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ASYMPTOTIC SOLUTIONS FOR LARGE-TIME OF HAMILTON-JACOBI EQUATIONS IN EUCLIDEAN n SPACE

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Abstract. Following [I2] we discuss the large time behavior of solutions of the Cauchy problem for the Hamilton-Jacobi equation $u_t + H(x, Du) = 0$ in $\mathbf{R}^n \times (0, \infty)$, where $H(x,p)$ is continuous on $\mathbf{R}^n \times \mathbf{R}^n$ and strictly convex in p. We present a general convergence result for viscosity solutions $u(x,t)$ of the Cauchy problem as $t \to \infty$.

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1. Introduction

In the last decade, there has been much interest on the asymptotic behavior of viscosity solutions of the Cauchy problem for Hamilton-Jacobi equations or viscous Hamilton-Jacobi equations. Namah and Roquejoffre [NR] and Fathi [F2] were the first those who established fairly general convergence results for the Hamilton-Jacobi equation $u_t(x,t)+H(x, Du(x,t)) = 0$ on a compact manifold M with smooth strictly convex Hamiltonian H. The approach by Fathi to this large time asymptotic problem is based on weak KAM theory [F1, F3, FS1] which is concerned with the Hamilton-Jacobi equation as well as with the Lagrangian or Hamiltonian dynamical structures behind it. Barles and Souganidis [BS1, BS2] took another approach, based on PDE techniques, to the same asymptotic problem. The weak KAM approach due to Fathi to the asymptotic problem has been developed and further improved by Roquejoffre [R] and Davini-Siconolfi [DS]. It should be remarked here that the same kind of asymptotic behavior of solutions of Hamilton-Jacobi equations has already been studied by Kruzkov [K], P.-L. Lions [L], and Barles [B1].

In this review we are concerned with the Cauchy problem for the Hamilton-Jacobi equation

$$u_t + H(x, Du) = 0 \quad \text{in } \mathbf{R}^n \times (0, \infty), \tag{1.1}$$

$$u(\cdot, 0) = u_0, \tag{1.2}$$

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where H is a scalar function on $\mathbf{R}^n \times \mathbf{R}^n$, $u = u(x,t)$ is the unknown scalar function on $\mathbf{R}^n \times [0,\infty)$, and u_0 is a given function on $\mathbf{R}^n$.

The function $H(x,p)$ is assumed here to be convex in p, and we call H the *Hamiltonian* and then the function L, defined by $L(x,\xi) = \sup_{p\in\mathbf{R}^n}(\xi \cdot p - H(x,p))$, the *Lagrangian.*

We are also concerned with the *additive eigenvalue* problem:

$$H(x, Dv) = c \quad \text{in } \mathbf{R}^n, \tag{1.3}$$

where the unknown is a pair $(c,v) \in \mathbf{R} \times C(\mathbf{R}^n)$ for which v is a viscosity solution of (1.3). This problem is also called the *ergodic control* problem due to the fact that PDE (1.3) appears as the dynamic programming equation in ergodic control of deterministic optimal control. We remark that the additive eigenvalue problem (1.3) appears as well in the homogenization of Hamilton-Jacobi equations. See for this [LPV].

For notational simplicity, given $\phi \in C^1(\mathbf{R}^n)$, we will write $H[\phi](x)$ for $H(x, D\phi(x))$ or $H[\phi]$ for the function: $x \mapsto H(x, D\phi(x))$ on $\mathbf{R}^n$. For instance, (1.3) may be written as $H[v] = c$ in $\mathbf{R}^n$. Also, we denote by $\mathcal{S}_H^+$ (resp., $\mathcal{S}_H^-$, and $\mathcal{S}_H$) the space of all viscosity supersolutions (resp., subsolutions, and solutions) u of $H[u] = 0$ in $\mathbf{R}^n$.

The paper is organized as follows: in Section 2 we state our assumptions on H and then the main result in [I2] (Theorem 1 below). In Section 3 we present an outline of the proof of Theorem 1. In Section 4 we discuss basic properties of Aubry sets. In Section 5 we give examples of H to which Theorem 1 applies, an example and two propositions related to equilibrium points in Aubry sets, and an example for which the desirable asymptotic behavior does not hold.

2. Main results

We make throughout the following assumptions on the Hamiltonian H.

(A1) $H \in C(\mathbf{R}^n \times \mathbf{R}^n)$.

(A2) H is *coercive*, that is, for any $R > 0$,

$$\lim_{r\to\infty} \inf\{H(x,p) \mid x \in B(0,R),\ p \in \mathbf{R}^n \setminus B(0,r)\} = \infty.$$

(A3) For any $x \in \mathbf{R}^n$, the function: $p \mapsto H(x,p)$ is strictly convex in $\mathbf{R}^n$.

(A4) There are functions $\phi_i \in C^{0+1}(\mathbf{R}^n)$ and $\sigma_i \in C(\mathbf{R}^n)$, with $i = 0,1$, such that for $i = 0,1$,

$$H(x, D\phi_i(x)) \le -\sigma_i(x) \quad \text{almost every } x \in \mathbf{R}^n,$$
$$\lim_{|x|\to\infty} \sigma_i(x) = \infty, \qquad \lim_{|x|\to\infty} (\phi_0 - \phi_1)(x) = \infty.$$

By adding a constant to the function ϕ_0, we assume henceforth that

$$\phi_0(x) \ge \phi_1(x) \quad \text{for } x \in \mathbf{R}^n.$$

We introduce the classes Φ_0 and Ψ_0 of functions defined, respectively, by

$$\Phi_0 = \{u \in C(\mathbf{R}^n) \mid \inf_{\mathbf{R}^n}(u - \phi_0) > -\infty\},$$
$$\Psi_0 = \{u \in C([0,\infty) \times \mathbf{R}^n) \mid \inf_{(x,t)\in\mathbf{R}^n\times[0,T]}(u(x,t) - \phi_0(x)) > -\infty \text{ for any } T > 0\}.$$

We call a function $m : [0,\infty) \to [0,\infty)$ a *modulus* if it is continuous and nondecreasing on $[0,\infty)$ and satisfies $m(0) = 0$. The space of all absolutely continuous functions $\gamma : [S,T] \to \mathbf{R}^n$ will be denoted by $\mathrm{AC}([S,T],\mathbf{R}^n)$. For $x, y \in \mathbf{R}^n$ and $t > 0$, $\mathcal{C}(x,t)$ (resp., $\mathcal{C}(x,t;y,0)$) will denote the spaces of all curves $\gamma \in \mathrm{AC}([0,t],\mathbf{R}^n)$ satisfying $\gamma(t) = x$ (resp., $\gamma(t) = x$ and $\gamma(0) = y$). For any interval $I \subset \mathbf{R}$ and $\gamma : I \to \mathbf{R}^n$, we call γ a curve if it is absolutely continuous on any compact subinterval of I.

We have established the following theorem in [I2].

Theorem 1. (a) *Let $u_0 \in \Phi_0$ and assume that* (A1)–(A4) *hold. Then there is a unique viscosity solution $u \in \Psi_0$ of* (1.1) *and* (1.2)*and the function u is represented as*

$$u(x,t) = \inf\left\{\int_0^t L(\gamma(s),\dot\gamma(s))\mathrm{d}s + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x,t)\right\} \tag{2.1}$$

for $(x,t) \in \mathbf{R}^n \times (0,\infty)$.

(b) *There is a solution $(c,v) \in \mathbf{R} \times \Phi_0$ of* (1.3). *Moreover the constant c is unique in the sense that if $(d,w) \in \mathbf{R} \times \Phi_0$ is another solution of* (1.3), *then $d = c$.*

(c) *Let $u \in \Psi_0$ be the viscosity solution of* (1.1) *and* (1.2). *Then there is a solution $(c,v) \in \mathbf{R} \times \Phi_0$ of* (1.3) *for which, as $t \to \infty$,*

$$u(x,t) + ct - v(x) \to 0 \quad \textit{in } C(\mathbf{R}^n).$$

Motivated by recent developments due to [BS1, BS2, F2, R, DS] concerning the large time behavior of solutions of Hamilton-Jacobi equations, the author jointly with Y. Fujita and P. Loreti (see [FIL1, FIL2]) has recently investigated the asymptotic problem for viscous Hamilton-Jacobi equations with Ornstein-Uhlenbeck operator and the corresponding Hamilton-Jacobi equations. The above theorem generalizes main results of [FIL2]. The new feature in [FIL1, FIL2, I2] is that we deal with Hamilton-Jacobi equation (1.1) on $\mathbf{R}^n \times (0,\infty)$ and the domain $\mathbf{R}^n$ is noncompact while in [BS1, BS2, F2, R, DS] the auhtors studied (1.1) on $\Omega \times (0,\infty)$ with Ω being compact. Barles and Roquejoffre [BR] have recently studied the large time behavior of solutions of (1.1) and (1.2) and obtained, among other results, a generalization of the main result in [NR] to unbounded solutions. See also [II] for results in the same direction. The large time behavior of solutions of Hamilton-Jacobi equations with boundary conditions has been studied by [B1, R, M].

We will see in Example 4 of Section 5 that if $H(x,p)$ does not satisfy strict convexity (A3) and is just convex in p, then in general assertion (d) does not hold.

Assertion (b) of the above theorem determines uniquely a constant c, which we will denote by c_H, for which (1.3) has a viscosity solution in the class Φ_0. The constant c_H is called the *additive eigenvalue* (or simply *eigenvalue*) or *critical value* for the Hamiltonian H. This definition may suggest that c depends on the choice of (ϕ_0,ϕ_1). Actually, it depends only on H, but not on the choice of (ϕ_0,ϕ_1), as the characterization of c_H in Proposition 9 below shows. It is clear that if (c,v) is a solution of (1.3), then $(c, v+K)$ is a solution of (1.3) for any $K \in \mathbf{R}$. As is well-known (see [LPV]), the structure of solutions of (1.3) is, in general, much more complicated than this one-dimensional structure.

For any solution $(c, v) \in \mathbf{R} \times \Phi_0$ of (1.3), we call the function $v(x) - ct$ an *asymptotic solution* of (1.1). It is clear that any asymptotic solution of (1.1) is a viscosity solution of (1.1) and (1.2). On the other hand, if u is a viscosity solution of (1.1) and (1.2), $(c, v) \in \mathbf{R} \times \Phi_0$, and, as $t \to \infty$, we have

$$u(\cdot, t) + ct - v \to 0 \quad \text{in } C(\mathbf{R}^n),$$

then (c, v) is a solution of (1.3) and hence an asymptotic solution of (1.1).

Note that $L(x, \xi) \geq -H(x, 0)$ for all $x \in \mathbf{R}^n$ and hence $\inf\{L(x, \xi) \mid (x, \xi) \in B(0, R) \times \mathbf{R}^n\} > -\infty$ for all $R > 0$. Note as well that for any $(x, t) \in \mathbf{R}^n \times (0, \infty)$ and $\gamma \in \mathcal{C}(x, t)$ the function: $s \mapsto L(\gamma(s), \dot\gamma(s))$ is measurable. Therefore it is natural and standard to set

$$\int_0^t L(\gamma(s), \dot\gamma(s))\mathrm{d}s = \infty,$$

with $\gamma \in \mathcal{C}(x, t)$, if the function: $s \mapsto L(\gamma(s), \dot\gamma(s))$ on $[0, t]$ is not integrable. In this sense the integral in formula (2.1) always makes sense.

In order to prove (c) of Theorem 1, we take an approach close to and inspired by the generalized dynamical approach introduced by Davini and Siconolfi [DS] (see also [R]). However our approach does not depend on the Aubry set for H and is much simpler than the generalized dynamical approach by [DS].

In the following we *always assume* unless otherwise stated that (A1)–(A4) hold.

3. Outline of proof of Theorem 1.

We give here a brief description of the proof of Theorem 1. We begin with a lemma (see [I2, Proposition 2.4]).

Lemma 2. *Let Ω be an open subset of $\mathbf{R}^n$, $\phi \in C^{0+1}(\Omega)$, and $\gamma \in \mathrm{AC}([a, b], \mathbf{R}^n)$, where $a, b \in \mathbf{R}$ satisfy $a < b$. Assume that $\gamma([a, b]) \subset \Omega$. Then there is a function $q \in L^\infty(a, b, \mathbf{R}^n)$ such that*

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\phi \circ \gamma(t) &= q(t) \cdot \dot\gamma(t) \qquad a.e.\ t \in (a, b), \\ q(t) &\in \partial_c\phi(\gamma(t)) \qquad a.e.\ t \in (a, b). \end{aligned}$$

Here $\partial_c\phi$ denotes the Clarke differential of ϕ (see [C]), that is,

$$\partial_c\phi(x) = \bigcap_{r>0} \overline{\mathrm{co}}\,\{D\phi(y) \mid y \in B(x, r),\ \phi \text{ is differentiable at } y\} \quad \text{for } x \in \Omega.$$

Lemma 3 ([I2, Proposition 2.5]). *Let Ω be an open subset of $\mathbf{R}^n$ and $w \in C(\Omega)$ a viscosity solution of $H[w] \leq 0$ in Ω. Let $a, b \in \mathbf{R}$ satisfy $a < b$ and let $\gamma \in \mathrm{AC}([a, b], \mathbf{R}^n)$. Assume that $\gamma([a, b]) \subset \Omega$. Then*

$$w(\gamma(b)) - w(\gamma(a)) \leq \int_a^b L(\gamma(s), \dot\gamma(s))\mathrm{d}s.$$

Proof. By Lemma 2, there is a function $q \in L^\infty(a, b, \mathbf{R}^n)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s} w(\gamma(s)) = q(s) \cdot \dot\gamma(s) \quad \text{and} \quad q(s) \in \partial_c w(\gamma(s)) \quad \text{a.e. } s \in (a, b).$$

Noting that $H(x, p) \leq 0$ for all $p \in \partial_c w(x)$ and all $x \in \Omega$, we calculate that

$$\begin{aligned} w(\gamma(b)) - w(\gamma(a)) &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}s} w(\gamma(s))\mathrm{d}s = \int_a^b q(s) \cdot \dot\gamma(s)\mathrm{d}s \\ &\leq \int_a^b [L(\gamma(s), \dot\gamma(s)) + H(\gamma(s), q(s))]\, \mathrm{d}s \leq \int_a^b L(\gamma(s), \dot\gamma(s))\mathrm{d}s. \end{aligned}$$ □

Proof of (a). A way of proving the existence of a viscosity solution $u \in \Psi_0$ of (1.1) and (1.2) is to show that the function u on $\mathbf{R}^n \times (0, \infty)$ given by

$$u(x, t) = \inf\Big\{\int_0^t L(\gamma(s), \dot\gamma(s))\mathrm{d}s + u_0(\gamma(0)) \mid \gamma \in \mathcal{C}(x, t)\Big\} \tag{3.1}$$

is a viscosity solution of (1.1) by using the dynamic programming principle.

In the proof of (a), u denotes always the function given by (3.1).

Lemma 4. *There exists a constant $C_0 > 0$ such that*

$$u(x, t) \geq \phi_0(x) - C_0(1 + t) \quad \textit{for all } (x, t) \in \mathbf{R}^n \times [0, \infty).$$

Proof. We choose $C_0 > 0$ so that $u_0(x) \geq \phi_0(x) - C_0$ and $H(x, D\phi_0(x)) \leq C_0$ a.e. $x \in \mathbf{R}^n$. Fix any $(x, t) \in \mathbf{R}^n \times (0, \infty)$. For each $\varepsilon > 0$ there is a curve $\gamma \in \mathcal{C}(x, t)$ such that

$$u(x, t) + \varepsilon > \int_0^t L(\gamma(s), \dot\gamma(s))\mathrm{d}s + u_0(\gamma(0)).$$

By Lemma 3, we have

$$\phi_0(\gamma(t)) - \phi_0(\gamma(0)) \leq \int_0^t [L(\gamma(s), \dot\gamma(s)) + C_0]\, \mathrm{d}s,$$

and hence

$$u(x, t) + \varepsilon > \phi_0(\gamma(t)) - \phi_0(\gamma(0)) - C_0 t + u_0(\gamma(0)) \geq \phi_0(x) - C_0(1 + t),$$

which shows that $u(x, t) \geq \phi_0(x) - C_0(1 + t)$. □

Lemma 5. *We have*

$$u(x, t) \leq u_0(x) + L(x, 0)t \quad \textit{for all } (x, t) \in \mathbf{R}^n \times (0, \infty).$$

We remark here that, thanks to (A1) and (A2), for each $R > 0$ there is an $\varepsilon > 0$ such that $\sup_{B(0,R)\times B(0,\varepsilon)} L < \infty$.

Proof. For $\gamma(s) := x$, we have

$$u(x,t) \le \int_0^t L(\gamma(s),\dot\gamma(s))\,ds + u_0(\gamma(0)) = u_0(x) + L(x,0)t. \qquad \square$$

Lemma 6. *For each $R > 0$ there exists a modulus m_R such that*

$$u(x,t) \ge u_0(x) - m_R(t) \qquad \textit{for all } (x,t) \in B(0,R)\times[0,\infty).$$

Proof. Let $C_0 > 0$ be as in the proof of Lemma 4. We choose $C_1 > 0$ so that $H(x, D\phi_1(x)) \le C_1$ a.e. $x \in \mathbf{R}^n$. Fix $R > 0$, $(x,t) \in B(0,R)\times(0,1)$, and $\varepsilon \in (0,1)$. There is a curve $\gamma \in \mathcal{C}(x,t)$ such that

$$u(x,t) + \varepsilon > \int_0^t L(\gamma(s),\dot\gamma(s))ds + u_0(\gamma(0)). \tag{3.2}$$

By the dynamic programming principle, for any $\tau \in [0,t]$, we have

$$u(x,t) + \varepsilon > \int_\tau^t L(\gamma(s),\dot\gamma(s))ds + u(\gamma(\tau),\tau).$$

Fix $\tau \in [0,t]$. Using Lemmas 3 and 4, we get

$$\begin{aligned} u(x,t) + 1 &> \phi_1(\gamma(t)) - \phi_1(\gamma(\tau)) - C_1(t-\tau) + u(\gamma(\tau),\tau) \\ &\ge \phi_1(x) - \phi_1(\gamma(\tau)) - C_1(t-\tau) + \phi_0(\gamma(\tau)) - C_0(\tau+1). \end{aligned}$$

Consequently, using Lemma 5, we have

$$\phi_0(\gamma(\tau)) - \phi_1(\gamma(\tau)) < u_0(x) + |L(x,0)| + 1 - \phi_1(x) + C_1 + 2C_0.$$

From this we see that there is a $C_R > 0$ depending only on R, C_0, C_1, ϕ_0, ϕ_1, u_0, and $L(\cdot,0)$ such that $|\gamma(\tau)| \le C_R$ for all $\tau \in [0,t]$.

There is an $A_\varepsilon > 0$, depending only on ε, u_0, and C_R, such that

$$|u_0(y) - u_0(z)| \le \varepsilon + A_\varepsilon|y-z| \qquad \text{for all } y,z \in B(0,C_R).$$

Observe by (A1) that for any $r > 0$,

$$\lim_{|\xi|\to\infty} \inf_{x\in B(0,r)} \frac{L(x,\xi)}{|\xi|} = \infty.$$

Hence there is a $B_\varepsilon > 0$, depending only on C_R, A_ε, and L, such that $L(x,\xi) \ge A_\varepsilon|\xi| - B_\varepsilon$ for all $(x,\xi) \in B(0,C_R)\times\mathbf{R}^n$.

From (3.2), we get

$$u(x,t) > -\varepsilon + \int_0^t (A_\varepsilon|\dot{\xi}(s)| - B_\varepsilon)\,\mathrm{d}s + u_0(x) - \varepsilon - A_\varepsilon|\gamma(0) - x| \geq -2\varepsilon - B_\varepsilon t,$$

from which we conclude that for any $R > 0$ we have $u(x,t) \geq u_0(x) - m_R(t)$ for all $(x,t) \in B(0,R) \times [0,\infty)$ and for some modulus m_R. □

By the dynamic programming principle, we infer (see [I2, Appendix] for the details) that u is a viscosity solution of (1.1) in the sense that its upper (resp., lower) semi-continuous envelope u^* (resp., u_*) is a viscosity subsolution (resp., supersolution) of (1.1).

Setting $u(x,0) = u_0(x)$ for $x \in \mathbf{R}^n$, we extend the domain of definition of u to $\mathbf{R}^n \times [0,\infty)$. The resulting u is continuous at every point $(x,0)$ with $x \in \mathbf{R}^n$.

We have the following comparison theorem for solutions of (1.1) and (1.2).

Theorem 7. *Let $T \in (0,\infty)$ and Ω be an open subset of $\mathbf{R}^n$. Let $u,\, v : \overline{\Omega} \times [0,T) \to \mathbf{R}$. Assume that u, $-v$ are upper semicontinuous on $\overline{\Omega} \times [0,T)$ and that u and v are, respectively, a viscosity subsolution and a viscosity supersolution of*

$$u_t + H(x, Du) = 0 \qquad \text{in } \Omega \times (0,T). \tag{3.3}$$

Moreover, assume that

$$\lim_{r\to\infty} \inf\{v(x,t) - \phi_1(x) \mid (x,t) \in (\Omega \setminus B(0,r)) \times [0,T)\} = \infty, \tag{3.4}$$

and that $u \leq v$ on $(\Omega \times \{0\}) \cup (\partial\Omega \times [0,T))$. Then $u \leq v$ in $\overline{\Omega} \times [0,T)$.

Proof. We choose a $C > 0$ so that

$$H(x, D\phi_1(x)) \leq C \qquad \text{a.e. } x \in \mathbf{R}^n,$$

and define the function $w \in C(\mathbf{R}^n \times \mathbf{R})$ by $w(x,t) := \phi_1(x) - Ct$. Observe that $w_t + H(x, Dw(x,t)) \leq 0$ a.e. $(x,t) \in \mathbf{R}^{n+1}$.

We need only to show that for all $(x,t) \in \overline{\Omega}$ and all $A > 0$,

$$\min\{u(x,t),\, w(x,t) + A\} \leq v(x,t). \tag{3.5}$$

Fix any $A > 0$. We set $w_A(x,t) = w(x,t) + A$ for $(x,t) \in \mathbf{R}^{n+1}$. The function w_A is a viscosity subsolution of (3.3). By the convexity of $H(x,p)$ in p, the function $\bar{u}$ defined by $\bar{u}(x,t) := \min\{u(x,t),\, w_A(x,t)\}$ is a viscosity subsolution of (3.3). Because of assumption (3.4), we see that there is a $R > 0$ such that $\bar{u}(x,t) \leq v(x,t)$ for all $(x,t) \in (\overline{\Omega} \setminus B(0,R)) \times [0,T)$. We set $\Omega_R := \Omega \cap \operatorname{int} B(0,2R)$, so that $\bar{u}(x,t) \leq v(x,t)$ for all $x \in \partial\Omega_R \times [0,T)$. Also, we have $\bar{u}(x,0) \leq u(x,0) \leq v(x,0)$ for all $x \in \Omega_R$.

Next we wish to use standard comparison results. However, H does not satisfy the usual assumptions for comparison. We thus take the sup-convolution of $\bar{u}$ in the variable t and take advantage of the coercivity of H. That is, for each $\varepsilon \in (0,1)$ we set

$$u^{\varepsilon}(x,t) := \sup_{s\in[0,T)} \left(\overline{u}(x,s) - \frac{(t-s)^2}{2\varepsilon} \right) \quad \text{for all } (x,t) \in \overline{\Omega}_R \times \mathbf{R}.$$

For each $\delta > 0$, there is a $\gamma \in (0, \min\{\delta, T/2\})$ such that $\bar{u}(x,t) - \delta \leq v(x,t)$ for all $(x,t) \in \overline{\Omega}_R \times [0,\gamma]$. As is well-known, there is an $\varepsilon \in (0,\delta)$ such that u^{ε} is a viscosity subsolution of (3.3) in $\Omega_R \times (\gamma, T-\gamma)$ and $u^{\varepsilon}(x,t) - 2\delta \leq v(x,t)$ for all $(x,t) \in (\overline{\Omega}_R \times [0,\gamma]) \cup (\partial\Omega_R \times [\gamma, T-\gamma])$. Observe that the family of functions: $t \mapsto u^{\varepsilon}(x,t)$ on $[\gamma, T-\gamma]$, with $x \in \overline{\Omega}_R$, is equi-Lipschitz continuous, with a Lipschitz bound $C_\varepsilon > 0$, and therefore that for each $t \in [\gamma, T-\gamma]$, the function $z : x \mapsto u^{\varepsilon}(x,t)$ in Ω_R satisfies $H(x, Dz(x)) \leq C_\varepsilon$ a.e., which implies that the family of functions: $x \mapsto u^{\varepsilon}(x,t)$, with $t \in [\gamma, T-\gamma]$, is equi-Lipschitz continuous in Ω_R.

Now, we may apply a standard comparison theorem, to get $u^{\varepsilon}(x,t) \leq v(x,t)$ for all $(x,t) \in \Omega_R \times [\gamma, T-\gamma]$, from which we get $\bar{u}(x,t) \leq v(x,t)$ for all $(x,t) \in \overline{\Omega} \times [0,T)$. This completes the proof. □

Using the above comparison theorem, we conclude that $u \in C(\mathbf{R}^n \times [0,\infty))$ and hence $u \in \Psi_0$. We have thus proved assertion (a). □

Proof of (b). In order to show the existence of a solution of (1.3), we let $\lambda > 0$ and consider the problem

$$\lambda v_\lambda(x) + H(x, Dv_\lambda(x)) = \lambda\phi_0(x) \quad \text{in } \mathbf{R}^n. \tag{3.6}$$

Thanks to the coercivity of H, it is not hard to construct a function $\psi_0 \in C^1(\mathbf{R}^n)$ such that

$$H(x, D\psi_0(x)) \geq -C_0 \quad \text{and} \quad \psi_0(x) \geq \phi_0(x) \quad \text{in } \mathbf{R}^n$$

for some constant $C_0 > 0$. We may assume that $H[\phi_0] \leq C_0$ in $\mathbf{R}^n$ in the viscosity sense.

We define the functions $v_\lambda^{\pm}$ on $\mathbf{R}^n$ by

$$v_\lambda^+(x) = \psi_0(x) + \lambda^{-1}C_0 \quad \text{and} \quad v_\lambda^-(x) = \phi_0(x) - \lambda^{-1}C_0.$$

It is easily seen that v_λ^+ and v_λ^- are viscosity supersolution and a viscosity subsolution of (3.6). Since $\phi_0 \leq \psi_0$ in $\mathbf{R}^n$, we have $v_\lambda^-(x) < v_\lambda^+(x)$ for all $x \in \mathbf{R}^n$. By the Perron method, we find a viscosity solution v_λ of (3.6) such that

$$v_\lambda^-(x) \leq v_\lambda(x) \leq v_\lambda^+(x) \quad \text{for all } x \in \mathbf{R}^n. \tag{3.7}$$

We formally compute that

$$\lambda\phi_0(x) = \lambda v_\lambda(x) + H(x, Dv_\lambda(x)) \geq \lambda\phi_0(x) - C_0 + H(x, Dv_\lambda(x)),$$

and hence $H(x, Dv_\lambda(x)) \le C_0$. This together with the coercivity of H yields the local equi-Lipschitz continuity of the family $\{v_\lambda\}_{\lambda>0}$. As a consequence, the family $\{v_\lambda - v_\lambda(0)\}_{\lambda>0} \subset C(\mathbf{R}^n)$ is uniformly bounded and equi-Lipschitz continuous on bounded subsets of $\mathbf{R}^n$.

By (3.7), we have $\lambda\phi_0(x) - C_0 \le \lambda v_\lambda(x) \le \lambda\psi_0(x) + C_0$ for all $x \in \mathbf{R}^n$. In particular, the set $\{\lambda v_\lambda(0)\}_{\lambda\in(0,1)} \subset \mathbf{R}$ is bounded. Thus we may choose a sequence $\{\lambda_j\}_{j\in\mathbf{N}} \subset (0, 1)$ such that, as $j \to \infty$,

$$\begin{aligned} \lambda_j \to 0, \qquad & -\lambda_j v_{\lambda_j}(0) \to c, \\ v_{\lambda_j} - v_{\lambda_j}(0) \to v \quad & \text{in } C(\mathbf{R}^n) \end{aligned}$$

for some $c \in \mathbf{R}$ and some function $v \in C^{0+1}(\mathbf{R}^n)$. Since

$$|\lambda(v_\lambda(x) - v_\lambda(0))| \le \lambda L_R|x| \quad \text{for all } x \in B(0, R),\ R > 0$$

and for some constant $L_R > 0$, we find that $-\lambda_j v_{\lambda_j} \to c$ in $C(\mathbf{R}^n)$ as $j \to \infty$. By the stability of the viscosity property, we deduce that (c, v) is a solution of (1.3). We need to show that $v \in \Phi_0$. For this we just refer to [I2].

It remains to prove the uniqueness of the constant c. We have the following comparison theorem.

Theorem 8 (**[I2, Theorem 3.2]**). *Let Ω be an open subset of $\mathbf{R}^n$ and $\varepsilon > 0$. Let $u, v : \overline{\Omega} \to \mathbf{R}$ be, respectively, an upper semicontinuous viscosity subsolution of $H[u] \le -\varepsilon$ in Ω and a lower semicontinuous viscosity supersolution of $H[v] \ge 0$ in Ω. Assume that $v \in \Phi_0$ and $u \le v$ on $\partial\Omega$. Then $u \le v$ on Ω.*

We skip the proof of the above theorem. Using the above theorem, it is easy to conclude the uniqueness of the constant c. □

The following characterization of c_H is valid.

Proposition 9. *We have: $c_H = \inf\{a \in \mathbf{R} \mid \mathcal{S}^-_{H-a} \ne \emptyset\}$, where $H - a$ denotes the function: $(x,p) \mapsto H(x,p) - a$.*

Proof. We write c temporarily for the right hand side of the above equality. It is clear that $c \le c_H$.

To complete the proof, we suppose that $c < c_H$ and will get a contradiction. By (b) of Theorem 1, there is a function $v \in \Phi_0 \cap \mathcal{S}_{H-c_H}$. It is obvious that $v \in \mathcal{S}^+_{H-c}$. Note by the stability of the viscosity property that $\mathcal{S}^-_{H-c} \ne \emptyset$. Fix $w \in \mathcal{S}^-_{H-c}$. We may choose a $C > 0$ so that the function $u(x) := \min\{w(x), \phi_1(x) + C\}$ is a viscosity solution of $H[u] \le c$ in $\mathbf{R}^n$. Moreover we may assume by replacing C by a larger constant if necessary that $u - C \le v$ in $\mathbf{R}^n$. We apply the Perron method to find a $\phi \in \mathcal{S}_{H-c}$, but this contradicts the uniqueness assertion of (b) of Theorem 1. □

Proof of (c). We assume that $c_H = 0$ in the following proof. Indeed, this condition can be achieved by replacing H and L by $H - c_H$ and $L + c_H$, respectively.

Let $\{S_t\}_{t\geq 0}$ be the semi-group of mappings on Φ_0 defined by $S_t u_0 = u(\cdot, t)$, where $u \in \Psi_0$ is the unique viscosity solution of (1.1) and (1.2).

Let $I \subset \mathbf{R}$ be an interval and $\phi \in \Phi_0$ a viscosity subsolution of $H[\phi] = 0$ in $\mathbf{R}^n$. We denote by $\mathcal{E}(I, \phi)$ the space of all curves $\gamma \in C(I, \mathbf{R}^n)$ such that for any $[a, b] \subset I$,

$$\gamma \in \mathrm{AC}([a, b], \mathbf{R}^n) \quad \text{and} \quad \int_a^b L(\gamma(t), \dot\gamma(t))\, dt \leq \phi(\gamma(b)) - \phi(\gamma(a)).$$

Such an element $\gamma \in \mathcal{E}(I, \phi)$ is called an *extremal* curve.

We need the following lemma.

Lemma 10 ([I2, Corollary 6.2]). *Let $x \in \mathbf{R}^n$ and $\phi \in \mathcal{S}_H \cap \Phi_0$. Then there exists a curve $\gamma \in \mathcal{E}((-\infty, 0], \phi)$ such that $\gamma(0) = x$.*

The following lemma is a variant of [DS, Lemma 5.2].

Lemma 11 ([I2, Proposition 7.1]). *Let K be a compact subset of $\mathbf{R}^n$. Then there exist a constant $\delta \in (0, 1)$ and a modulus ω for which if $u_0 \in \Phi_0$, $\phi \in \mathcal{S}_H^-$, $\gamma \in \mathcal{E}([0, T], \phi)$, $\gamma([0, T]) \subset K$, $T > \tau \geq 0$ and $\frac{\tau}{T-\tau} \leq \delta$, then*

$$S_T u_0(\gamma(T)) - S_\tau u_0(\gamma(0)) \leq \phi(\gamma(T)) - \phi(\gamma(0)) + \frac{\tau T}{T-\tau}\omega\left(\frac{\tau}{T-\tau}\right).$$

We skip here the proof of the above two lemmas.

We fix any $u_0 \in \Phi_0$ and define the functions $u^\pm : \mathbf{R}^n \to \mathbf{R}$ by

$$u^+(x) = \limsup_{t\to\infty} S_t u_0(x), \qquad u^-(x) = \liminf_{t\to\infty} S_t u_0(x).$$

It is not hard to see that the function $u(x, t) := S_t u_0(x)$ is bounded and uniformly continuous on $B(0, R) \times [0, \infty)$ for any $R > 0$, the proof of which we refer to [I2, Lemmas 5.1, 5.6, and 5.7]. From this, we see that $u^\pm \in C(\mathbf{R}^n)$ and that $u^+(x) = \limsup^*_{t\to\infty} u(x, t)$ and $u^-(x) = \liminf_{*t\to\infty} u(x, t)$ for all $x \in \mathbf{R}^n$. As is standard in viscosity solutions theory, we have $u^+ \in \mathcal{S}_H^-$ and $u^- \in \mathcal{S}_H^+$. Moreover, by the convexity of $H(x, \cdot)$, we have $u^- \in \mathcal{S}_H^-$ (and hence $u^- \in \mathcal{S}_H$). Also, we have $u^\pm \in \Phi_0$ (see [I2, Lemma 5.1]).

To conclude the proof, it is enough to show that $u^+(x) = u^-(x)$ for all $x \in \mathbf{R}^n$.

We fix any $x \in \mathbf{R}^n$. By Lemma 10, we find an extremal curve $\gamma \in \mathcal{E}((-\infty, 0], u^-)$ such that $\gamma(0) = x$.

We show that $\gamma((-\infty, 0])$ is bounded in $\mathbf{R}^n$. To see this, let $C > 0$ be a constant and set $\psi(x) = \min\{\phi_1(x) + C, u^-(x)\}$ for $x \in \mathbf{R}^n$. We then fix C so that $H(x, D\psi(x)) \leq 0$ a.e. $x \in \mathbf{R}^n$. Using Lemma 3, we get

$$\psi(\gamma(0)) - \psi(\gamma(-t)) \leq \int_{-t}^0 L(\gamma(s), \dot\gamma(s))\, ds \leq u^-(\gamma(0)) - u^-(\gamma(-t)) \quad \text{for all } t \geq 0.$$

Hence we have $u^-(\gamma(-t)) - \psi(\gamma(-t)) \leq u^-(x) - \psi(x)$ for all $t \geq 0$. Since $\lim_{|y|\to\infty}(u^-(y) - \psi(y)) = \infty$, we see that $\gamma((-\infty, 0])$ is a bounded subset of $\mathbf{R}^n$.

By the definition of u^+, we may choose a divergent sequence $\{t_j\} \subset (0, \infty)$ such that $\lim_{j\to\infty} u(x, t_j) = u^+(x)$. Since the sequence $\{\gamma(-t_j)\}$ is bounded in $\mathbf{R}^n$, we may assume by replacing $\{t_j\}$ by one of its subsequences if necessary that $\gamma(-t_j) \to y$ as $j \to \infty$ for some $y \in \mathbf{R}^n$.

Fix any $\varepsilon > 0$, and choose a $\tau > 0$ so that $u^-(y) + \varepsilon > u(y, \tau)$. Let $\delta \in (0, 1)$ and ω be those from Lemma 11. Let $j \in \mathbf{N}$ be so large that $\tau(t_j - \tau)^{-1} \leq \delta$. We now apply Lemma 11, to get

$$u(x, t_j) = u(\gamma(0), t_j) \leq u(\gamma(-t_j), \tau) + u^-(\gamma(0)) - u^-(\gamma(-t_j)) + \frac{\tau t_j}{t_j - \tau}\omega\left(\frac{\tau}{t_j - \tau}\right).$$

Sending $j \to \infty$ yields

$$u^+(x) \leq u(y, \tau) + u^-(x) - u^-(y) < u^-(y) + \varepsilon + u^-(x) - u^-(y) = u^-(x) + \varepsilon,$$

from which we conclude that $u^+(x) \leq u^-(x)$. This completes the proof. □

4. Aubry sets

Let $c = c_H$. Following [FS2], we introduce the Aubry set for $H[u] = c$. We define the function $d_H \in C(\mathbf{R}^n \times \mathbf{R}^n)$ by

$$d_H(x, y) = \sup\{v(x) \mid v \in \mathcal{S}^-_{H-c},\ v(y) = 0\} \tag{4.1}$$

and $\mathcal{A}_H$ as the set of those $y \in \mathbf{R}^n$ for which the function $d_H(\cdot, y)$ is a viscosity solution of $H[u] = c$ in $\mathbf{R}^n$. We call $\mathcal{A}_H$ the *Aubry set* for H or for $H[u] = c$.

Unless otherwise stated, we henceforth *assume* as in the proof of (c) of Theorem 1 that $c = 0$.

The following proposition describes some of basic properties of d_H (see [I2, Section 8]).

Proposition 12. *We have:*
(a) *d_H is locally Lipschitz continuous in $\mathbf{R}^n \times \mathbf{R}^n$.*
(b) *$d_H(y, y) = 0$ for all $y \in \mathbf{R}^n$.*
(c) *$d_H(\cdot, y) \in \mathcal{S}^-_H$ for all $y \in \mathbf{R}^n$.*
(d) *$d_H(\cdot, y)$ is a viscosity solution of $H = 0$ in $\mathbf{R}^n \setminus \{y\}$ for all $y \in \mathbf{R}^n$.*
(e) *$d_H(x, z) \leq d_H(x, y) + d_H(y, z)$ for all $x, y, z \in \mathbf{R}^n$.*

We see from (d) of the above proposition that

$$y \in \mathbf{R}^n \setminus \mathcal{A}_H \iff \exists p \in D^-_1 d_H(y, y) \text{ such that } H(y, p) < 0, \tag{4.2}$$

where $D^-_1 d(x, y)$ denotes the subdifferential at x of the function: $x \mapsto d(x, y)$.

We have the following variational formula for d_H.

Proposition 13 (**[I2, Proposition 8.2]**). *The following formula is valid for all x, $y \in \mathbf{R}^n$:*

$$d_H(x,y) = \inf\Big\{ \int_0^t L(\gamma(s), \dot\gamma(s))ds \;\Big|\; t>0,\ \gamma \in C(x,t;y,0) \Big\}. \tag{4.3}$$

We skip here the proof of the above proposition.

Proposition 14. *We have $y \in \mathbf{R}^n \setminus \mathcal{A}_H$ if and only if there are functions $\phi, \sigma \in C(\mathbf{R}^n)$ such that $\sigma \geq 0$ in $\mathbf{R}^n$, $\sigma(y) > 0$, and $H[\phi] \leq -\sigma$ in $\mathbf{R}^n$ in the viscosity sense.*

Proof. Assume that $y \in \mathbf{R}^n \setminus \mathcal{A}_H$. Set $u = d_H(\cdot, y)$. In view of (4.2), there is a function $\psi \in C^1(\mathbf{R}^n)$ such that $u(y) = \psi(y)$, $u(x) > \psi(x)$ for all $x \in \mathbf{R}^n \setminus \{y\}$, and $H(y, D\psi(y)) < 0$. We may moreover assume that $\lim_{|x|\to\infty}(u-\psi)(x) = \infty$. If we choose $\varepsilon > 0$ sufficiently small and set $\phi(x) = \max\{u(x), \psi(x)+\varepsilon\}$ for $x \in \mathbf{R}^n$, then $\phi \in \mathcal{S}_H^-$ and moreover there is a function $\sigma \in C(\mathbf{R}^n)$ satisfying $\sigma \geq 0$ in $\mathbf{R}^n$ and $\sigma(y) > 0$ such that $H(x, D\phi(x)) \leq -\sigma(x)$ in $\mathbf{R}^n$ in the viscosity sense.

Next, assume that there are functions $\phi, \sigma \in C(\mathbf{R}^n)$ such that $\sigma \geq 0$ in $\mathbf{R}^n$, $\sigma(y) > 0$, and $H(x, D\phi(x)) \leq -\sigma(x)$ in $\mathbf{R}^n$ in the viscosity sense. We may choose a compact neighborhood V of y so that $\sigma(x) > 0$ in V. By a small perturbation of ϕ if necessary, we may assume that $d_H(x,y) > \phi(x) - \phi(y)$ for all $x \in V \setminus \{y\}$. We need to show that $y \in \mathbf{R}^n \setminus \mathcal{A}_H$. For this, we suppose that $y \in \mathcal{A}_H$ and will get a contradiction. Let $\{\phi_k\}_{k\in\mathbf{N}} \subset C^1(\mathbf{R}^n)$ be a sequence converging to ϕ in $C(\mathbf{R}^n)$ such that $H(x, D\phi_k(x)) \leq -\sigma(x)/2$ in V. Let $y_k \in V$ be a minimum point of $d_H(\cdot, y) - \phi_k$ over V. Since $d_H(\cdot,y) - \phi$ has a strict minimum at y over V, we deduce that $y_k \to y$ as $k \to \infty$. Consequently, for sufficiently large k, we have $H(y_k, D\phi(y_k)) \geq 0$, which is a contradiction. □

Proposition 15. *The Aubry set $\mathcal{A}_H$ is a nonempty compact subset of* $\mathbf{R}^n$.

Proof. By Proposition 14, it is easy to see that $\mathbf{R}^n \setminus \mathcal{A}_H$ is an open subset of $\mathbf{R}^n$, which says that $\mathcal{A}_H$ is a closed subset of $\mathbf{R}^n$.

Since $c_H = 0$, by (b) of Theorem 1, there is a function $\phi \in \mathcal{S}_H \cap \Phi_0$. Since $\lim_{|x|\to\infty} \sigma_1(x) = \infty$, we may choose a $C > 0$ so that the function $\psi(x) := \min\{\phi(x), \phi_1(x) + C\}$ is a viscosity subsolution of $H[\psi] = 0$ in $\mathbf{R}^n$. Since $\lim_{|x|\to\infty}(\phi - \phi_1)(x) = \infty$, we see that $H(x, D\psi(x)) \leq -\sigma_1(x)$ in $\mathbf{R}^n \setminus B(0,R)$ in the viscosity sense for some $R > 0$. We choose $r > R$ so that $\sigma_1(x) > 0$ for $\mathbf{R}^n \setminus B(0,r)$ and conclude by Proposition 14 that $\mathcal{A}_H \subset B(0,r)$.

It remains to show that $\mathcal{A}_H \neq \emptyset$. To do so, we suppose that $\mathcal{A}_H = \emptyset$ and will get a contradiction. Let ψ and $r > 0$ be as above. In view of Proposition 14, there are finite sequences $\{y_j\}_{j=1}^N \subset B(0,r)$ and $\{\psi_j\}_{j=1}^N$, $\{f_j\}_{j=1}^N \subset C(\mathbf{R}^n)$ such that $f_j \geq 0$ in $\mathbf{R}^n$ for all j, $H[\psi_j] \leq -f_j$ in $\mathbf{R}^n$ in the viscosity sense for all j, and $B(0,r) \subset \bigcup_{j=1}^N \{x \in \mathbf{R}^n \mid f_j(x) > 0\}$. Set $u = \frac{1}{N+1}\left(\psi + \sum_{j=1}^N \psi_j\right)$, and observe by the convexity of H that u is a viscosity solution of $H[u] \leq -\frac{1}{N+1}\left(\sigma + \sum_{j=1}^n f_j\right)$ in $\mathbf{R}^n$, from which we deduce that there is a $\varepsilon > 0$ such that $u \in \mathcal{S}_{H-\varepsilon}^-$. This is a contradiction in view of Proposition 9. □

In the PDE viewpoint, the following uniqueness property features Aubry sets.

Theorem 16. *Let $v \in \mathcal{S}_H^-$ and $w \in \mathcal{S}_H^+ \cap \Phi_0$. Assume that $v \le w$ on $\mathcal{A}_H$. Then $v \le w$ on $\mathbf{R}^n$.*

Proof. Fix any $\varepsilon > 0$. Choose a compact neighborhood V of $\mathcal{A}_H$ so that $v(x) \le w(x) + \varepsilon$ for all $x \in V$. As in the proof of Proposition 9, we may find a $\psi \in C(\mathbf{R}^n)$ and a $\delta > 0$ such that $H[\psi] \le -\delta$ in $\mathbf{R}^n \setminus V$ in the viscosity sense and $\psi(x) = \phi_1(x)$ for all x, with $|x|$ sufficiently large. Let $\lambda \in (0,1)$ and set $v_\lambda(x) = (1-\lambda)v(x) + \lambda\psi(x) - 2\varepsilon$ for $x \in \mathbf{R}^n$. Observe that $H[v_\lambda] \le -\lambda\delta$ in $\mathbf{R}^n \setminus V$ and that for $\lambda \in (0,1)$ sufficiently small, $v_\lambda(x) \le w(x)$ for all $x \in V$. We may apply standard comparison results, to get $v_\lambda(x) \le w(x)$ for all $x \in \mathbf{R}^n \setminus V$ and all λ sufficiently small. Hence, for $\lambda \in (0,1)$ sufficiently small, we have $v_\lambda(x) \le w(x)$ for all $x \in \mathbf{R}^n$. From this, we obtain $v(x) \le w(x)$ for all $x \in \mathbf{R}^n$. □

The above theorem has the following corollary.

Corollary 17. *Let $u \in \mathcal{S}_H \cap \Phi_0$. Then*

$$u(x) = \inf\{u(y) + d_H(x,y) \mid y \in \mathcal{A}_H\} \quad \textit{for all } x \in \mathbf{R}^n. \tag{4.4}$$

5. Examples

We give two sufficient conditions for H to satisfy (A4).

Example 1. Let $H_0 \in C(\mathbf{R}^n \times \mathbf{R}^n)$ and $f \in C(\mathbf{R}^n)$. Set $H(x,p) = H_0(x,p) - f(x)$ for $(x,p) \in \mathbf{R}^n \times \mathbf{R}^n$. We assume that

$$\lim_{|x|\to\infty} f(x) = \infty, \tag{5.1}$$

and that there exists a $\delta > 0$ such that

$$\sup_{\mathbf{R}^n \times B(0,\delta)} |H_0| < \infty. \tag{5.2}$$

Fix such a $\delta > 0$ and set $C_\delta = \sup_{\mathbf{R}^n \times B(0,\delta)} |H_0|$. Then we define $\phi_i \in C^{0+1}(\mathbf{R}^n)$, with $i = 0,1$, by setting $\phi_0(x) = -\frac{\delta}{2}|x|$ and $\phi_1(x) = -\delta|x|$, and observe that for $i = 0,1$,

$$H_0(x, D\phi_i(x)) \le C_\delta \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}.$$

Hence, for $i = 0,1$, we have

$$H_0(x, D\phi_i(x)) \le \frac{1}{2}f(x) + C_\delta - \frac{1}{2}\min_{\mathbf{R}^n} f \quad \text{for all } x \in \mathbf{R}^n \setminus \{0\}.$$

If we set

$$\sigma_i(x) = \frac{1}{2}f(x) - C_\delta + \frac{1}{2}\min_{\mathbf{R}^n} f \quad \text{for } x \in \mathbf{R}^n \text{ and } i = 0,1,$$

then H satisfies (A4) with these ϕ_i and σ_i, $i = 0, 1$. It is clear that if H_0 satisfies (A1)–(A3), then so does H.

Example 2. Let $\alpha > 0$ and let $H_0 \in C(\mathbf{R}^n)$ be a strictly convex function satisfying the superlinear growth condition

$$\lim_{|p|\to\infty} \frac{H_0(p)}{|p|} = \infty.$$

Let $f \in C(\mathbf{R}^n)$. We set

$$H(x,p) = \alpha x \cdot p + H_0(p) - f(x) \quad \text{for } (x,p) \in \mathbf{R}^n \times \mathbf{R}^n.$$

This class of Hamiltonians H is very close to that treated in [FIL2].

Clearly, this function H satisfies (A1)–(A3). Let L_0 denote the convex conjugate H_0^* of H_0. By the strict convexity of H_0, we see that $L_0 \in C^1(\mathbf{R}^n)$. Define the function $\psi \in C^1(\mathbf{R}^n)$ by $\psi(x) = -\frac{1}{\alpha}L_0(-\alpha x)$. Then we have $D\psi(x) = DL_0(-\alpha x)$ and therefore, by the convex duality, $H_0(D\psi(x)) = D\psi(x)\cdot(-\alpha x) - L_0(-\alpha x)$ for all $x \in \mathbf{R}^n$. Consequently, for all $x \in \mathbf{R}^n$, we have

$$H(x, D\psi(x)) = \alpha x \cdot D\psi(x) + H_0(D\psi(x)) - f(x) = -L_0(-\alpha x) - f(x).$$

Now we assume that there is a convex function $l \in C(\mathbf{R}^n)$ such that

$$\lim_{|x|\to\infty} (l(-\alpha x) + f(x)) = \infty, \tag{5.3}$$

$$\lim_{|\xi|\to\infty} (L_0 - l)(\xi) = \infty. \tag{5.4}$$

Let h denote the convex conjugate of l. We define $\phi \in C^{0+1}(\mathbf{R}^n)$ by $\phi(x) = -\frac{1}{\alpha}l(-\alpha x)$ for $x \in \mathbf{R}^n$. This function ψ is almost everywhere differentiable. Let $x \in \mathbf{R}^n$ be any point where ϕ is differentiable. By a computation similar to the above for ψ, we get

$$\alpha x \cdot D\phi(x) + h(D\phi(x)) - f(x) \le -l(-\alpha x) - f(x). \tag{5.5}$$

By assumption (5.4), there is a $C > 0$ such that $L_0(\xi) \ge l(\xi) - C$ for all $\xi \in \mathbf{R}^n$. This inequality implies that $H_0 \le h + C$ in $\mathbf{R}^n$. Hence, from (5.5), we get

$$H(x, D\phi(x)) \le -l(-\alpha x) - f(x) + C.$$

We now conclude that the function H satisfies (A4), with the functions $\phi_0 = \phi$, $\phi_1 = \psi$, $\sigma_0(x) = l(-\alpha x) + f(x) - C$, and $\sigma_1(x) = L(-\alpha x) + f(x)$.

It is assumed here that H_0 is strictly convex in $\mathbf{R}^n$, while it is only assumed in [FIL2] that H_0 is just convex in $\mathbf{R}^n$, so that L_0 may not be a C^1 function.

The reason why the strict convexity of H_0 is not needed in [FIL2] is in the fact that Hamiltonians H in this class have a simple structure of the Aubry sets. Indeed, if c is the additive eigenvalue of H, then $\min_{p\in\mathbf{R}^n} H(x,p) = c$ for all $x \in \mathcal{A}_H$. Given such a property of the Aubry set, the proof of (c) of Theorem 1 can be simplified greatly and does not require the C^1 regularity of L_0 (see [FIL2]), while such a regularity is needed in the proof of Lemma 11 in the general case. Any $x \in \mathcal{A}_H$ is called an *equilibrium point* if $\min_{p\in\mathbf{R}^n} H(x,p) = c$. A characterization of an equilibrium point $x \in \mathcal{A}_H$ is given by the condition that $L(x,0) = -c$. The property of Aubry sets $\mathcal{A}_H$ mentioned above can be stated that the set $\mathcal{A}_H$ comprises only of equilibrium points.

The following example illustrates the fact that Aubry sets may not contain any equilibrium point.

Example 3. We consider the two-dimensional case. We fix $\alpha, \beta \in \mathbf{R}$ so that $0 < \alpha < \beta$ and choose a function $g \in C([0,\infty))$ so that $g(r) = 0$ for all $r \in [\alpha,\beta]$, $g(r) > 0$ for all $r \in [0,\alpha) \cup (\beta,\infty)$, and $\lim_{r\to\infty} g(r)/r^2 = \infty$. We define the functions $H_0, H \in C(\mathbf{R}^4)$ by

$$\begin{aligned} H_0(x,p) &= (p_1 - x_2)^2 + (p_2 + x_1)^2 - |x|^2, \\ H(x,p) &= H_0(x,p) - g(|x|). \end{aligned}$$

It is easily seen that the function H satisfies (A1)–(A3). Let $\delta > 0$ and set $\psi(x) = -\delta|x|^2$ for $x \in \mathbf{R}^2$. We observe that $D\psi(x) = -2\delta x$ and $H_0(x, D\psi(x)) = 4\delta^2|x|^2$ for all $x \in \mathbf{R}^2$. Therefore, for any $\delta > 0$, if we set $\phi_0(x) = -\delta|x|^2$ and $\phi_1(x) = -2\delta|x|^2$ for $x \in \mathbf{R}^2$, then (A4) holds with these ϕ_0 and ϕ_1.

Noting that the zero function $z = 0$ is a viscosity subsolution of $H[z] = 0$ in $\mathbf{R}^2$, we find that the additive eigenvalue c_H is nonpositive. We fix any $r \in [\alpha,\beta]$ and consider the curve $\gamma \in \mathrm{AC}([0,2\pi])$ given by $\gamma(t) := r(\cos t, \sin t)$. We denote by U the open annulus int $B(0,\beta) \setminus B(0,\alpha)$ for simplicity of notation. Let $\phi \in C^{0+1}(\mathbf{R}^2)$ be a viscosity solution of $H[\phi] = c_H$ in $\mathbf{R}^n$. Such a viscosity solution indeed exists according to (b) of Theorem 1. Due to Lemma 2, there is a function $q = (q_1, q_2) \in L^\infty(0, 2\pi, \mathbf{R}^2)$ such that for almost all $t \in (0, 2\pi)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi(\gamma(t)) = r(-q_1(t)\sin t + q_2(t)\cos t) \quad \text{and} \quad q(t) \in \partial_c\phi(\gamma(t)).$$

The last inclusion guarantees that $H(x(t), q(t)) \le c_H$ a.e. $t \in (0,2\pi)$. Hence, recalling that $\alpha \le r \le \beta$, we get

$$c_H \ge H_0(x(t), q(t)) = |q(t)|^2 - 2\gamma_2(t)q_1(t) + 2\gamma_1(t)q_2(t) \quad \text{a.e. } t \in (0, 2\pi).$$

We calculate that for all $T \in [0, 2\pi]$,

$$\begin{aligned} &\phi(\gamma(T)) - \phi(\gamma(0)) = r\int_0^T (-q_1(t)\sin t + q_2(t)\cos t)\,\mathrm{d}t \\ &= \int_0^T (-q_1(t)\gamma_2(t) + q_2(t)\gamma_1(t))\,\mathrm{d}t \le \frac{1}{2}\int_0^T \left(c_H - |q(t)|^2\right)\mathrm{d}t \le \frac{1}{2}c_H T. \end{aligned}$$

This clearly implies that $c_H = 0$ and also that the function: $t \mapsto \phi(\gamma(t))$ is a constant. Thus we find that $\phi(x) = h(|x|^2)$ for some function $h \in C^{0+1}([\alpha,\beta])$.

Next, we show that ϕ is a constant function in U. For any $r \in (\alpha, \beta)$ and any $x \in \partial B(0,r)$, we have $D\phi(x) = 2h'(|x|^2)x$, and, in particular, $x_2\partial\phi/\partial x_1 - x_1\partial\phi/\partial x_2 = 0$. Therefore, for almost all $x \in U$, we have

$$0 \geq H_0(x, D\phi(x)) = (\partial\phi/\partial x_1 - x_2)^2 + (\partial\phi/\partial x_2 + x_1)^2 - |x|^2 = |D\phi|^2.$$

That is, we have $D\phi(x) = 0$ a.e. $x \in U$, which assures that ϕ is a constant in U.

Now we know that for any $y \in U$, the function: $x \mapsto d_H(x,y)$ is a constant in a neighborhood of y, which guarantees that $U \subset \mathcal{A}_H$ and moreover that $\overline{U} \subset \mathcal{A}_H$. For the function $z = 0$, we have $H[z] = -g(|x|)$ in $\mathbf{R}^n$ in the viscosity sense, which shows that $\mathcal{A}_H \subset \overline{U}$ and hence $\mathcal{A}_H = \overline{U}$.

Finally, we note that $H(x,(x_2,-x_1)) = H_0(x,(x_2,-x_1)) = -|x|^2 < 0$ for all $x \in \overline{U}$, and conclude that any $x \in \mathcal{A}_H = \overline{U}$ is not an equilibrium points.

The following two propositions give sufficient conditions for points of the Aubry set $\mathcal{A}_H$ to be equilibrium points.

Proposition 18. *If y is an isolated point of $\mathcal{A}_H$, then it is an equilibrium point.*

Proof. Let y be an isolated point of $\mathcal{A}_H$. Since $d_H(\cdot,y) \in \mathcal{S}_H$, according to Lemma 10, there exists a curve $\gamma \in \mathcal{E}((-\infty,0], d_H(\cdot,y))$ such that $\gamma(0) = y$.

We show that $\gamma(t) \in \mathcal{A}_H$ for all $t \leq 0$, which guarantees that

$$\gamma(t) = y \qquad \text{for all } t \leq 0. \tag{5.6}$$

For this purpose we fix any $z \in \mathbf{R}^n \setminus \mathcal{A}_H$. By Proposition 14 there are two functions $\phi \in \mathcal{S}_H^- \cap \Phi_0$ and $\sigma \in C(\mathbf{R}^n)$ such that $H[\phi] \leq -\sigma$ in $\mathbf{R}^n$ in the viscosity sense, $\sigma \geq 0$ in $\mathbf{R}^n$, and $\sigma(z) > 0$. By Lemma 3, for any fixed $t > 0$, we have

$$\begin{aligned}\phi(y) - \phi(\gamma(-t)) &\leq \int_{-t}^0 L(\gamma(s),\dot\gamma(s))\mathrm{d}s - \int_{-t}^0 \sigma(\gamma(s))\,\mathrm{d}s \\ &= d_H(y,y) - d_H(\gamma(-t),y) - \int_{-t}^0 \sigma(\gamma(s))\mathrm{d}s.\end{aligned}$$

Accordingly we have

$$\int_{-t}^0 \sigma(\gamma(s))\mathrm{d}s + d_H(\gamma(-t),y) \leq \phi(\gamma(-t)) - \phi(\gamma(0)) \leq d_H(\gamma(-t),y).$$

Hence we get $\int_{-t}^0 \sigma(\gamma(s))\mathrm{d}s \leq 0$, which implies that $\gamma(s) \neq z$ for all $s \leq 0$. Thus we conclude that (5.6) holds.

Now we have

$$0 = d_H(y,y) - d_H(\gamma(-1),y) = \int_{-1}^0 L(\gamma(t),\dot\gamma(t))\mathrm{d}t = L(y,0),$$

which shows that y is an equilibrium point. $\square$

Proposition 19. *Assume that there exists a viscosity solution $w \in C(\mathbf{R}^n)$ of $H(x, Dw) = \min_{p\in\mathbf{R}^n} H(x,p)$ in $\mathbf{R}^n$. Then $\mathcal{A}_H$ consists only of equilibrium points.*

For instance, if $H(x,0) \le H(x,p)$ for all $(x,p) \in \mathbf{R}^{2n}$, then $w = 0$ satisfies $H(x, Dw(x)) = \min_{p\in\mathbf{R}^n} H(x,p)$ for all $x \in \mathbf{R}^n$ in the viscosity sense. If H has the form $H(x,p) = \alpha x\cdot p + H_0(p) - f(x)$ as before, then H attains a minimum as a function of p at a unique point q satisfying $\alpha x + D^- H_0(q) \ni 0$, or equivalently $q = DL_0(-\alpha x)$, that is,

$$\min_{p\in\mathbf{R}^n} H(x,p) = \alpha x\cdot q + H_0(q) - f(x),$$

where L_0 denotes the convex conjugate H_0^* of H_0. Therefore, in this case, the function $w(x) := -(1/\alpha)L_0(-\alpha x)$ is a viscosity solution of $H[w] = \min_{p\in\mathbf{R}^n} H(x,p)$ in $\mathbf{R}^n$. In these two cases, the Aubry sets consist only of equilibrium points.

Proof. Let $c_H = 0$ as usual. We have $\min_{p\in\mathbf{R}^n} H(x,p) \le 0$ for all $x \in \mathbf{R}^n$. Note that the function $\sigma(x) := -\min_{p\in\mathbf{R}^n} H(x,p)$ is continuous on $\mathbf{R}^n$ and that w is a viscosity solution of $H[w] = -\sigma$ in $\mathbf{R}^n$. Applying Proposition 14, we see that if $y \in \mathbf{R}^n$ and $\min_{p\in\mathbf{R}^n} H(y,p) < 0$, then $y \notin \mathcal{A}_H$. That is, if $y \in \mathcal{A}_H$, then $\min_{p\in\mathbf{R}^n} H(y,p) = 0$, which is equivalent that y is an equilibrium point. □

The following example shows that one cannot replace the strict convexity (A3) in (c) of Theorem 1 by the convexity of $H(x,p)$ in p.

Example 4. Consider the Hamiltonian $H \in C(\mathbf{R}^2 \times \mathbf{R}^2)$ given by

$$H(x,p) = H_0(x,p) - \big||x| - 1\big|,$$

where $H_0(x,p) = \sqrt{(p_1 - x_2)^2 + (p_2 + x_1)^2} - |x|$. It is clear that H satisfies (A1) and (A2). Also, H satisfies (A4) with $\phi_0(x) = 0$ and $\phi_1(x) = -|x|$. Moreover, $H(x,p)$ is convex in p on $\mathbf{R}^2$. However, it is not strictly convex in p, i.e., (A4) does not hold.

It is easily checked that the function $\phi_0(x) = 0$ is indeed a viscosity subsolution of $H(x, D\phi_0(x)) = 0$ in $\mathbf{R}^2$, which implies that $c_H \le 0$ by Proposition 9.

Let L denote the Lagrangian of H, and we observe that

$$\begin{aligned}
L(x,\xi) &= L_0(x,\xi) + \big|1 - |x|\big|,\\
L_0(x,\xi) &:= \sup_{p\in\mathbf{R}^2} (p\cdot\xi - H_0(x,p)) = \delta_{B(0,1)}(\xi) + x_2\xi_1 - x_1\xi_2 + |x|,\\
L_0(x,\xi) &\ge \delta_{B(0,1)}(\xi) \ge 0,
\end{aligned}$$

where δ_B denotes the indicator function of the set B, i.e., $\delta_B(\xi) = 0$ for $\xi \in B$ and $= \infty$ for $\xi \in \mathbf{R}^n \setminus B$.

Let $\phi \in C(\mathbf{R}^2)$ be a subsolution of $H(x, D\phi(x)) \le c_H$ in $\mathbf{R}^2$. Consider the curve $\gamma(t) = (\cos t, \sin t)$, with $t \in [0, 2\pi]$, and observe that

$$0 = \phi(\gamma(2\pi)) - \phi(\gamma(0)) \le \int_0^{2\pi} (L(\gamma(t), \dot\gamma(t)) + c_H)\,dt = 2\pi c_H,$$

from which we see that $c_H \ge 0$. We now conclude that $c_H = 0$.

Let $u_0 \in \mathrm{BUC}(\mathbf{R}^n)$ be such that $u_0(e_1) = 0$, where $e_1 = (1, 0)$, and $u_0(x) > 0$ for all $x \in \mathbf{R}^2 \setminus \{e_1\}$, and we consider the Cauchy problem

$$u_t(x,t) + H(x, Du(x,t)) = 0 \quad \text{in } \mathbf{R}^2 \times (0,\infty) \quad \text{and} \quad u(\cdot, 0) = u_0. \tag{5.7}$$

The formula (2.1) for the solution u of (5.7) tells us that $u(x,t) \geq 0$ for all $(x,t) \in \mathbf{R}^2 \times [0,\infty)$, and for any $k \in \mathbf{N}$,

$$u(e_1, 2k\pi) \leq \int_0^{2k\pi} L(\gamma(t), \dot{\gamma}(t))\, dt + u_0(\gamma(0)) = 0,$$

where $\gamma(t) = (\cos t, \sin t)$ for all $t \geq 0$. In particular, we have $u(e_1, 2k\pi) = 0$ for all $k \in \mathbf{N}$.

We show that there is a $\varepsilon > 0$ such that

$$u(e_1, (2k+1)\pi) \geq \varepsilon \quad \text{for all } k \in \mathbf{N}. \tag{5.8}$$

Indeed, as we will show, (5.8) holds with $\varepsilon = \min\{1/8, m/2\}$, where $m = \min\{u_0(x) \mid x \in K\}$ and $K = \{(x_1, x_2) \in B(0, 3/2) \mid x_1 \leq 0\}$.

Let $k \in \mathbf{N}$. We set $\varepsilon = \min\{1/8, m/2\}$ and $T = (2k+1)\pi$. We argue by contradiction that $u(e_1, T) \geq \varepsilon$, and thus suppose that $u(e_1, T) < \varepsilon$. We can choose a $\gamma \in \mathcal{C}(x, T)$ so that

$$\varepsilon > \int_0^T L(\gamma(t), \dot{\gamma}(t))\, dt + u_0(\gamma(0)). \tag{5.9}$$

Next, noting that $\dot{\gamma}(t) \in B(0,1)$ and hence $|(d/dt)|\gamma(t)|| \leq 1$ a.e. $t \in (0,T)$, we compute that for any $t \in [0,T]$,

$$(|\gamma(t)| - 1)^2 = -2\int_t^T (|\gamma(s)| - 1)\frac{d|\gamma(s)|}{ds}\, ds,$$

and therefore, by (5.9),

$$(|\gamma(t)| - 1)^2 \leq 2\int_0^T ||\gamma(s)| - 1|\, ds < 2\varepsilon.$$

Hence we have $||\gamma(t)| - 1| < (2\varepsilon)^{\frac{1}{2}} \leq 1/2$ for all $t \in [0,T]$. That is, we have $1/2 < |\gamma(t)| < 3/2$ for all $t \in [0,T]$.

We now use the polar coordinates, that is, we choose functions $r, \theta \in \mathrm{AC}([0,T])$ so that $\gamma(t) = (r(t)\cos\theta(t), r(t)\sin\theta(t))$ and $r(t) \geq 0$ for all $t \in [0,T]$ and $\theta(T) = 0$. Such functions r and θ exist because $\gamma(t) \neq 0$ for all $t \in [0,T]$. Note that $|\dot{\gamma}(t)|^2 = \dot{r}(t)^2 + r(t)^2\dot{\theta}(t)^2 \leq 1$ and $L_0(\gamma(t), \dot{\gamma}(t)) = r(t)(1 - r(t)\dot{\theta}(t))$ a.e. $t \in [0,T]$. Inequality (5.9) reads

$$\varepsilon > \int_0^T \Big[r(t)(1 - r(t)\dot{\theta}(t)) + |r(t) - 1|\Big]\, dt + u_0(\gamma(0)).$$

From this, since $r(t) > 1/2$ for all $t \in [0, T]$, we get $\varepsilon > \frac{1}{2}\int_0^T (1 - r(t)\dot\theta(t))\,\mathrm{d}t$. Note also that $|\dot\theta(t)| \le 1/r(t) < 2$ a.e. $t \in (0, T)$. Combining these observations, we get

$$\left|\int_0^T (1-\dot\theta(t))\,\mathrm{d}t\right| \le \left|\int_0^T (1-r(t)\dot\theta(t))\,\mathrm{d}t\right| + \left|\int_0^T (r(t)-1)\dot\theta(t)\,\mathrm{d}t\right|$$
$$< 2\varepsilon + \int_0^T |r(t)-1||\dot\theta(t)|\,\mathrm{d}t \le 2\varepsilon + 2\int_0^T |r(t)-1|\,\mathrm{d}t < 4\varepsilon,$$

from which we obtain $|T + \theta(0)| < 4\varepsilon \le 1/2 \le \pi/2$. Hence we have $\theta(0) \in [-2k\pi - 3\pi/2, -2k\pi - \pi/2]$. Thus we get $\gamma(0) \in K$ and moreover $\varepsilon > u_0(\gamma(0)) \ge m$, but this contradicts our choice of ε. We conclude that (5.8) holds and that the limit, as $t \to \infty$, of $u(e_1, t)$ does not exist.

6. Characterizations of the asymptotic solutions

The function v in assertion (c) of Theorem 1 is characterized as follows.

Theorem 19 ([I2, Theorem 8.1]). *Let $v \in C(\mathbf{R}^n)$ be the function from (c) of Theorem 1. Then, for any $x \in \mathbf{R}^n$,*

$$v(x) = \inf\{d_H(x,y) + d_H(y,z) + u_0(z) \mid y \in \mathcal{A}_H,\ z \in \mathbf{R}^n\}. \tag{6.1}$$

We do not give here the proof of the above theorem.

Theorem 20. *Let u_0 and v be from Theorem 1. Assume that $c_H = 0$. Then*

$$v(x) = \inf\{\phi(x) \mid \phi \in \mathcal{S}_H,\ \phi \ge u_0^- \ \text{ in } \mathbf{R}^n\} \quad \text{for all } x \in \mathbf{R}^n, \tag{6.2}$$

where u_0^- is the function on $\mathbf{R}^n$ given by

$$u_0^-(x) = \sup\{\psi(x) \mid \psi \in \mathcal{S}_H^-,\ \psi \le u_0 \ \text{ in } \mathbf{R}^n\}.$$

The formula (6.2) has been obtained in [I1, Theorem 2.2] under slightly different assumptions.

Proof. We write temporarily

$$f(x) = \inf\{d_H(x,y) + u_0(y) \mid y \in \mathbf{R}^n\} \quad \text{for } x \in \mathbf{R}^n,$$
$$g(x) = \inf\{\phi(x) \mid \phi \in \mathcal{S}_H,\ \phi \ge u_0^- \text{ in } \mathbf{R}^n\} \quad \text{for } x \in \mathbf{R}^n.$$

By Theorem 19, we have $v(x) = \inf\{d_H(x,y) + f(y) \mid y \in \mathcal{A}_H\}$ for all $x \in \mathbf{R}^n$. Thus, we need to show that

$$g(x) = \inf\{d_H(x,y) + f(y) \mid y \in \mathcal{A}_H\} \quad \text{for all } x \in \mathbf{R}^n. \tag{6.3}$$

We write $h(x)$ for the right hand side of (6.3).

We first observe that $f = u_0^-$. Indeed, since $f \in \mathcal{S}_H^-$ and $f \le u_0$ in $\mathbf{R}^n$, we see that $f \le u_0^-$ in $\mathbf{R}^n$. On the other hand, since $u_0^- \in \mathcal{S}_H^-$ and $u_0^- \le u_0$ in $\mathbf{R}^n$, we see that $u_0^-(x) \le d_H(x,y) + u_0(y)$ for all $x, y \in \mathbf{R}^n$ and therefore $u_0^- \le f$ in $\mathbf{R}^n$. Thus we have $u_0^- = f$ in $\mathbf{R}^n$.

Next we observe that $d_H(x,y) + u_0^-(y) \ge u_0^-(x)$ for all $x, y \in \mathbf{R}^n$, $d_H(\cdot, y) + u_0^-(y) \in \mathcal{S}_H$ for all $y \in \mathcal{A}_H$ and hence $g(x) \le \inf\{d_H(x,y) + u_0^-(y) \mid y \in \mathcal{A}_H\} = h(x)$ for all $x \in \mathbf{R}^n$. In particular, we have $u_0^-(x) \le g(x) \le h(x) \le u_0^-(x)$ for all $x \in \mathcal{A}_H$. Hence, $g(x) = h(x)$ for all $x \in \mathcal{A}_H$. Since $u_0 \in \Phi_0$, we may choose a $C > 0$ so that $u_0 \ge \phi_0 - C$ in $\mathbf{R}^n$. We may assume without loss of generality that $\phi_0 \in \mathcal{S}_H^-$. By the definition of u_0^-, we see that $u_0^- \ge \phi_0 - C$ in $\mathbf{R}^n$. This ensures that $g \ge \phi_0 - C$ and therefore $g, h \in \Phi_0$. Finally, noting that $g, h \in \mathcal{S}_H$, we apply Theorem 16, to conclude that $g = h$ in $\mathbf{R}^n$. □

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