数理解析研究所講究録
1538巻 2007年 11-19



# Note on a class of convex functions

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ABSTRACT. In this paper we define a general class of convex functions, denoted by $SL_{\beta}^{c}(q)$, with respect to a convex domain $D$ ($q(z)\in\mathcal{H}_u(U)$, $q(0)=1$, $q(U)=D$) contained in the right half plane by using the linear operator $D_\lambda^\beta$ defined by

$$D_\lambda^\beta : A \to A,$$

$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1+(j-1)\lambda)^\beta a_j z^j ,$$

where $\beta,\lambda\in\mathbb{R}$, $\beta\geq 0$, $\lambda\geq 0$ and $f(z)=z+\sum\limits_{j=2}^{\infty}a_jz^j$. This operator generalize the Sălăgean operator and the Al-Oboudi operator. Regarding the class $SL_{\beta}^{c}(q)$ we give a inclusion theorem, a preserving theorem (we use the Libera-Pascu integral operator) and many particular results.

2000 Mathematical Subject Classification: 30C45

Key words and phrases: Convex functions, Libera-Pascu integral operator, Briot-Bouquet differential subordination, generalized Sălăgean operator

## 1 Introduction

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$, $A=\{f\in\mathcal{H}(U):f(0)=f'(0)-1=0\}$, $\mathcal{H}_u(U)=\{f\in\mathcal{H}(U): f \text{ is univalent in } U\}$ and $S=\{f\in A: f \text{ is univalent in } U\}$.

Let $D^n$ be the Sălăgean differential operator (see [13]) defined as:

$$D^n : A\to A , \quad n\in\mathbb{N} \text{ and } D^0f(z)=f(z)$$

$$D^1f(z)=Df(z)=zf'(z) , \quad D^nf(z)=D(D^{n-1}f(z)).$$

**Remark 1.1** *If $f\in S, f(z)=z+\sum\limits_{j=2}^{\infty}a_jz^j$, $z\in U$ then $D^nf(z)=z+\sum\limits_{j=2}^{\infty}j^na_jz^j$.*

Let $n\in\mathbb{N}$ and $\lambda\geq 0$. Let denote with $D_\lambda^n$ the Al-Oboudi operator (see [7]) defined by

$$D_\lambda^n : A\to A,$$

$$D_\lambda^0 f(z) = f(z)\ ,\ D_\lambda^1 f(z) = (1-\lambda) f(z) + \lambda z f'(z) = D_\lambda f(z)\,,$$
$$D_\lambda^n f(z) = D_\lambda \left(D_\lambda^{n-1} f(z)\right)\ .$$

We observe that $D_\lambda^n$ is a linear operator and for $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$ we have

$$D_\lambda^n f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^n a_j z^j\ .$$

The aim of this paper is to define a general class of convex functions with respect to a convex domain $D$, contained in the right half plane, by using a operator which generalize the Sălăgean operator and the Al-Oboudi operator and to obtain some properties of this class.

## 2 Preliminary results

We recall here the definition of the well - known class of convex functions

$$S^c = CV = K = \left\{ f \in H(U);\ f(0) = f'(0) - 1 = 0, Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,\ z \in U \right\}\ .$$

**Remark 2.1** *By using the subordination relation, we may define the class $S^c$ thus if $f(z) = z + a_2 z^2 + \ldots$, $z \in U$, then $f \in S^c$ if and only if $\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec \frac{1+z}{1-z}$, $z \in U$, where by "$\prec$" we denote the subordination relation.*

Let consider the Libera-Pascu integral operator $L_a : A \to A$ defined as:

(1) $$f(z) = L_a F(z) = \frac{1+a}{z^a} \int_0^z F(t) \cdot t^{a-1} dt\ ,\quad a \in \mathbb{C}\ ,\quad Re\ a \geq 0.$$

In the case $a = 1$ this operator was introduced by R.J.Libera and it was studied by many authors in different general cases. In this general form ($a \in \mathbb{C}$ , $Re\ a \geq 0$) was used first time by N.N. Pascu in [12].

**Definition 2.1** *[6] Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda \geq 0$ and $f(z) = z + \sum\limits_{j=2}^{\infty} a_j z^j$. We denote by $D_\lambda^\beta$ the linear operator defined by*

$$D_\lambda^\beta : A \to A\,,$$
$$D_\lambda^\beta f(z) = z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^\beta a_j z^j\ .$$

**Remark 2.2** *It is easy to observe that for $\beta = n \in \mathbb{N}$ we obtain the Al-Oboudi operator and for $\beta = n \in \mathbb{N}$, $\lambda = 1$ we obtain the Sălăgean operator.*

The next theorem is result of the so called "admissible functions method" introduced by P.T. Mocanu and S.S. Miller (see [9], [10], [11]).

**Theorem 2.1** *Let $h$ convex in $U$ and $Re[\beta h(z) + \gamma] > 0$, $z \in U$. If $p \in H(U)$ with $p(0) = h(0)$ and $p$ satisfied the Briot-Bouquet differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z), \quad \text{then } p(z) \prec h(z).$$

# 3 Main results

**Definition 3.1** *Let $q(z) \in \mathcal{H}_u(U)$, with $q(0) = 1$ and $q(U) = D$, where $D$ is a convex domain contained in the right half plane, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 0$. We say that a function $f(z) \in A$ is in the class $SL^c_\beta(q)$ if $\dfrac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} \prec q(z)$, $z \in U$.*

**Remark 3.1** *Geometric interpretation: $f(z) \in SL^c_\beta(q)$ if and only if $\dfrac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)}$ take all values in the convex domain $D$ contained in the right half-plane.*

**Remark 3.2** *It is easy to observe that if we choose different function $q(z)$ we obtain variously classes of convex functions, such as (for example), for $\beta = n \in \mathbb{N}$ the class $SL^c_n(q)$ (see [3]), for $\lambda = 1$ and $\beta = 0$, the class of convex functions, the class of convex functions of order $\gamma$ (see [8]), the class of convex functions with respect to a hyperbola (see [5]) , and, for $\beta = n \in \mathbb{N}$ and $\lambda = 1$, the class of $n$-convex functions (see [2]), the class of $n$-convex functions with respect to a hyperbola (see [1]), the class of $n$-convex functions with respect to a convex domain contained in the right half-plane(see [2]), for $\beta \in \mathbb{R}$ and $\lambda = 1$, the class $S^c_\beta(q)$ of the $\beta$-$q$-convex functions (see [4]).*

**Remark 3.3** *For $q_1(z) \prec q_2(z)$ we have $SL^c_\beta(q_1) \subset SL^c_\beta(q_2)$. From the above we obtain $SL^c_\beta(q) \subset SL^c_\beta\left(\dfrac{1+z}{1-z}\right)$.*

**Theorem 3.1** *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. We have*

$$SL^c_{\beta+1}(q) \subset SL^c_\beta(q).$$

**Proof.** Let $f(z) \in SL^c_{\beta+1}(q)$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$.

With notation
$$p(z) = \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)}, \, p(0) = 1,$$
we obtain

$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+2} f(z)} = \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} \cdot \frac{D_\lambda^{\beta+1} f(z)}{D_\lambda^{\beta+2} f(z)} = \frac{1}{p(z)} \cdot \frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} \tag{2}$$

We have
$$\frac{D_\lambda^{\beta+3} f(z)}{D_\lambda^{\beta+1} f(z)} = \frac{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+3} a_j z^j}{z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j}$$

and
$$zp'(z) = \frac{z\left(D_\lambda^{\beta+2} f(z)\right)'}{D_\lambda^{\beta+1} f(z)} - \frac{D_\lambda^{\beta+2} f(z)}{D_\lambda^{\beta+1} f(z)} \cdot \frac{z\left(D_\lambda^{\beta+1} f(z)\right)'}{D_\lambda^{\beta+1} f(z)}$$
$$= \frac{z\left(D_\lambda^{\beta+2} f(z)\right)'}{D_\lambda^{\beta+1} f(z)} - p(z) \cdot \frac{z\left(D_\lambda^{\beta+1} f(z)\right)'}{D_\lambda^{\beta+1} f(z)}$$
$$= \frac{z\left(z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} a_j z^j\right)'}{D_\lambda^{\beta+1} f(z)} - p(z) \cdot \frac{z\left(z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j\right)'}{D_\lambda^{\beta+1} f(z)}$$
$$= \frac{z\left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+2} j a_j z^{j-1}\right)}{D_\lambda^{\beta+1} f(z)} - p(z) \cdot \frac{z\left(1 + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} j a_j z^{j-1}\right)}{D_\lambda^{\beta+1} f(z)}$$
or

$$zp'(z) = \frac{z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+2} a_j z^j}{D_\lambda^{\beta+1} f(z)} - p(z) \cdot \frac{z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+1} a_j z^j}{D_\lambda^{\beta+1} f(z)}. \tag{3}$$

Also, we have
$$z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+1} a_j z^j = z + \sum_{j=2}^{\infty} ((j-1)+1)\,(1 + (j-1)\lambda)^{\beta+1} a_j z^j$$
$$= z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} a_j z^j + \sum_{j=2}^{\infty} (j-1)\,(1 + (j-1)\lambda)^{\beta+1} a_j z^j$$

$$= D_\lambda^{\beta+1}f(z) + \sum_{j=2}^{\infty}(j-1)\,(1+(j-1)\lambda)^{\beta+1}\,a_jz^j$$

$$= D_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty}((j-1)\lambda)\,(1+(j-1)\lambda)^{\beta+1}\,a_jz^j$$

$$= D_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}\sum_{j=2}^{\infty}(1+(j-1)\lambda-1)\,(1+(j-1)\lambda)^{\beta+1}\,a_jz^j$$

$$= D_\lambda^{\beta+1}f(z) - \frac{1}{\lambda}\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+1}\,a_jz^j + \frac{1}{\lambda}\sum_{j=2}^{\infty}(1+(j-1)\lambda)^{\beta+2}\,a_jz^j$$

$$= D_\lambda^{\beta+1}f(z) - \frac{1}{\lambda}\left(D_\lambda^{\beta+1}f(z) - z\right) + \frac{1}{\lambda}\left(D_\lambda^{\beta+2}f(z) - z\right)$$

$$= D_\lambda^{\beta+1}f(z) - \frac{1}{\lambda}D_\lambda^{\beta+1}f(z) + \frac{z}{\lambda} + \frac{1}{\lambda}D_\lambda^{\beta+2}f(z) - \frac{z}{\lambda}$$

$$= \frac{\lambda-1}{\lambda}D_\lambda^{\beta+1}f(z) + \frac{1}{\lambda}D_\lambda^{\beta+2}f(z)$$

$$= \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)\right).$$

Similarly we have

$$z + \sum_{j=2}^{\infty} j\,(1+(j-1)\lambda)^{\beta+2}\,a_jz^j = \frac{1}{\lambda}\left((\lambda-1)D_\lambda^{\beta+2}f(z) + D_\lambda^{\beta+3}f(z)\right).$$

From (3) we obtain

$$zp'(z) = \frac{1}{\lambda}\left(\frac{(\lambda-1)D_\lambda^{\beta+2}f(z) + D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} - p(z)\frac{(\lambda-1)D_\lambda^{\beta+1}f(z) + D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}\right)$$

$$= \frac{1}{\lambda}\left((\lambda-1)p(z) + \frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} - p(z)\left((\lambda-1)+p(z)\right)\right)$$

$$= \frac{1}{\lambda}\left(\frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} - p(z)^2\right)$$

Thus

$$\lambda zp'(z) = \frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} - p(z)^2$$

or

$$\frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} = p(z)^2 + \lambda zp'(z).$$

From (2) we obtain

$$\frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+2}f(z)} = \frac{1}{p(z)}\left(p(z)^2 + \lambda zp'(z)\right) = p(z) + \lambda\frac{zp'(z)}{p(z)},$$

where $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$, $\lambda > 0$.

From $f(z) \in SL^c_{\beta+1}(q)$ we have

$$p(z) + \lambda \frac{zp'(z)}{p(z)} \prec q(z),$$

with $p(0) = q(0) = 1$, $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda > 0$. In this conditions from Theorem 2.1, we obtain

$$p(z) \prec q(z)$$

or

$$\frac{D^{\beta+2}_\lambda f(z)}{D^{\beta+1}_\lambda f(z)} \prec q(z).$$

This means $f(z) \in SL^c_\beta(q)$.

**Corollary 3.1** *For every $\beta \in \mathbb{N}^*$ we have $SL^c_\beta(q) \subset SL^c_0(q) \subset S^c$.*

**Theorem 3.2** *Let $\beta, \lambda \in \mathbb{R}$, $\beta \geq 0$ and $\lambda \geq 1$. If $F(z) \in SL^c_\beta(q)$ then $f(z) = L_a F(z) \in SL^c_\beta(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).*

**Proof.** From (1) we have

$$(1+a)F(z) = af(z) + zf'(z)$$

and, by using the linear operator $D^{\beta+1}_\lambda$, we obtain

$$(1+a)D^{\beta+1}_\lambda F(z) = aD^{\beta+1}_\lambda f(z) + D^{\beta+1}_\lambda \left(z + \sum_{j=2}^{\infty} ja_j z^j\right)$$

$$= aD^{\beta+1}_\lambda f(z) + z + \sum_{j=2}^{\infty} (1 + (j-1)\lambda)^{\beta+1} ja_j z^j$$

We have (see the proof of the above theorem)

$$z + \sum_{j=2}^{\infty} j\,(1 + (j-1)\lambda)^{\beta+1} a_j z^j = \frac{1}{\lambda}\left((\lambda - 1)D^{\beta+1}_\lambda f(z) + D^{\beta+2}_\lambda f(z)\right)$$

Thus

$$(1+a)D^{\beta+1}_\lambda F(z) = aD^{\beta+1}_\lambda f(z) + \frac{1}{\lambda}\left((\lambda-1)D^{\beta+1}_\lambda f(z) + D^{\beta+2}_\lambda f(z)\right)$$
$$= \left(a + \frac{\lambda - 1}{\lambda}\right) D^{\beta+1}_\lambda f(z) + \frac{1}{\lambda} D^{\beta+2}_\lambda f(z)$$

or

$$\lambda(1+a)D^{\beta+1}_\lambda F(z) = ((a+1)\lambda - 1)\, D^{\beta+1}_\lambda f(z) + D^{\beta+2}_\lambda f(z).$$

Similarly, we obtain

$$\lambda(1+a)D_\lambda^{\beta+2}F(z) = ((a+1)\lambda-1)\, D_\lambda^{\beta+2}f(z) + D_\lambda^{\beta+3}f(z)\,.$$

Then

$$\frac{D_\lambda^{\beta+2}F(z)}{D_\lambda^{\beta+1}F(z)} = \frac{\dfrac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} + ((a+1)\lambda-1)\cdot\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)}}{\dfrac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} + ((a+1)\lambda-1)}\,.$$

With notation

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} = p(z)\,,\ p(0)=1\,,$$

we obtain

(4) $$\frac{D_\lambda^{\beta+2}F(z)}{D_\lambda^{\beta+1}F(z)} = \frac{\dfrac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} + ((a+1)\lambda-1)\cdot p(z)}{p(z)+((a+1)\lambda-1)}$$

We have (see the proof of the above theorem)

$$\frac{D_\lambda^{\beta+3}f(z)}{D_\lambda^{\beta+1}f(z)} = p(z)^2 + \lambda z p'(z)\,.$$

From (4), we obtain

$$\frac{D_\lambda^{\beta+2}F(z)}{D_\lambda^{\beta+1}F(z)} = \frac{p(z)^2+\lambda z p'(z) + ((a+1)\lambda-1)\,p(z)}{p(z)+((a+1)\lambda-1)} = p(z) + \lambda\frac{zp'(z)}{p(z)+((a+1)\lambda-1)}\,,$$

where $a\in\mathbb{C}$, $Re\,a\ge 0$, $\beta,\lambda\in\mathbb{R}$, $\beta\ge 0$ and $\lambda\ge 1$. From $F(z)\in SL_\beta^c(q)$ we have

$$p(z) + \frac{zp'(z)}{\frac{1}{\lambda}\left(p(z)+((a+1)\lambda-1)\right)} \prec q(z)\,,$$

where $a\in\mathbb{C}$, $Re\,a\ge 0$, $\beta,\lambda\in\mathbb{R}$, $\beta\ge 0$, $\lambda\ge 1$, and from her construction, we have $Re\,q(z)>0$. In this conditions we have from Theorem 2.1 we obtain

$$p(z)\prec q(z)$$

or

$$\frac{D_\lambda^{\beta+2}f(z)}{D_\lambda^{\beta+1}f(z)} \prec q(z)\,.$$

This means $f(z) = L_aF(z)\in SL_\beta^c(q)$.

For $\beta=n\in\mathbb{N}$ and $\lambda=1$ we obtain

**Corollary 3.2** *If $F(z) \in CV_n(q)$ then $f(z) = L_aF(z) \in CV_n(q)$, where $L_a$ is the Libera-Pascu integral operator and by $CV_n(q)$ we denote the class of $n$-convex functions subordinate to the function $q(z)$ (see [2]).*

For $\beta = n \in \mathbb{N}$ we obtain

**Corollary 3.3** *[3] Let $n \in \mathbb{N}$ and $\lambda \geq 1$. If $F(z) \in SL_n^c(q)$ then $f(z) = L_aF(z) \in SL_n^c(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).*

For $\beta \in \mathbb{R}$ and $\lambda = 1$ we obtain

**Corollary 3.4** *[4] If $F(z) \in S_\beta^c(q)$ then $f(z) = L_aF(z) \in S_\beta^c(q)$, where $L_a$ is the Libera-Pascu integral operator defined by (1).*

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