$ under σ does not necessarily coincide with Σ in (Σ, ρ). For example,

$$(A, A, A, \dots) \notin \overline{\bigcup_{k=1}^{\infty} \sigma^{-k}((B, B, B, \dots))},$$

where the closure is taken in (Σ, ρ).

Corollary 4.5. *Let $(K(f)^*, \mathcal{G})$ be as in Corollary 1.8 and let $K \in K(f)^*$. Then*

$$K(f)^* = \overline{\bigcup_{k=1}^{\infty} F^{-k}(K)},$$

where the closure is taken in $(K(f)^*, \mathcal{G})$.

Theorem 4.6. *Let $(\Sigma, \mathcal{O})$ be as in Theorem 1.7. Then*

$$\Sigma = \overline{\{\text{periodic point of } \sigma \text{ in } \Sigma\}},$$

where the closure is taken in $(\Sigma, \mathcal{O})$.

Remark 4.7. The closure of the set of all periodic points of Σ does not coincide with Σ in (Σ, ρ) since $t = (t_0, t_1, \ldots, t_l, A, A, A, \ldots)$ with $t_l \neq A$ is an isolated point in (Σ, ρ).

Corollary 4.8. *Let* $(K(f)^*, \mathcal{G})$ *be as in Corollary* 1.8. *Then*

$$K(f)^* = \overline{\{\text{periodic point of } F \text{ in } K(f)^*\}},$$

where the closure is taken in $(K(f)^*, \mathcal{G})$.

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