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A Lotka-Volterra Cross-Diffusion Model in Spatially Heterogeneous Environments

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1 SKT Model in Heterogeneous Environments

This article is concerned with the Lotka-Volterra reaction-diffusion system :

$$(\mathrm{P})\begin{cases} u_t = \Delta[(1+k\rho(x)v)u] + u(a-u-c(x)v) & \text{in } \Omega\times(0,\infty),\\ v_t = \Delta v + v(b+d(x)u-v) & \text{in } \Omega\times(0,\infty),\\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega\times(0,\infty),\\ u(\cdot,0)=u_0\ge 0,\ \ v(\cdot,0)=v_0\ge 0 & \text{in } \Omega. \end{cases}$$

Here, Ω is a bounded domain in $\mathbb{R}^N (N \le 3)$, a and k are positive constants, b is a real constant, $c(x)$ and $d(x)$ are smooth nonnegative functions, and $\rho(x)$ is a smooth positive function with $\partial_\nu\rho = 0$ on $\partial\Omega$. From the ecological viewpoint, the unknown functions u and v, respectively, represent the population densities of the prey and predator, which interact and migrate in Ω. In the reaction terms, a and b denote the birth rates of the prey and predator, respectively. While a and b are spatially homogeneous, the prey-predator interactions $c(x)$ and $d(x)$ are assumed to be spatially dependent nonnegative functions. In the diffusion terms, the linear part represents the natural dispersive force of the movement of an individual. On the other hand, the nonlinear diffusion

$$\Delta[\rho(x)vu] = \nabla[u\nabla(\rho(x)v) + \rho(x)v\nabla u] \tag{1.1}$$

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yields the most characteristic term in (P). In the homogeneous case when ρ is a positive constant, this term models the tendency such that the prey escapes to a low density region of the predator. In this case, $\rho\Delta(vu)$ is usually referred as the *cross-diffusion* term whose nonlinear effect has attracted attention in the field of reaction diffusion systems (e.g., [1], [6]–[17], [21]) after the research by Shigesada-Kawasaki-Teramoto [20]. (In honor of their pioneering research, such a class of Lotka-Volterra cross-diffusion systems is often called *SKT* model.) In the heterogeneous case when $\rho(x)$ is spatially dependent on x, (1.1) reveals that $\rho(x)$ represents a type of the environment potential. Further, $\Delta(\rho(x)vu)$ describes the spatially and density dependent diffusion such that the prey moves to the low value region of $\rho(x)v$. We refer to [18] for a further ecological background.

Our aim is to derive the spatially heterogeneous effects of $c(x)$, $d(x)$ and $\rho(x)$ on the stationary solution set of (P). Then we study the following strongly coupled elliptic system:

$$\text{(SP)}\quad \begin{cases} \Delta[(1+k\rho(x)v)u] + u(a-u-c(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(b+d(x)u-v) = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

We are interested in positive solutions of (SP). Here it is said that (u, v) is a positive solution if both of u and v are positive in Ω. Ecologically, a positive solution corresponds to a *coexistence* steady-state of the prey and the predator. We study the positive solution set of (SP) by considering b to be the bifurcation parameter. In order to obtain the bifurcation branch of the positive solution set, we define the *semitrivial* solution sets with the bifurcation parameter b by

$$\Gamma_u := \{(u,v,b) = (a,0,b) \,:\, b \in \mathbb{R}\},\quad \Gamma_v := \{(u,v,b) = (0,b,b) \,:\, b \in \mathbb{R}\}.$$

In [10], we prove that the positive solution branch which connects Γ_u with Γ_v. More precisely, for any fixed $(a, k, \rho(x), c(x), d(x))$, we find a negative number $b = b_* < 0$ and a positive number $b = b^* > 0$ (both depend on $(a, k, \rho(x), c(x), d(x))$) such that the positive solution set contains a bounded continuum Γ_p which bifurcates from $(a, 0, b_*) \in \Gamma_u$ and joins $(0, b^*, b^*) \in \Gamma_v$. Hence, (SP) has at least one positive solution if $b_* < b < b^*$. In the spatially homogeneous case when c, d and ρ are constants, it is easily verified that Γ_p forms a bounded line of the positive constant solutions (see Figure 1). This line is expressed as

$$\Gamma_p = \left\{\left(\frac{a-bc}{1+cd}, \frac{b+ad}{1+cd}, b\right) \,:\, -da < b < \frac{a}{c}\right\}.$$

Now, the change in the shape of Γ_p according to the spatial heterogeneity of $(\rho(x), c(x), d(x))$ needs to be studied. Among other things, we prove that a spatial heterogeneity can cause Γ_p to form a $\subset$-shaped branch with respect to b when the birth rate a is small and the cross-diffusion k is large. As a rough sketch of the main result (Theorem 4.1), we can give the following theorem. Throughout this article, we denote the average of $f(x)$ over Ω by $⨍_\Omega f(x)\,\mathrm{d}x\,(:= |\Omega|^{-1}\int_\Omega f(x)\,\mathrm{d}x)$.

Theorem 1.1. *If $a > 0$ is sufficiently small and k is sufficiently large, the positive solution set of* (SP) *(with the bifurcation parameter b) forms the bounded smooth curve*

$$\Gamma_p = \{(u(x;s), v(x;s), b(s)) \;:\; 0 < s < C\}.$$

Here $(u(x;s), v(x;s), b(s))$ satisfies

$$(u(x;0), v(x;0), b(0)) = (a, 0, b_*) \quad \textit{and} \quad (u(x;C), v(x;C), b(C)) = (0, b^*, b^*)$$

for a negative number b_* and a positive number b^*. Furthermore, in the case when

$$⨍_\Omega \rho(x)\,\mathrm{d}x \, ⨍_\Omega d(x)\,\mathrm{d}x > ⨍_\Omega \rho(x)d(x)\,\mathrm{d}x, \tag{1.2}$$

the following (i) *and* (ii) *hold true for a small number $a^* = a^*(k, \rho(x), c(x), d(x))$.*

(i) *If $0 < a \le a^*/3$, then $b'(0) > 0$, that is, Γ_p supercritically bifurcates from $(a, 0, b_*)$.*

(ii) *If $2a^*/3 \le a \le a^*$, then $b'(0) < 0$, that is, Γ_p subcritically bifurcates from $(a, 0, b_*)$. In this case, for $\underline{b} := \min_{0\le s\le C} b(s)$,* (SP) *possesses at least two positive solutions if $b \in (\underline{b}, b_*)$, at one positive solution if $b \in \{\underline{b}\} \cup [b_*, b^*)$, no positive solution if $b \in (-\infty, \underline{b}) \cup [b^*, \infty)$.*

Obviously, both sides of (1.2) are equal if either d or ρ is constant. If both $d(x)$ and $\rho(x)$ are spatially heterogeneous, (1.2) may hold. For any fixed positive number $\varepsilon < ⨍_\Omega \rho(x)\mathrm{d}x$, if $\operatorname{supp}(\rho - \varepsilon)_+$ $((\rho - \varepsilon)_+ := \max\{\rho - \varepsilon, 0\})$ and $\operatorname{supp} d$ are disjoint, (1.2) holds true. This is because

$$⨍_\Omega \rho(x)\,\mathrm{d}x \, ⨍_\Omega d(x)\,\mathrm{d}x > \varepsilon ⨍_\Omega d(x)\,\mathrm{d}x \ge ⨍_\Omega \rho(x)d(x)\,\mathrm{d}x.$$

For such a case, Theorem 1.1 asserts that if a belongs to a certain range, Γ_p subcritially bifurcates from $(a, 0, b_*)$. Then, (SP) has at least two positive solutions even if b is slightly lesser than $b_*(< 0)$. It is observed that $\operatorname{supp}(\rho - \varepsilon)_+$ provides a favorable domain for the prey in which the cross-diffusion (escape) effect from the predator is comparatively strong, whereas $\operatorname{supp} d$ gives a favorable domain for the predator in which the increase of individuals due to preying is expected. From the ecological viewpoint, our result implies that the spatial segregation of $\operatorname{supp}(\rho - \varepsilon)_+$ and $\operatorname{supp} d$ can produce the coexistence steady-states even if the death rate of the predator is comparatively high.

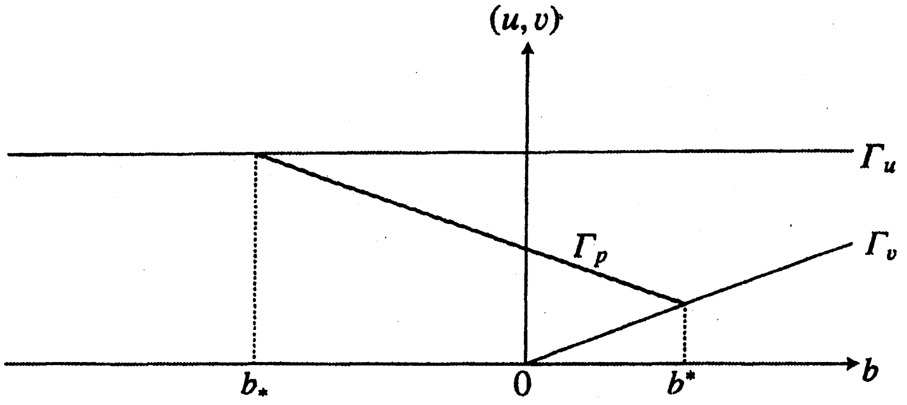

Figure 1 Positive solution branch in spatially homogeneous case.

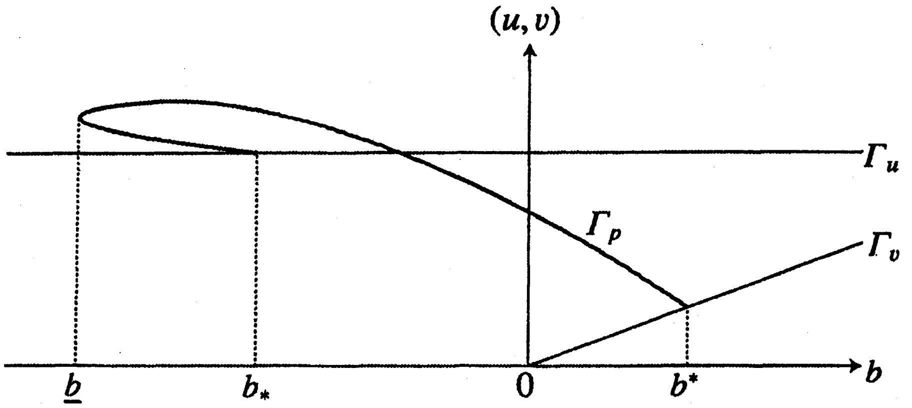

Figure 2 Positive solution branch in case (ii) of Theorem 1.1.

In this article, the usual norms of the spaces $L^p(\Omega)$ for $p \in [1, \infty)$ and $C(\overline{\Omega})$ are, respectively, defined as

$$\|u\|_p := \left(\int_\Omega |u(x)|^p dx \right)^{1/p} \quad \text{and} \quad \|u\|_\infty := \max_{x \in \overline{\Omega}} |u(x)|.$$

As the functional framework for our analysis, we introduce the following Banach spaces:

$$X := W_\nu^{2,p}(\Omega) \times W_\nu^{2,p}(\Omega), \quad Y := L^p(\Omega) \times L^p(\Omega) \quad (p > N), \tag{1.3}$$

where $W^{2,p}_\nu(\Omega) := \{u \in W^{2,p}(\Omega)\ :\ \partial_\nu u|_{\partial\Omega} = 0\}$. Here we note that $X \subset C^1(\overline{\Omega}) \times C^1(\overline{\Omega})$ for $p > N$ by the Sobolev embedding theorem.

2 Finite Dimensional Limiting System

2.1 Bounded Bifurcation Continuum

As a preliminary result, we obtain the following bounded bifurcation continuum of positive solutions of (SP) with the bifurcation parameter b:

Theorem 2.1. *For any fixed $(a, k, \rho(x), c(x), d(x))$, there exist $b_* = b_*(a, d) < 0$ and $b^* = b^*(a, k\rho, c) > 0$ such that the positive solution set of* (SP) (*with the bifurcation parameter b*) *forms a bounded continuum Γ_p ($\subset X\times\mathbb{R}$), which bifurcates from $(u, v, b) = (a, 0, b_*) \in \Gamma_u$ and joins $(u, v, b) = (0, b^*, b^*) \in \Gamma_v$.*

Owing to Theorem 2.1, we find at least one positive solution when $b_* < b < b^*$. Theorem 2.1 can be proved by the combination of the local and global bifurcation theorems ([2], [19]), the a priori estimates and the nonexistence region of the positive solutions. We refer to [10] for the proof.

2.2 Lyapunov-Schmidt Reduction

We now study the spatial heterogeneous effect of $\rho(x), c(x)$ and $d(x)$ on the positive solution branch Γ_p introduced in Theorem 2.1. More concretely, we derive a heterogeneous effect that enables Γ_p to form a $\subset$-shaped branch, when the cross-diffusion k is large and the birth (or death) rates a and $|b|$ are small. For the derivation, we employ the following scalings in (EP):

$$U = \varepsilon w,\ v = \varepsilon z,\ a = \varepsilon\alpha,\ b = \varepsilon\beta,\ k = \frac{1}{\varepsilon}. \tag{2.1}$$

Here $\varepsilon > 0$ is a small perturbation parameter. Furthermore, α is a positive number and β is a real number. Hereafter, β will function as the bifurcation parameter. From (2.1), it is observed that the new unknown function (w, z) satisfies the next perturbed elliptic system:

$$\begin{cases} \Delta w + \varepsilon F(w, z) = 0 & \text{in } \Omega, \\ \Delta z + \varepsilon G(w, z, \beta) = 0 & \text{in } \Omega, \\ \partial_\nu w = \partial_\nu z = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where

$$\begin{cases} F(w,z) := \dfrac{w}{1+\rho(x)z}\left(\alpha - \dfrac{w}{1+\rho(x)z} - c(x)z\right), \\ G(w,z,\beta) := z\left(\beta + \dfrac{d(x)w}{1+\rho(x)z} - z\right). \end{cases} \tag{2.3}$$

Hence, (2.2) has the semitrivial solutions

$$(w,z) = (\alpha, 0) \quad \text{and} \quad (w,z) = (0,\beta),$$

in addition to the trivial solution $w = z = 0$.

We solve (EP) by the Lyapunov-Schmidt reduction. For the Banach spaces X and Y in (1.3), we introduce the linear operator $H : X \to Y$ and the nonlinear operator $B : X \times \mathbb{R} \to Y$ by

$$H(w,z) := (\Delta w, \Delta z), \quad B(w,z,\beta) := (F(w,z), G(w,z,\beta)). \tag{2.4}$$

Hence, (2.2) is equivalent to the equation

$$H(w,z) + \varepsilon B(w,z,\beta) = \mathbf{0}. \tag{2.5}$$

By considering $\operatorname{Ker} H = \mathbb{R}^2$, X and Y can be decomposed as

$$X = \mathbb{R}^2 \oplus X_1, \quad Y = \mathbb{R}^2 \oplus Y_1,$$

where X_1 (resp. Y_1) denotes the L^2 orthogonal space of $\mathbb{R}^2$ in X (resp. Y). Let $P : X \to X_1$ and $Q : Y \to Y_1$ be the orthogonal projections. Hence, the unknown function $(w,z) \in X$ of (2.5) can also be decomposed as

$$(w,z) = (r,s) + \boldsymbol{u}, \quad \boldsymbol{u} = P(w,z).$$

Since $H(r,s) = \mathbf{0}$ and $(I-Q)H(X_1) = \mathbf{0}$, (2.5) is consequently reduced to

$$QH(\boldsymbol{u}) + \varepsilon QB((r,s) + \boldsymbol{u}, \beta) = \mathbf{0} \tag{2.6}$$

and

$$(I-Q)B((r,s) + \boldsymbol{u}, \beta) = \mathbf{0}.$$

In order to solve (2.5), we first construct the solution set of (2.6) near $\varepsilon = 0$.

Lemma 2.2. *For any $C > 0$, there exist a small $\varepsilon_0 > 0$ and a neighborhood N of $\{(w,z,\beta,\varepsilon) = (r,s,\beta,0) \in X \times \mathbb{R}^2 : |r|,|s|,|\beta| \le C\}$ such that all solutions of* (2.6) *contained in N can be parameterized by*

$$K := \{((r,s) + \varepsilon U(r,s,\beta,\varepsilon),\beta,\varepsilon) : |r|,|s|,|\beta| \le C + \varepsilon_0, |\varepsilon| \le \varepsilon_0\}.$$

Here, $U(r,s,\beta,\varepsilon)$ is an X_1-valued smooth function defined in $|r|,|s|,|\beta| \le C+\varepsilon_0$, $|\varepsilon| \le \varepsilon_0$. Hence, an element $(w,z,\beta,\varepsilon) = ((r,s) + \varepsilon U(r,s,\beta,\varepsilon),\beta,\varepsilon) \in K$ is a solution of (2.5) *if and only if*

$$\Phi^{\varepsilon}(r,s,\beta) := (I-Q)B((r,s) + \varepsilon U(r,s,\beta,\varepsilon),\beta) = \mathbf{0}.$$

The proof of Lemma 2.2 can be carried out by using the implicit function theorem and a compactness argument. We refer to [11, Lemma 3.1] for the proof.

2.3 Exact Expression of the Limiting Solution Set

Lemma 2.2 asserts that for each $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$, the solution set of (2.5) (equivalently (2.2)) in N coincides with Ker Φ^{ε}. Since $(I-Q)(w,z) = (⨍_\Omega w\,\mathrm{d}x, ⨍_\Omega z\,\mathrm{d}x)$, we obtain

$$\begin{aligned}\Phi^0(r,s,\beta) &= \left(⨍_\Omega F(r,s)\,\mathrm{d}x, ⨍_\Omega G(r,s,\beta)\,\mathrm{d}x\right)\\ &= \begin{pmatrix} r ⨍_\Omega \dfrac{1}{1+sp(x)}\left(\alpha - \dfrac{r}{1+sp(x)} - sc(x)\right)\mathrm{d}x \\ s\left(\beta - s + r ⨍_\Omega \dfrac{d(x)}{1+sp(x)}\mathrm{d}x\right)\end{pmatrix}\end{aligned} \tag{2.7}$$

in the extreme case $\varepsilon = 0$. Therefore, Ker Φ^0 comprises the union of the following four sets in $\mathbb{R}^3$;

$$\mathcal{L}_0 = \{(0,0,\beta) : \beta \in \mathbb{R}\}, \qquad \mathcal{L}_w = \{(\alpha,0,\beta) : \beta \in \mathbb{R}\},$$
$$\mathcal{L}_z = \{(0,\beta,\beta) : \beta \in \mathbb{R}\}, \qquad \mathcal{L}_p = \{(f(s),s,g(s)) : s \in \mathbb{R}\},$$

where

$$f(s) := ⨍_\Omega \frac{\alpha - sc(x)}{1+sp(x)}\mathrm{d}x \Big/ ⨍_\Omega \frac{\mathrm{d}x}{(1+sp(x))^2}, \quad g(s) := s - f(s) ⨍_\Omega \frac{d(x)}{1+sp(x)}\mathrm{d}x. \tag{2.8}$$

It should be remarked that $\mathcal{L}_p$ contains the limiting set of the positive solution set of (2.2) as $\varepsilon \to 0$. Then it is important to study the profiles of $f(s)$ and $g(s)$.

Lemma 2.3. *Let $f(s)$ and $g(s)$ be the functions defined by (2.8). The following profiles of $f(s)$ and $g(s)$ hold true:*

(i) *There exists $s_0 = s_0(\alpha, c(x), \rho(x)) > 0$ such that*

$$\begin{cases} f(s) > 0 \ \text{for} \ s \in [0, s_0), \\ f(s) < 0 \ \text{for} \ s \in (s_0, \infty). \end{cases} \tag{2.9}$$

Additionally, $f(0) = \alpha$ also holds true.

(ii) *It follows that $g(0) = -\alpha ⨍_\Omega d(x)\,\mathrm{d}x < 0$ and*

$$g'(0) = 1 + ⨍_\Omega c(x)\,\mathrm{d}x ⨍_\Omega d(x)\,\mathrm{d}x - \alpha\left(⨍_\Omega \rho(x)\,\mathrm{d}x ⨍_\Omega d(x)\,\mathrm{d}x - ⨍_\Omega \rho(x)d(x)\,\mathrm{d}x\right). \tag{2.10}$$

For the zero point s_0 of $f(s)$, $g(s)$ satisfies

$$\max_{0\le s\le s_0} g(s) = g(s_0) = s_0 > 0 \quad \text{and} \quad g'(s_0) > 0.$$

Proof. In view of (2.8), it is easy to verify $f(0) = \alpha$ and $\lim_{s\to\infty} f(s) = -\infty$. Then, f possesses at least one zero point. After direct calculations, we know

$$\begin{aligned} f'(s) = &\left\{2 ⨍_\Omega \frac{\rho(x)}{(1+s\rho(x))^3}\mathrm{d}x ⨍_\Omega \frac{\alpha - sc(x)}{1+s\rho(x)}\mathrm{d}x \right. \\ &\left. - ⨍_\Omega \frac{\mathrm{d}x}{(1+s\rho(x))^2} ⨍_\Omega \frac{c(x)+\alpha\rho(x)}{(1+s\rho(x))^2}\mathrm{d}x\right\} \Big/ \left(⨍_\Omega \frac{\mathrm{d}x}{(1+s\rho(x))^2}\right)^2. \end{aligned} \tag{2.11}$$

For any zero point s_0 of f, (2.8) yields

$$⨍_\Omega \frac{\alpha - s_0 c(x)}{1+s_0\rho(x)}\mathrm{d}x = 0. \tag{2.12}$$

Then, letting $s = s_0$ in (2.11) implies

$$f'(s_0) = -⨍_\Omega \frac{c(x)+\alpha\rho(x)}{(1+s_0\rho(x))^2}\mathrm{d}x \Big/ ⨍_\Omega \frac{\mathrm{d}x}{(1+s_0\rho(x))^2} < 0.$$

Consequently, we know that s_0 is a unique zero point of $f(s)$, which gives (2.9). It immediately follows from (2.8) that $g(0) = -\alpha ⨍_\Omega d(x)\mathrm{d}x < 0$ and $g(s_0) = s_0 > 0$. Since

$$g'(s) = 1 - f'(s) ⨍_\Omega \frac{d(x)}{1+s\rho(x)}\mathrm{d}x + f(s) ⨍_\Omega \frac{\rho(x)d(x)}{(1+s\rho(x))^2}\mathrm{d}x, \tag{2.13}$$

letting $s = s_0$ in (2.13) yields $g'(s_0) > 0$. Furthermore, (2.10) can be obtained by substituting (2.8) and (2.11) into (2.13) and letting $s = s_0$ in the resulting expression. Thus the proof of Lemma 2.3 is complete. □

It follows from Lemma 2.3 that

$$(f(0), 0, g(0)) = \left(\alpha, 0, -\alpha \fint_{\Omega} d(x)\,\mathrm{d}x\right) \in \mathcal{L}_w, \quad (f(s_0), s_0, g(s_0)) = (0, s_0, s_0) \in \mathcal{L}_z.$$

Hence, the bounded curve $\{(f(s), s, g(s)) : 0 < s < s_0\} \subset \mathcal{L}_p$ implies the limiting solution set of the positive solution set of (2.2) as $\varepsilon \to 0$.

3 Construction of the Perturbed Solution Branch

3.1 Perturbed Solution Branch of (2.2)

In this section, for small $\varepsilon > 0$, we construct the positive solution set of (2.2) by perturbing the limiting solution set $\{(f(s), s, g(s)) : 0 < s < s_0\}$ obtained in Lemma 2.3. More concretely, we aim to prove the following theorem.

Theorem 3.1. *For any fixed $(\alpha, \rho(x), c(x), d(x))$, there exist a small $\varepsilon_0 > 0$ and a family of bounded curves*

$$\{S(\xi, \varepsilon) = (r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon)) \in \mathbb{R}^3 : (\xi, \varepsilon) \in [0, C_\varepsilon] \times [0, \varepsilon_0]\}$$

such that for each $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of (2.2) are parameterized as

$$\Gamma_\varepsilon = \left\{ \begin{array}{l} (w(\xi, \varepsilon), z(\xi, \varepsilon), \beta(\xi, \varepsilon)) = ((r, s) + \varepsilon U(r, s, \beta, \varepsilon), \beta) : \\ (r, s, \beta) = (r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon)), \;\; \xi \in (0, C_\varepsilon) \end{array} \right\}, \tag{3.1}$$

where $U(r, s, \beta, \varepsilon)$ is the X_1-valued smooth function defined in Lemma 2.2 and $S(\xi, \varepsilon)$ is a certain smooth function which satisfies

$$S(\xi, 0) = (f(\xi), \xi, g(\xi)), \quad S(0, \varepsilon) = (\alpha, 0, \beta_*(\varepsilon)), \quad S(C_\varepsilon, \varepsilon) = (0, \beta^*(\varepsilon), \beta^*(\varepsilon))$$

for the functions f and g defined by (2.8). Here, $\beta_*(\varepsilon)$ and $\beta^*(\varepsilon)$ are defined by

$$\beta_*(\varepsilon) = \frac{b_*(\varepsilon\alpha)}{\varepsilon} < 0 \quad \textit{and} \quad \beta^*(\varepsilon) = \frac{b^*(\varepsilon\alpha)}{\varepsilon} > 0 \tag{3.2}$$

for the functions b_* and b^* obtained in Theorem 2.1. Additionally, C_ε is a certain smooth function in $\varepsilon \in [0, \varepsilon_0]$ such that $C_0 = s_0$, which is obtained in (2.9).

The sketch of the proof of Theorem 3.1 will be given in the next subsection.

We now observe $\beta(\xi, \varepsilon)$ in (3.1). Since $\beta(\xi, 0) = g(\xi)$ when $\varepsilon = 0$, (2.10) reveals that

$$I(\rho, d) := \fint_{\Omega} \rho(x)\,\mathrm{d}x \fint_{\Omega} d(x)\,\mathrm{d}x - \fint_{\Omega} \rho(x) d(x)\,\mathrm{d}x$$

is an important term for the direction of the bifurcation of Γ_ε at $(\alpha, 0, \beta_*(\varepsilon))$ when $\varepsilon > 0$ is sufficiently small. In the case when $\varepsilon > 0$ is sufficiently small and $I(\rho, d) > 0$, the direction changes according to the value of α.

Theorem 3.2. *Let $I(\rho, d) > 0$ and*

$$\alpha_* := \left(1 + \fint_\Omega c(x)\,\mathrm{d}x \fint_\Omega d(x)\,\mathrm{d}x\right) I(\rho, d)^{-1} > 0.$$

For any small positive number η, there exists a small $\varepsilon_0 > 0$ such that if $(\alpha, \varepsilon) \in (0, \alpha_* - \eta] \times [0, \varepsilon_0]$, then $\beta_\xi(0, \varepsilon) > 0$, that is, the bifurcation of Γ_ε at $(\alpha, 0, \beta_*(\varepsilon))$ is supercritical. On the other hand, if $(\alpha, \varepsilon) \in [\alpha_* + \eta, \eta^{-1}] \times [0, \varepsilon_0]$, then $\beta_\xi(0, \varepsilon) < 0$, that is, the bifurcation of Γ_ε at $(\alpha, 0, \beta_*(\varepsilon))$ is subcritical. Furthermore, if $(\alpha, \varepsilon) \in [\alpha_* + \eta, \eta^{-1}] \times [0, \varepsilon_0]$, then $\beta(\xi, \varepsilon)$ satisfies

$$\underline{\beta}(\varepsilon) := \min_{\xi \in [0, C_\varepsilon]} \beta(\xi, \varepsilon) < \beta_*(\varepsilon),$$

and the following properties hold true:

(i) *If $\beta < \underline{\beta}(\varepsilon)$ or $\beta \geq \beta^*(\varepsilon)$, (2.2) has no positive solution.*

(ii) *If $\beta = \underline{\beta}(\varepsilon)$ or $\beta_*(\varepsilon) \leq \beta < \beta^*(\varepsilon)$, (2.2) has at least one positive solution.*

(iii) *If $\underline{\beta}(\varepsilon) < \beta < \beta_*(\varepsilon)$, (2.2) has at least two positive solutions.*

Proof. (Assuming Theorem 3.1) Let $S(\xi, \varepsilon) = (r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon))$ be the smooth curve defined by (3.1). We observe that $S(\xi, 0) = (f(\xi), \xi, g(\xi))$. Additionally, it is possible to prove that

$$\lim_{\varepsilon \to 0} (r(\xi, \varepsilon), \beta(\xi, \varepsilon)) = (f(\xi), g(\xi)) \text{ in } C^1([0, s_0]) \times C^1([0, s_0]), \tag{3.3}$$

where s_0 is the number obtained in (2.9). When $I(\rho, d) > 0$, it follows from (2.10) that

$$\begin{cases} g'(0) > 0 & \text{if } \ 0 < \alpha < \alpha_*, \\ g'(0) < 0 & \text{if } \ \alpha > \alpha_*. \end{cases} \tag{3.4}$$

Let η be any fixed small positive number. Hence, (3.3) and (3.4) enable us to find a small $\varepsilon_0 > 0$ such that if $(\alpha, \varepsilon) \in (0, \alpha_* - \eta] \times [0, \varepsilon_0]$, then $\beta_\xi(0, \varepsilon) > 0$, that is, Γ_ε supercritically bifurcates from $(\alpha, 0, \beta_*(\varepsilon))$. On the other hand, if $(\alpha, \varepsilon) \in [\alpha_* + \eta, \eta^{-1}] \times [0, \varepsilon_0]$, then $\beta_\xi(0, \varepsilon) < 0$, that is, Γ_ε subcritically bifurcates from $(\alpha, 0, \beta_*(\varepsilon))$. In the latter case, $g_\varepsilon(\xi) (:= \beta(\xi, \varepsilon))$ satisfies $g'_\varepsilon(0) < 0$, $g(\xi) < g(C_\varepsilon)$ $(0 \leq \xi \leq C_\varepsilon)$ and attains its minimum value at a certain $\xi = \underline{\xi}(\varepsilon) \in (0, C_\varepsilon)$. We denote the minimum value by $\underline{\beta}(\varepsilon)(:= g_\varepsilon(\underline{\xi}(\varepsilon)))$ and set

$$K_\varepsilon(\beta) := \{\xi \in (0, C_\varepsilon) : g_\varepsilon(\xi) = \beta\}.$$

Then, in the case when $\varepsilon > 0$ is sufficiently small, $K_\varepsilon(\beta)$ is empty if $\beta < \underline{\beta}(\varepsilon)$ or $\beta \geq \beta^*(\varepsilon)$, contains at least one element if $\beta = \underline{\beta}(\varepsilon)$ or $\beta_*(\varepsilon) \leq \beta < \beta^*(\varepsilon)$, and contains at least two elements if $\underline{\beta}(\varepsilon) < \beta < \beta_*(\varepsilon)$. In view of (3.1), we know that for any fixed β, the number of elements of $K_\varepsilon(\beta)$ is equal to that of positive solutions of (2.2). Thus the proof of Theorem 3.2 is complete. □

3.2 Sketch of the Proof of Theorem 3.1

As the first step of the proof of Theorem 3.1, we construct the local branches of positive solutions of (2.5) near the bifurcation points.

Lemma 3.3. *There exist a neighborhood* $\mathcal{U}_*$ $(\subset \mathbb{R}^3)$ *of* $(\alpha, 0, -\alpha \fint_\Omega d(x)\,\mathrm{d}x)$ *and a positive number* δ_* *such that for any* $\varepsilon \in [0, \delta_*]$,

$$\mathrm{Ker}\, \Phi^\varepsilon \cap \mathcal{U}_* \cap \overline{\mathbb{R}}^3_+ = \{(r(\xi,\varepsilon), s(\xi,\varepsilon), \beta(\xi,\varepsilon)) : \xi \in [0, \delta_*]\} \cup \{(\alpha, 0, \beta) \in \mathcal{U}_*\}, \quad (3.5)$$

where $(r(\xi,\varepsilon), s(\xi,\varepsilon), \beta(\xi,\varepsilon))$ *is a certain smooth function, which satisfies*

$$(r(\xi,0), s(\xi,0), \beta(\xi,0)) = (f(\xi), \xi, g(\xi)), \quad (r(0,\varepsilon), s(0,\varepsilon), \beta(0,\varepsilon)) = (\alpha, 0, \beta_*(\varepsilon)).$$

Lemma 3.4. *Let* s_0 *be the positive number in* (2.9). *There exist a neighborhood* $\mathcal{U}^*(\subset \mathbb{R}^3)$ *of* $(0, s_0, s_0)$ *and a positive number* δ^* *such that for any* $\varepsilon \in [0, \delta^*]$,

$$\mathrm{Ker}\, \Phi^\varepsilon \cap \mathcal{U}^* \cap \overline{\mathbb{R}}^3_+ = \{(\hat{r}(\xi,\varepsilon), \hat{s}(\xi,\varepsilon), \hat{\beta}(\xi,\varepsilon)) \in \mathbb{R}^3 : \xi \in [0, \delta^*]\} \cup \{(0, \beta, \beta) \in \mathcal{U}^*\},$$

where $\hat{S}(\xi,\varepsilon) := (\hat{r}(\xi,\varepsilon), \hat{s}(\xi,\varepsilon), \hat{\beta}(\xi,\varepsilon))$ *is a smooth function with*

$$\hat{S}(\xi,0) = (f(s_0 - \xi), s_0 - \xi, g(s_0 - \xi)), \quad \hat{S}(0,\varepsilon) = (0, \beta^*(\varepsilon), \beta^*(\varepsilon)).$$

With the aid of the local bifurcation theorem [2], we can prove Lemmas 3.3 and 3.4 (see [10]). The next lemma is the most crucial part of the proof of Theorem 3.1.

Lemma 3.5. *There exists a neighborhood* $\mathcal{U}$ $(\subset X \times \mathbb{R})$ *of* $\{(f(s), s, g(s)) : 0 \leq s \leq s_0\}$ *such that if* $\varepsilon > 0$ *is sufficiently small, then all positive solutions of* (3.1) *contained in* $\mathcal{U}$ *can be parameterized by* (3.1).

Proof. We use the perturbation theory of Du-Lou [4, Appendix] in the proof. For the positive numbers δ_* and δ^* introduced in Lemmas 3.3 and 3.4, we put $\mathcal{L}_p([\delta_*/2, s_0 - \delta^*/2]) := \{(f(s), s, g(s)) : s \in [\delta_*/2, s_0 - \delta^*/2]\}$. Here, s_0 represents the zero point of f obtained in Lemma 2.3. From (2.7) and (2.8), some direct calculations yield

$$\det \Phi^0_{(r,s)}(f(s), s, g(s)) = sf(s)g'(s) \int_\Omega \frac{dx}{(1 + s\rho(x))^2}. \tag{3.6}$$

From (2.9), we know that $f(\overline{s}) > 0$ for any $(f(\overline{s}), \overline{s}, g(\overline{s})) \in \mathcal{L}_p([\delta_*/2, s_0 - \delta^*/2])$. Therefore, (3.6) reveals that $\Phi^0_{(r,s)}(f(\overline{s}), \overline{s}, g(\overline{s}))$ is invertible if and only if $g'(\overline{s}) \neq 0$. In such a case, the implicit function theorem ensures a certain positive number $\delta = \delta(\overline{s})$ and a neighborhood $\mathcal{W}_{\overline{s}}$ of $(f(\overline{s}), \overline{s})$ such that for each $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} \Phi^\varepsilon \cap \mathcal{U}_{\overline{s}} = \{(r(\beta, \varepsilon), s(\beta, \varepsilon), \beta) : \beta \in (g(\overline{s}) - \delta, g(\overline{s}) + \delta)\}. \tag{3.7}$$

Here, $\mathcal{U}_{\overline{s}} := \mathcal{W}_{\overline{s}} \times (g(\overline{s}) - \delta, g(\overline{s}) + \delta)$ and $(r(\beta, \varepsilon), s(\beta, \varepsilon))$ is a smooth function with $(r(g(\overline{s}), 0), s(g(\overline{s}), 0)) = (f(\overline{s}), \overline{s})$.

On the other hand, if $g'(\overline{s}) = 0$, (3.6) implies $\operatorname{rank}\Phi^0_{(r,s)}(f(\overline{s}), \overline{s}, g(\overline{s})) = 1$; therefore

$$\dim \operatorname{Ker} \Phi^0_{(r,s)}(f(\overline{s}), \overline{s}, g(\overline{s})) = \operatorname{codim} \operatorname{Range} \Phi^0_{(r,s)}(f(\overline{s}), \overline{s}, g(\overline{s})) = 1. \tag{3.8}$$

After some computations, we can verify

$$\Phi^0_\beta(f(\overline{s}), \overline{s}, g(\overline{s})) = \begin{pmatrix} \overline{s} \\ 0 \end{pmatrix} \notin \operatorname{Range} \Phi^0_{(r,s)}(f(\overline{s}), \overline{s}, g(\overline{s})) \tag{3.9}$$

By virtue of (3.8) and (3.9), we can use the spontaneous bifurcation theory of Crandall-Rabinowitz [3, Theorem 3.2 and Remark 3.3] to obtain a positive number $\delta = \delta(\overline{s})$ and a neighborhood $\mathcal{U}_{\overline{s}}$ of $(f(\overline{s}), \overline{s}, g(\overline{s}))$ such that, for any $\varepsilon \in [0, \delta]$,

$$\operatorname{Ker} \Phi^\varepsilon \cap \mathcal{U}_{\overline{s}} = \{(r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon)) : \xi \in (-\delta, \delta)\}. \tag{3.10}$$

Here, $(r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon))$ is a smooth function in $(\xi, \varepsilon) \in [-\delta, \delta] \times [0, \delta]$ with

$$(r(0,0), s(0,0), \beta(0,0)) = (g(\overline{s}), \overline{s}, f(\overline{s})).$$

Since (3.7) or (3.10) holds true for each $\mathcal{U}_{\overline{s}}$,

$$\mathcal{L}_p([\delta_*/2, s_0 - \delta^*/2]) \subset \bigcup \{\mathcal{U}_{\overline{s}} : \overline{s} \in [\delta_*/2, s_0 - \delta^*/2]\}.$$

Furthermore, the compactness of $\mathcal{L}_p([\delta_*/2, s_0 - \delta^*/2])$ enables us to find finitely many points $\{s_j\}_{j=1}^n$ such that

$$\begin{cases} (f(s_j), s_j, g(s_j)) \in \mathcal{L}_p([\delta_*/2, s_0 - \delta^*/2]) \ \text{ for } \ 1 \le j \le n, \\ \mathcal{L}_p([\delta_*/2, s_0 - \delta^*/2]) \subset \bigcup_{j=1}^n \mathcal{U}_j, \ \ (\text{where } \mathcal{U}_j := \mathcal{U}_{s_j}). \end{cases}$$

With regard to Lemmas 3.3 and 3.4, we put $\mathcal{U}_0 := \mathcal{U}_*$ and $\mathcal{U}_{n+1} := \mathcal{U}^*$. Hence, we can assume $\mathcal{U}_j \cap \mathcal{U}_{j+1} \neq \emptyset$ $(j = 0, 1, 2, \ldots, n)$ without loss of generality. By virtue of (3.7) and (3.10), we put $\delta_j := \delta(s_j)$. Then, for any $\varepsilon \in [0, \delta_j]$ $(1 \le j \le n)$, there exists a smooth function $(r_j(\xi, \varepsilon), s_j(\xi, \varepsilon), \beta_j(\xi, \varepsilon))$ such that

$$\operatorname{Ker} \Phi^\varepsilon \cap \mathcal{U}_j = \{(r_j(\xi, \varepsilon), s_j(\xi, \varepsilon), \beta_j(\xi, \varepsilon)) : \xi \in (-\delta_j, \delta_j)\} =: J_j^\varepsilon. \tag{3.11}$$

Here $(r_j(\xi, \varepsilon), s_j(\xi, \varepsilon), \beta_j(\xi, \varepsilon))$ satisfies

$$(r_j(0,0), s_j(0,0), \beta_j(0,0)) = (f(s_j), s_j, g(s_j))$$

for each $1 \le j \le n$. Furthermore, by considering Lemmas 3.3 and 3.4, we put

$$\begin{aligned} J_0^\varepsilon &:= \{(r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon)) : \xi \in (0, \delta_*]\}, \\ J_{n+1}^\varepsilon &:= \{(\hat{r}(\xi, \varepsilon), \hat{s}(\xi, \varepsilon), \hat{\beta}(\xi, \varepsilon)) : \xi \in (0, \delta^*]\}, \end{aligned}$$

and $\mathcal{U} := \bigcup_{j=0}^{n+1} \mathcal{U}_j$. Consequently, it follows from (3.11) and Lemmas 3.3 and 3.4 that

$$\operatorname{Ker} \Phi^\varepsilon \cap \mathcal{U} \cap \mathbb{R}_+^3 = \bigcup_{j=0}^{n+1} J_j^\varepsilon \ \ \text{for any} \ \ \varepsilon \in \left[0, \min\nolimits_{0 \le j \le n+1} \delta_j\right]. \tag{3.12}$$

Here, we put $\delta_0 := \delta_*$ and $\delta_{n+1} := \delta^*$. Hence, (3.12) implies that Ker $\Phi^\varepsilon \cap \mathcal{U} \cap \mathbb{R}_+^3$ forms a one-dimensional sub-manifold. Indeed, with the aid of the perturbation theory of Du-Lou [4, Proposition A3], we can construct a bounded smooth curve $S(\xi, \varepsilon) = (r(\xi, \varepsilon), s(\xi, \varepsilon), \beta(\xi, \varepsilon))$ which satisfies

$$\begin{cases} \bigcup_{j=0}^{n+1} J_j^\varepsilon = S((0, C_\varepsilon), \varepsilon), \\ (r(\xi, 0), s(\xi, 0), \beta(\xi, 0)) = (f(\xi), \xi, g(\xi)), \\ (r(0, \varepsilon), s(0, \varepsilon), \beta(0, \varepsilon)) = (\alpha, 0, \beta_*(\varepsilon)), \\ (r(C_\varepsilon, \varepsilon), s(C_\varepsilon, \varepsilon), \beta(C_\varepsilon, \varepsilon)) = (0, \beta^*(\varepsilon), \beta^*(\varepsilon)). \end{cases}$$

Thus, we have proved Lemma 3.5. □

In order to prove Theorem 3.1, we have to show that (2.2) does not admit any positive solution outside of $\mathcal{U}$ obtained in Lemma 3.5. The next lemma can be shown by a contradiction argument. We refer [10] for the proof.

Lemma 3.6. *Let $\mathcal{V}(\subset \mathbb{R}^3)$ be any fixed neighborhood of $\{(f(s), s, g(s)) : 0 \le s \le s_0\}$. If $\varepsilon > 0$ is sufficiently small, then for any positive solution (w, z) of (2.2), there exists $(r, s, \beta) \in \mathcal{V}$ such that*

$$(w, z) = (r, s) + \varepsilon U(r, s, \beta, \varepsilon).$$

Here, $U(r, s, \beta, \varepsilon)$ is the X_1 valued function defined in Lemma 2.2.

Consequently, together with Lemmas 3.5 and 3.6, we obtain Theorem 3.1.

4 Stationary Solution Set of the SKT Model

In view of (2.1), it is convenient to state our result on the positive solution set of the following system, which is obtained from (SP) with $(a, k) = (\varepsilon\alpha, \varepsilon^{-1})$:

$$(\mathrm{SP})_\varepsilon \begin{cases} \Delta[(1 + \varepsilon^{-1}\rho(x)v)u] + u(\varepsilon\alpha - u - c(x)v) = 0 & \text{in } \Omega, \\ \Delta v + v(b + d(x)u - v) = 0 & \text{in } \Omega, \\ \partial_\nu u = \partial_\nu v = 0 & \text{on } \partial\Omega. \end{cases}$$

The following theorem is the main result obtained in this paper.

Theorem 4.1. *Let $(\alpha, \rho(x), c(x), d(x))$ be fixed arbitrarily. Then, if $\varepsilon > 0$ is sufficiently small, the positive solution set of $(\mathrm{SP})_\varepsilon$ (with the bifurcation parameter b) forms a bounded smooth curve*

$$\Gamma_p = \{(u, v, b) = (u(\xi, \varepsilon), v(\xi, \varepsilon), b(\xi, \varepsilon)) \in X \times \mathbb{R} : \xi \in (0, C_\varepsilon)\},$$

where $(u(\xi, \varepsilon), v(\xi, \varepsilon), b(\xi, \varepsilon))$ satisfies

$$(u(0, \varepsilon), v(0, \varepsilon), b(0, \varepsilon)) = (\varepsilon\alpha, 0, b_*(\varepsilon\alpha)) \in \Gamma_u,$$
$$(u(C_\varepsilon, \varepsilon), v(C_\varepsilon, \varepsilon), b(C_\varepsilon, \varepsilon)) = (0, b^*(\varepsilon\alpha), b^*(\varepsilon\alpha)) \in \Gamma_v,$$

and $b(\xi, \varepsilon) < b^*(\varepsilon\alpha)$ $(0 \le \xi < C_\varepsilon)$. Here, $b_*(\varepsilon\alpha)$ and $b^*(\varepsilon\alpha)$ are the functions obtained in Theorem 2.1 and C_ε is the positive function introduced in Theorem 3.1.

In the case when

$$I(\rho, d) := \fint_\Omega \rho(x)\,\mathrm{d}x \fint_\Omega d(x)\,\mathrm{d}x - \fint_\Omega \rho(x)d(x)\,\mathrm{d}x > 0,$$

for the positive number

$$\alpha_* := \left(1 + \fint_\Omega c(x)\,\mathrm{d}x \fint_\Omega d(x)\,\mathrm{d}x\right) I(\rho, d)^{-1}$$

and any small positive number η, there exists a small positive number $\varepsilon_0 = \varepsilon_0(\alpha_*, \eta)$ such that if $0 < \alpha \leq \alpha_* - \eta$ and $0 < \varepsilon \leq \varepsilon_0$, then $b_\xi(0, \varepsilon) > 0$ and the bifurcation of Γ_p from $(\varepsilon\alpha, 0, b_*(\varepsilon\alpha))$ is supercritical. On the other hand, if $\alpha_* + \eta \leq \alpha \leq \eta^{-1}$ and $0 < \varepsilon \leq \varepsilon_0$, then $b_\xi(0, \varepsilon) < 0$ and the bifurcation of Γ_p from $(\varepsilon\alpha, 0, b_*(\varepsilon\alpha))$ is subcritical. In such a case, for the minimum value of b;

$$\underline{b} := \min_{\xi \in [0, C_\varepsilon]} b(\xi, \varepsilon) < b_*(\varepsilon\alpha),$$

the following properties hold true:

(i) *If $b < \underline{b}$ or $b \geq b^*(\varepsilon\alpha)$, then $(\mathrm{SP})_\varepsilon$ possesses no positive solution.*

(ii) *If $b = \underline{b}$ or $b_*(\varepsilon\alpha) \leq b < b^*(\varepsilon\alpha)$, then $(\mathrm{SP})_\varepsilon$ possesses at least one positive solution.*

(iii) *If $\underline{b} < b < b_*(\varepsilon\alpha)$, then $(\mathrm{SP})_\varepsilon$ possesses at least two positive solutions.*

Proof. It follows from (2.1) that (w, z) is a positive solution of (2.2) if and only if

$$(u, v, b) = \varepsilon\left(\frac{w}{1 + \rho(x)z}, z, \beta\right) \tag{4.1}$$

is a positive solution of $(\mathrm{SP})_\varepsilon$. Therefore, Theorem 3.1 implies that for any fixed $(\alpha, \rho(x), c(x), d(x))$, there exists a small positive number ε_0 such that if $\varepsilon \in (0, \varepsilon_0]$, all positive solutions of $(\mathrm{SP})_\varepsilon$ can be parameterized by

$$\Gamma_p = \{(u(\xi, \varepsilon), v(\xi, \varepsilon), b(\xi, \varepsilon)) \in X \times \mathbb{R} : (\xi, \varepsilon) \in [0, C_\varepsilon] \times [0, \varepsilon_0]\}.$$

Hence by (4.1), $(u(\xi, \varepsilon), v(\xi, \varepsilon), b(\xi, \varepsilon))$ satisfies

$$(u(\xi, \varepsilon), v(\xi, \varepsilon), b(\xi, \varepsilon)) = \varepsilon\left(\frac{w(\xi, \varepsilon)}{1 + \rho(x)z(\xi, \varepsilon)}, z(\xi, \varepsilon), \beta(\xi, \varepsilon)\right)$$

for the function $(w(\xi, \varepsilon), z(\xi, \varepsilon), \beta(\xi, \varepsilon))$ obtained in (3.1). In view of (3.2), we know that

$$b(0, \varepsilon) = \varepsilon\beta_*(\varepsilon) = b_*(\varepsilon\alpha) < 0, \quad b(C_\varepsilon, \varepsilon) = \varepsilon\beta^*(\varepsilon) = b^*(\varepsilon\alpha) > 0.$$

In the case when $I(\rho, d) > 0$, from Theorem 3.2 and the one-to-one relationships of (4.1), we obviously obtain the desired $\subset$-shaped bifurcation curve of Γ_p. Thus the proof of Theorem 4.1 is complete. □

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