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INTRODUCTION TO A THEORY OF b-FUNCTIONS

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We give an introduction to a theory of b-functions, i.e. Bernstein-Sato polynomials. After reviewing some facts from D-modules, we introduce b-functions including the one for arbitrary ideals of the structure sheaf. We explain the relation with singularities, multiplier ideals, etc., and calculate the b-functions of monomial ideals and also of hyperplane arrangements in certain cases.

1. D-modules.

1.1. Let X be a complex manifold or a smooth algebraic variety over $\mathbf{C}$. Let $\mathcal{D}_X$ be the ring of partial differential operators. A local section of $\mathcal{D}_X$ is written as

$$\textstyle\sum_{\nu\in\mathbf{N}^n} a_\nu\partial_1^{\nu_1}\cdots\partial_n^{\nu_n}\in\mathcal{D}_X \quad \text{with } a_\nu\in\mathcal{O}_X,$$

where $\partial_i=\partial/\partial x_i$ with $(x_1,\dots,x_n)$ a local coordinate system.

Let F be the filtration by the order of operators i.e.

$$F_p\mathcal{D}_X=\{\textstyle\sum_{|\nu|\le p}a_\nu\partial_1^{\nu_1}\cdots\partial_n^{\nu_n}\},$$

where $|\nu|=\sum_i\nu_i$. Let $\xi_i=\mathrm{Gr}_1^F\partial_i\in\mathrm{Gr}_1^F\mathcal{D}_X$. Then

$$\begin{aligned}&\mathrm{Gr}^F\mathcal{D}_X:=\textstyle\bigoplus_p\mathrm{Gr}_p^F\mathcal{D}_X=\bigoplus_p\mathrm{Sym}^p\Theta_X\ (=\mathcal{O}_X[\xi_1,\dots,\xi_n]\ \text{locally}),\\ &\mathcal{S}pec_X\mathrm{Gr}^F\mathcal{D}_X=T^*X.\end{aligned}\tag{1.1.1}$$

1.2 Definition. We say that a left $\mathcal{D}_X$-module M is *coherent* if it has locally a finite presentation

$$\textstyle\bigoplus\mathcal{D}_X\to\bigoplus\mathcal{D}_X\to M\to 0.$$

1.3. Remark. A left $\mathcal{D}_X$-module M is coherent if and only if it is quasi-coherent over $\mathcal{O}_X$ and locally finitely generated over $\mathcal{D}_X$. (It is known that $\mathrm{Gr}^F\mathcal{D}_X$ is a noetherian ring, i.e. an increasing sequence of locally finitely generated $\mathrm{Gr}^F\mathcal{D}_X$-submodules of a coherent $\mathrm{Gr}^F\mathcal{D}_X$-module is locally stationary.)

1.4. Definition. A filtration F on a left $\mathcal{D}_X$-module M is *good* if (M,F) is a coherent filtered $\mathcal{D}_X$-module, i.e. if $F_p\mathcal{D}_XF_qM\subset M_{p+q}$ and $\mathrm{Gr}^FM:=\bigoplus_p\mathrm{Gr}_p^FM$ is coherent over $\mathrm{Gr}^F\mathcal{D}_X$.

1.5. Remark. A left $\mathcal{D}_X$-module M is coherent if and only if it has a good filtration locally.

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1.6. Characteristic varieties. For a coherent left $\mathcal{D}_X$-module M, we define the characteristic variety $\mathrm{CV}(M)$ by

$$\mathrm{CV}(M) = \mathrm{Supp}\,\mathrm{Gr}^F M \subset T^*M, \tag{1.6.1}$$

taking locally a good filtration F of M.

1.7. Remark. The above definition is independent of the choice of F. If $M = \mathcal{D}_X/\mathcal{I}$ for a coherent left ideal $\mathcal{I}$ of $\mathcal{D}_X$, take $P_i \in F_{k_i}\mathcal{I}$ such that the $\rho_i := \mathrm{Gr}^F_{k_i} P_i$ generate $\mathrm{Gr}^F\mathcal{I}$ over $\mathrm{Gr}^F\mathcal{D}_X$. Then $\mathrm{CV}(M)$ is defined by the $\rho_i \in \mathcal{O}_X[\xi_1, \dots, \xi_n]$.

1.8. Theorem (Sato, Kawai, Kashiwara [39], Bernstein [2]). *We have the inequality* $\dim \mathrm{CV}(M) \ge \dim X$. (*More precisely,* $\mathrm{CV}(M)$ *is involutive, see* [39].)

1.9. Definition. We say that a left $\mathcal{D}_X$-module M is *holonomic* if it is coherent and $\dim \mathrm{CV}(M) = \dim X$.

2. De Rham functor.

2.1. Definition. For a left $\mathcal{D}_X$-module M, we define the de Rham functor $\mathrm{DR}(M)$ by

$$M \to \Omega^1_X \otimes_{\mathcal{O}_X} M \to \cdots \to \Omega^{\dim X}_X \otimes_{\mathcal{O}_X} M, \tag{2.1.1}$$

where the last term is put at the degree 0. In the algebraic case, we use analytic sheaves or replace M with the associated analytic sheaf $M^{\mathrm{an}} := M \otimes_{\mathcal{O}_X} \mathcal{O}_{X^{\mathrm{an}}}$ in case M is algebraic (i.e. M is an $\mathcal{O}_X$-module with $\mathcal{O}_X$ algebraic).

2.2. Perverse sheaves. Let $D^b_c(X, \mathbf{C})$ be the derived category of bounded complexes of $\mathbf{C}_X$-modules K with $\mathcal{H}^j K$ constructible. (In the algebraic case we use analytic topology for the sheaves although we use Zariski topology for constructibility.) Then the category of perverse sheaves $\mathrm{Perv}(X, \mathbf{C})$ is a full subcategory of $D^b_c(X, \mathbf{C})$ consisting of K such that

$$\dim \mathrm{Supp}\,\mathcal{H}^{-j}K \le j, \quad \dim \mathrm{Supp}\,\mathcal{H}^{-j}\mathbf{D}K \le j, \tag{2.2.1}$$

where $\mathbf{D}K := \mathbf{R}\mathcal{H}om(K, \mathbf{C}[2\dim X])$ is the dual of K, and $\mathcal{H}^j K$ is the j-th cohomology sheaf of K.

2.3. Theorem (Beilinson, Bernstein, Deligne [1]). $\mathrm{Perv}(X, \mathbf{C})$ *is an abelian category.*

2.4. Theorem (Kashiwara). *If* M *is holonomic, then* $\mathrm{DR}(M)$ *is a perverse sheaf.*

Outline of proof. By Kashiwara [19], we have $\mathrm{DR}(M) \in D^b_c(X, \mathbf{C})$, and the first condition of (2.2.1) is verified. Then the assertion follows from the commutativity of the dual $\mathbf{D}$ and the de Rham functor DR.

2.5. Example. $\mathrm{DR}(\mathcal{O}_X) = \mathbf{C}_X[\dim X]$.

2.6. Direct images. For a closed immersion $i : X \to Y$ such that X is defined by $x_i = 0$ in Y for $1 \le i \le r$, define the direct image of left $\mathcal{D}_X$-modules M by

$$i_+ M := M[\partial_1, \dots, \partial_r].$$

(Globally there is a twist by a line bundle.) For a projection $p : X \times Y \to Y$, define

$$p_+M = \mathbf{R}p_*\mathrm{DR}_X(M).$$

In general, $f_+ = p_+i_+$ using $f = pi$ with i graph embedding. See [4] for details.

2.7. Regular holonomic D-modules. Let M be a holonomic $\mathcal{D}_X$-module with support Z, and U be a Zariski-open of Z such that $\mathrm{DR}(M)|_U$ is a local system up to a shift. Then M is *regular* if and only if there exists locally a divisor D on X containing $Z \setminus U$ and such that $M(*D)$ is the direct image of a regular holonomic $\mathcal{D}$-module 'of Deligne-type' (see [11]) on a desingularization of $(Z, Z \cap D)$, and $\mathrm{Ker}(M \to M(*D))$ is regular holonomic (by induction on $\dim \mathrm{Supp}\, M$).

Note that the category $M_{rh}(\mathcal{D}_X)$ of regular holonomic $\mathcal{D}_X$-modules is stable by subquotients and extensions in the category $M_h(\mathcal{D}_X)$ of holonomic $\mathcal{D}_X$-modules.

2.8. Theorem (Kashiwara-Kawai [24], [22], Mebkhout [28]).
(i) The structure sheaf $\mathcal{O}_X$ is regular holonomic.
(ii) The functor DR induces an equivalence of categories

$$\mathrm{DR} : M_{rh}(\mathcal{D}_X) \xrightarrow{\sim} \mathrm{Perv}(X, \mathbf{C}). \tag{2.8.1}$$

(See [4] for the algebraic case.)

3. b-Functions.

3.1. Definition. Let f be a holomorphic function on X, or $f \in \Gamma(X, \mathcal{O}_X)$ in the algebraic case. Then we have

$$\mathcal{D}_X[s]f^s \subset \mathcal{O}_X[\tfrac{1}{f}][s]f^s \quad \text{where } \partial_i f^s = s(\partial_i f)f^{s-1},$$

and $b_f(s)$ is the monic polynomial of the least degree satisfying

$$b_f(s)f^s = P(x, \partial, s)f^{s+1} \quad \text{in } \mathcal{O}_X[\tfrac{1}{f}][s]f^s,$$

with $P(x, \partial, s) \in \mathcal{D}_X[s]$. Locally, it is the minimal polynomial of the action of s on

$$\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}.$$

We define $b_{f,x}(s)$ replacing $\mathcal{D}_X$ with $\mathcal{D}_{X,x}$.

3.2. Theorem (Sato [38], Bernstein [2], Bjork [3]). *The b-function exists at least locally, and exists globally in the case X affine variety with f algebraic.*

3.3. Observation. Let $i_f : X \to \widetilde{X} := X \times \mathbf{C}$ be the graph embedding. Then there are canonical isomorphisms

$$\widetilde{M} := i_{f+}\mathcal{O}_X = \mathcal{O}_X[\partial_t]\delta(f-t) = \mathcal{O}_{X\times\mathbf{C}}[\tfrac{1}{f-t}]/\mathcal{O}_{X\times\mathbf{C}}, \tag{3.3.1}$$

where the action of ∂_i on $\delta(f-t)$ $(= \frac{1}{f-t})$ is given by

$$\partial_i\delta(f-t) = -(\partial_i f)\partial_t\delta(f-t). \tag{3.3.2}$$

Moreover, f^s is canonically identified with $\delta(f-t)$ setting $s = -\partial_t t$, and we have a canonical isomorphism as $\mathcal{D}_X[s]$-modules

$$\mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f-t). \tag{3.3.3}$$

3.4. V-filtration. We say that V is a filtration of Kashiwara-Malgrange if V is exhaustive, separated, and satisfies for any $\alpha \in \mathbf{Q}$:

(i) $V^\alpha \widetilde{M}$ is a coherent $\mathcal{D}_X[s]$-submodule of $\widetilde{M}$.

(ii) $tV^\alpha \widetilde{M} \subset V^{\alpha+1}\widetilde{M}$ and $=$ holds for $\alpha \gg 0$.

(iii) $\partial_t V^\alpha \widetilde{M} \subset V^{\alpha-1}\widetilde{M}$.

(iv) $\partial_t t - \alpha$ is nilpotent on $\mathrm{Gr}_V^\alpha \widetilde{M}$.

If it exists, it is unique.

3.5. Relation with the *b*-function. If X is affine or Stein and relatively compact, then the multiplicity of a root α of $b_f(s)$ is given by the minimal polynomial of $s - \alpha$ on

$$\mathrm{Gr}_V^\alpha(\mathcal{D}_X[s]f^s/\mathcal{D}_X[s]f^{s+1}), \tag{3.5.1}$$

using $\mathcal{D}_X[s]f^s = \mathcal{D}_X[s]\delta(f-t)$ with $s = -\partial_t t$.

Note that $V^\alpha\widetilde{M}$ and $\mathcal{D}_X[s]f^{s+i}$ are 'lattices' of $\widetilde{M}$, i.e.

$$V^\alpha\widetilde{M} \subset \mathcal{D}_X[s]f^{s+i} \subset V^\beta\widetilde{M} \quad \text{for } \alpha \gg i \gg \beta, \tag{3.5.2}$$

and $V^\alpha\widetilde{M}$ is an analogue of the Deligne extension with eigenvalues in $[\alpha, \alpha+1)$. The existence of V is equivalent to the existence of $b_f(s)$ locally.

3.6. Theorem (Kashiwara [21], [23], Malgrange [27]). *The filtration V exists on $\widetilde{M} := i_{f+}M$ for any holonomic $\mathcal{D}_X$-module M.*

3.7. Remarks. (i) There are many ways to prove this theorem, since it is essentially equivalent to the existence of the *b*-function (in a generalized sense). One way is to use a resolution of singularities and reduce to the case where $\mathrm{CV}(M)$ has normal crossings, if M is regular.

(ii) The filtration V is indexed by $\mathbf{Q}$ if M is quasi-unipotent.

3.8. Relation with vanishing cycle functors. Let $\rho : X_t \to X_0$ be a 'good' retraction (using a resolution of singularities of (X, X_0)), where $X_t = f^{-1}(t)$ with $t \neq 0$ sufficiently near 0. Then we have canonical isomorphisms

$$\psi_f \mathbf{C}_X = \mathbf{R}\rho_* \mathbf{C}_{X_t}, \quad \varphi_f \mathbf{C}_X = \psi_f \mathbf{C}_X / \mathbf{C}_{X_0}, \tag{3.8.1}$$

where $\psi_f \mathbf{C}_X, \varphi_f \mathbf{C}_X$ are nearby and vanishing cycle sheaves, see [13].

Let F_x denote the Milnor fiber around $x \in X_0$. Then

$$(\mathcal{H}^j \psi_f \mathbf{C}_X)_x = H^j(F_x, \mathbf{C}), \quad (\mathcal{H}^j \varphi_f \mathbf{C}_X)_x = \widetilde{H}^j(F_x, \mathbf{C}). \tag{3.8.2}$$

For a $\mathcal{D}_X$-module M admitting the V-filtration on $\widetilde{M} = i_{*+}M$, we define $\mathcal{D}_X$-modules

$$\psi_f M = \textstyle\bigoplus_{0<\alpha\leq 1} \mathrm{Gr}_V^\alpha \widetilde{M}, \quad \varphi_f M = \bigoplus_{0\leq\alpha<1} \mathrm{Gr}_V^\alpha \widetilde{M}. \tag{3.8.3}$$

3.9. Theorem (Kashiwara [23], Malgrange [27]). *For a regular holonomic $\mathcal{D}_X$-module M, we have canonical isomorphisms*

$$\begin{aligned} \mathrm{DR}_X \psi_f(M) &= \psi_f \mathrm{DR}_X(M)[-1], \\ \mathrm{DR}_X \varphi_f(M) &= \varphi_f \mathrm{DR}_X(M)[-1], \end{aligned} \tag{3.9.1}$$

and $\exp(-2\pi i\partial_t t)$ *on the left-hand side corresponds to the monodromy* T *on the right-hand side.*

3.10. Definition. Let

$R_f = \{\text{roots of } b_f(-s)\}$,
$\alpha_f = \min R_f$,
m_α : the multiplicity of $\alpha \in R_f$.

(Similarly for $R_{f,x}$, etc. for $b_{f,x}(s)$.)

3.11. Theorem (Kashiwara [20]). $R_f \subset \mathbf{Q}_{>0}$.

(This is proved by using a resolution of singularities.)

3.12. Theorem (Kashiwara [23], Malgrange [27]).

(i) $e^{-2\pi i R_f} = \{$*the eigenvalues of* T *on* $H^j(F_x, \mathbf{C})$ *for* $x \in X_0, j \in \mathbf{Z}\}$,
(ii) $m_\alpha \le \min\{i \mid N^i \psi_{f,\lambda}\mathbf{C}_X = 0\}$ *with* $\lambda = e^{-2\pi i\alpha}$,
where $\psi_{f,\lambda} = \mathrm{Ker}(T_s - \lambda) \subset \psi_f$, $N = \log T_u$ *with* $T = T_s T_u$.

(This is a corollary of the above Theorem (3.9) of Kashiwara and Malgrange.)

4. Relation with other invariants.

4.1. Microlocal b-function. We define $\widetilde{R}_f, \widetilde{m}_\alpha, \widetilde{\alpha}_f$ with $b_f(s)$ replaced by the *microlocal* (or reduced) b-function

(4.1.1) $$\widetilde{b}_f(s) := b_f(s)/(s+1).$$

This $\widetilde{b}_f(s)$ coincides with the monic polynomial of the least degree satisfying

(4.1.2) $$\widetilde{b}_f(s)\delta(f-t) = \widetilde{P}\partial_t^{-1}\delta(f-t) \quad \text{with } \widetilde{P} \in \mathcal{D}_X[s, \partial_t^{-1}].$$

Put $n = \dim X$. Then

4.2. Theorem. $\widetilde{R}_f \subset [\widetilde{\alpha}_f, n - \widetilde{\alpha}_f]$, $\quad \widetilde{m}_\alpha \le n - \widetilde{\alpha}_f - \alpha + 1$.

(The proof uses the filtered duality for φ_f, see [35].)

4.3. Spectrum. We define the spectrum by $\mathrm{Sp}(f,x) = \sum_\alpha n_\alpha t^\alpha$ with

(4.3.1) $$n_\alpha := \textstyle\sum_j (-1)^{j-n+1} \dim \mathrm{Gr}_F^p \widetilde{H}^j(F_x, \mathbf{C})_\lambda,$$

where $p = [n-\alpha]$, $\lambda = e^{-2\pi i\alpha}$, and F is the Hodge filtration (see [12]) of the mixed Hodge structure on the Milnor cohomology, see [44]. We define

(4.3.2) $$E_f = \{\alpha \mid n_\alpha \ne 0\} \text{ (called the exponents)}.$$

4.4. Remarks. (i) If f has an isolated singularity at the origin, then $\widetilde{\alpha}_{f,x}$ coincides with the minimal exponent as a corollary of results of Malgrange [26], Varchenko [45], Scherk-Steenbrink [41].

(ii) If f is weighted-homogeneous with an isolated singularity at the origin, then by Kashiwara (unpublished)

(4.4.1) $$\widetilde{R}_f = E_f, \quad \max \widetilde{R}_f = n - \widetilde{\alpha}_f, \quad \widetilde{m}_\alpha = 1 \ (\alpha \in \widetilde{R}_f).$$

If $f = \sum_i x_i^2$, then $\widetilde{\alpha}_f = n/2$ and this follows from the above Theorem (4.2).
By Steenbrink [42], we have moreover

$$\text{(4.4.2)} \qquad \mathrm{Sp}(f, x) = \textstyle\prod_i (t - t^{w_i})/(t^{w_i} - 1),$$

where $(w_1, \ldots, w_n)$ is the weights of f, i.e. f is a linear combination of monomials $x_1^{m_1} \cdots x_n^{m_n}$ with $\sum_i w_i m_i = 1$.

4.5. Malgrange's formula (isolated singularities case). We have the Brieskorn lattice [5] and its saturation defined by

$$\text{(4.5.1)} \qquad H_f'' = \Omega_{X,x}^n / df \wedge d\Omega_{X,x}^{n-2}, \quad \widetilde{H}_f'' = \textstyle\sum_{i \geq 0} (t\partial_t)^i H_f'' \subset H_f''[t^{-1}].$$

These are finite $\mathbf{C}\{t\}$-modules with a regular singular connection.

4.6. Theorem (Malgrange [26]). *The reduced b-function $\widetilde{b}_f(s)$ coincides with the minimal polynomial of $-\partial_t t$ on $\widetilde{H}_f''/t\widetilde{H}_f''$.*

(The above formula of Kashiwara on b-function (4.4.1) can be proved by using this together with Brieskorn's calculation.)

4.7. Asymptotic Hodge structure (Varchenko [45], Scherk-Steenbrink [41]). *In the isolated singularity case we have*

$$\text{(4.7.1)} \qquad F^p H^{n-1}(F_x, \mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha H_f'',$$

using the canonical isomorphism

$$\text{(4.7.2)} \qquad H^{n-1}(F_x, \mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha H_f''[t^{-1}],$$

where $p = [n - \alpha]$, $\lambda = e^{-2\pi i\alpha}$, and V on $H_f''[t^{-1}]$ is the filtration of Kashiwara and Malgrange.

(This can be generalized to the non-isolated singularity case using mixed Hodge modules.)

4.8. Reformulation of Malgrange's formula. We define

$$\text{(4.8.1)} \qquad \widetilde{F}^p H^{n-1}(F_x, \mathbf{C})_\lambda = \mathrm{Gr}_V^\alpha \widetilde{H}_f'',$$

using the canonical isomorphism (4.7.2), where $p = [n - \alpha]$, $\lambda = e^{-2\pi i\alpha}$. Then

$$\text{(4.8.2)} \qquad \widetilde{m}_\alpha = \text{ the minimal polynomial of } N \text{ on } \mathrm{Gr}_{\widetilde{F}}^p H^{n-1}(F_x, \mathbf{C})_\lambda.$$

4.9. Remark. If f is weighted homogeneous with an isolated singularity, then

$$\text{(4.9.1)} \qquad \widetilde{F} = F, \quad \widetilde{R}_f = E_f \text{ (by Kashiwara).}$$

If f is not weighted homogeneous (but with isolated singularities), then

$$\text{(4.9.2)} \qquad \widetilde{R}_f \subset \textstyle\bigcup_{k \in \mathbf{N}} (E_f - k), \quad \widetilde{\alpha}_f = \min \widetilde{R}_f = \min E_f.$$

4.10. Example. If $f = x^5 + y^4 + x^3y^2$, then

$$E_f = \left\{ \frac{i}{5} + \frac{j}{4} : 1 \leq i \leq 4,\ 1 \leq j \leq 3 \right\}, \quad \widetilde{R}_f = E_f \cup \left\{ \frac{11}{20} \right\} \setminus \left\{ \frac{31}{20} \right\}.$$

More generally, if $f = g + h$ with g weighted homogeneous and h is a linear combination of monomials of higher degrees, then $E_f = E_g$ but $\widetilde{R}_f \neq \widetilde{R}_g$ if f is a non trivial deformation.

4.11. Relation with rational singularities [34]. *Assume $D := f^{-1}(0)$ is reduced. Then D has rational singularities if and only if $\widetilde{\alpha}_f > 1$. Moreover, $\omega_D/\rho_*\omega_{\widetilde{D}} \simeq F_{1-n}\varphi_f\mathcal{O}_X$, where $\rho : \widetilde{D} \to D$ is a resolution of singularities.*

In the isolated singularities case, this was proved in 1981 (see [31]) using the coincidence of $\widetilde{\alpha}_f$ and the minimal exponent.

4.12. Relation with the pole order filtration [34]. *Let P be the pole order filtration on $\mathcal{O}_X(*D)$, i.e. $P_i = \mathcal{O}_X((i+1)D)$ if $i \geq 0$, and $P_i = 0$ if $i < 0$. Let F be the Hodge filtration on $\mathcal{O}_X(*D)$. Then $F_i \subset P_i$ in general, and $F_i = P_i$ on a neighborhood of x for $i \leq \widetilde{\alpha}_{f,x} - 1$.*

(For the proof we need the theory of microlocal b-functions [35].)

4.13. Remark. In case $X = \mathbf{P}^n$, replacing $\widetilde{\alpha}_{f,x}$ with $[(n-r)/d]$ where $r = \dim \operatorname{Sing} D$ and $d = \deg D$, the assertion was obtained by Deligne (unpublished).

5. Relation with multiplier ideals.

5.1. Multiplier ideals. Let $D = f^{-1}(0)$, and $\mathcal{J}(X, \alpha D)$ be the multiplier ideals for $\alpha \in \mathbf{Q}$, i.e.

(5.1.1) $$\mathcal{J}(X, \alpha D) = \rho_*\omega_{\widetilde{X}/X}(-\textstyle\sum_i [\alpha m_i]\widetilde{D}_i)),$$

where $\rho : (\widetilde{X}, \widetilde{D}) \to (X, D)$ is an embedded resolution and $\widetilde{D} = \sum_i m_i\widetilde{D}_i := \rho^*D$. There exist jumping numbers $0 < \alpha_0 < \alpha_1 < \cdots$ such that

(5.1.2) $$\mathcal{J}(X, \alpha_j D) = \mathcal{J}(X, \alpha D) \neq \mathcal{J}(X, \alpha_{j+1} D) \quad \text{for} \quad \alpha_j \leq \alpha < \alpha_{j+1}.$$

Let V denote also the induced filtration on

$$\mathcal{O}_X \subset \mathcal{O}_X[\partial_t]\delta(f-t).$$

5.2. Theorem (Budur, S. [10]). *If α is not a jumping number,*

(5.2.1) $$\mathcal{J}(X, \alpha D) = V^\alpha\mathcal{O}_X.$$

For α general we have for $0 < \varepsilon \ll 1$

(5.2.2) $$\mathcal{J}(X, \alpha D) = V^{\alpha+\varepsilon}\mathcal{O}_X, \quad V^\alpha\mathcal{O}_X = \mathcal{J}(X, (\alpha-\varepsilon)D).$$

Note that V is left-continuous and $\mathcal{J}(X, \alpha D)$ is right-continuous, i.e.

(5.2.3) $$V^\alpha\mathcal{O}_X = V^{\alpha-\varepsilon}\mathcal{O}_X, \quad \mathcal{J}(X, \alpha D) = \mathcal{J}(X, (\alpha+\varepsilon)D).$$

The proof of (5.2) uses the theory of bifiltered direct images [32], [33] to reduce the assertion to the normal crossing case.

As a corollary we get another proof of the results of Ein, Lazarsfeld, Smith and Varolin [16], and of Lichtin, Yano and Kollár [25]:

5.3. Corollary.
(i) {Jumping numbers ≤ 1} $\subset R_f$, see [16].
(ii) α_f = minimal jumping number, see [25].

Define $\alpha'_{f,x} = \min_{y\neq x}\{\alpha_{f,y}\}$. Then

5.4. Theorem. *If $\xi f = f$ for a vector field ξ, then*

(5.4.1) $$R_f \cap (0, \alpha'_{f,x}) = \{\text{Jumping numbers}\} \cap (0, \alpha'_{f,x}).$$

(This does not hold without the assumption on ξ nor for $[\alpha'_{f,x}, 1)$.)

For the constantness of the jumping numbers under a topologically trivial deformation of divisors, see [14].

6. b-Functions for any subvarieties.

6.1. Let Z be a closed subvariety of a smooth X, and $f = (f_1, \dots, f_r)$ be generators of the ideal of Z (which is not necessarily reduced nor irreducible). Define the action of t_j on

$$\mathcal{O}_X\left[\tfrac{1}{f_1\cdots f_r}\right][s_1, \dots, s_r]\textstyle\prod_i f_i^{s_i},$$

by $t_j(s_i) = s_i + 1$ if $i = j$, and $t_j(s_i) = s_i$ otherwise. Put $s_{i,j} := s_i t_i^{-1} t_j$, $s = \sum_i s_i$. Then $b_f(s)$ is the monic polynomial of the least degree satisfying

(6.1.1) $$b_f(s)\textstyle\prod_i f_i^{s_i} = \sum_{k=1}^r P_k t_k \prod_i f_i^{s_i},$$

where P_k belong to the ring generated by $\mathcal{D}_X$ and $s_{i,j}$.

Here we can replace $\prod_i f_i^{s_i}$ with $\prod_i \delta(t_i - f_i)$, using the direct image by the graph of $f : X \to \mathbf{C}^r$. Then the existence of $b_f(s)$ follows from the theory of the V-filtration of Kashiwara and Malgrange. This b-function has appeared in work of Sabbah [30] and Gyoja [18] for the study of b-functions of several variables.

6.2. Theorem (Budur, Mustaţă, S. [8]). *Let $c = codim_X Z$. Then $b_Z(s) := b_f(s-c)$ depends only on Z and is independent of the choice of $f = (f_1, \dots, f_r)$ and also of r.*

6.3. Equivalent definition. The b-function $b_f(s)$ coincides with the monic polynomial of the least degree satisfying

(6.3.1) $$b_f(s)\textstyle\prod_i f^{s_i} \in \sum_{|c|=1} \mathcal{D}_X[\mathbf{s}] \prod_{c_i<0}\binom{s_i}{-c_i}\prod_i f_i^{s_i+c_i},$$

where $c = (c_1, \dots, c_r) \in \mathbf{Z}^r$ with $|c| := \sum_i c_i = 1$. Here $\mathcal{D}_X[\mathbf{s}] = \mathcal{D}_X[s_1, \cdots, s_r]$.

This is due to Mustaţă, and is used in the monomial ideal case. Note that the well-definedness does not hold without the term $\prod_{c_i<0}\binom{s_i}{-c_i}$.

We have the induced filtration V by

$$\mathcal{O}_X \subset i_{f+}\mathcal{O}_X = \mathcal{O}_X[\partial_1, \dots, \partial_r]\textstyle\prod_i \delta(t_i - f_i).$$

6.4. Theorem (Budur, Mustaţă, S. [8]). *If α is not a jumping number,*

(6.4.1) $$\mathcal{J}(X, \alpha Z) = V^\alpha \mathcal{O}_X.$$

For α general we have for $0 < \varepsilon \ll 1$

$$\mathcal{J}(X, \alpha Z) = V^{\alpha+\varepsilon}\mathcal{O}_X, \quad V^{\alpha}\mathcal{O}_X = \mathcal{J}(X, (\alpha - \varepsilon)Z). \tag{6.4.2}$$

6.5. Corollary (Budur, Mustaţă, S. [8]). *We have the inclusion*

$$\{\text{Jumping numbers}\} \cap [\alpha_f, \alpha_f + 1) \subset R_f. \tag{6.5.1}$$

6.6. Theorem (Budur, Mustaţă, S. [8]). *If Z is reduced and is a local complete intersection, then Z has only rational singularities if and only if $\alpha_f = r$ with multiplicity* 1.

7. Monomial ideal case.

7.1. Definition. Let $\mathfrak{a} \subset \mathbf{C}[x] := \mathbf{C}[x_1, \dots, x_n]$ a monomial ideal. We have the associated semigroup defined by

$$\Gamma_{\mathfrak{a}} = \{u \in \mathbf{N}^n \mid x^u \in \mathfrak{a}\}.$$

Let $P_{\mathfrak{a}}$ be the convex hull of $\Gamma_{\mathfrak{a}}$ in $\mathbf{R}^n_{\geq 0}$. For a face Q of $P_{\mathfrak{a}}$, define

M_Q : the subsemigroup of $\mathbf{Z}^n$ generated by $u - v$ with $u \in \Gamma_{\mathfrak{a}}$, $v \in \Gamma_{\mathfrak{a}} \cap Q$.

$M'_Q = v_0 + M_Q$ for $v_0 \in \Gamma_{\mathfrak{a}} \cap Q$ (this is independent of v_0).

For a face Q of $P_{\mathfrak{a}}$ not contained in any coordinate hyperplane, take a linear function with rational coefficients $L_Q : \mathbf{R}^n \to \mathbf{R}$ whose restriction to Q is 1. Let

V_Q : the linear subspace generated by Q.

$e = (1, \dots, 1)$.

$R_Q = \{L_Q(u) \mid u \in (e + (M_Q \setminus M'_Q)) \cap V_Q\}$,

$R_{\mathfrak{a}} = \{\text{roots of } b_{\mathfrak{a}}(-s)\}$.

7.2. Theorem (Budur, Mustaţă, S. [9]). *We have $R_{\mathfrak{a}} = \bigcup_Q R_Q$ with Q faces of $P_{\mathfrak{a}}$ not contained in any coordinate hyperplanes.*

Outline of the proof. Let $f_j = \prod_i x_i^{a_{i,j}}$, $\ell_i(\mathrm{s}) = \sum_j a_{i,j} s_j$. Define

$$g_c(\mathrm{s}) = \textstyle\prod_{c_i<0} \binom{s_i}{-c_i} \prod_{\ell_i(c)>0} \binom{\ell_i(\mathrm{s}) + \ell_i(c)}{\ell_i(c)}.$$

Let $I_{\mathfrak{a}} \subset \mathbf{C}[\mathrm{s}]$ be the ideal generated by $g_c(\mathrm{s})$ with $c \in \mathbf{Z}^r, \sum_i c_i = 1$. Then

7.3. Proposition (Mustaţă). *The b-function $b_{\mathfrak{a}}[s]$ of the monomial ideal $\mathfrak{a}$ is the monic generator of $\mathbf{C}[s] \cap I_{\mathfrak{a}}$, where $s = \sum_i s_i$.*

Using this, Theorem (7.2) follows from elementary computations.

7.4. Case $n = 2$. Here it is enough to consider only 1-dimensional Q by (7.2). Let Q be a compact face of $P_{\mathfrak{a}}$ with $\{v^{(1)}, v^{(2)}\} = \partial Q$, where $v^{(i)} = (v_1^{(i)}, v_2^{(i)})$ with $v_1^{(1)} < v_1^{(2)}$, $v_2^{(1)} > v_2^{(2)}$. Let

G : the subgroup generated by $u - v$ with $u, v \in Q \cap \Gamma_{\mathfrak{a}}$.

$v^{(3)} \in Q \cap \mathbf{N}^2$ such that $v^{(3)} - v^{(1)}$ generates G.

$S_Q = \{(i, j) \in \mathbf{N}^2 \mid i < v_1^{(3)}, j < v_2^{(1)}\}$.

$$S_Q^{[1]} = S \cap M'_Q, \quad S_Q^{[0]} = S_Q \setminus S_Q^{[1]}.$$

Then

$$R_Q = \{L_Q(u+e) - k \mid u \in S_Q^{[k]}\ (k = 0, 1)\}.$$

In the case $Q \subset \{x = m\}$, we have $R_Q = \{i/m \mid i = 1, \dots, m\}$.

7.5. Examples. (i) If $\mathfrak{a} = (x^a y, xy^b)$, with $a, b \geq 2$, then

$$R_\mathfrak{a} = \Big\{\frac{(b-1)i + (a-1)j}{ab-1}\ \Big|\ 1 \leq i \leq a,\ 1 \leq j \leq b\Big\}.$$

(ii) If $\mathfrak{a} = (xy^5, x^3y^2, x^5y)$, then $S_Q^{[1]} = \emptyset$ and

$$R_\mathfrak{a} = \Big\{\frac{5}{13}, \frac{i}{13}\ (7 \leq i \leq 17), \frac{19}{13}, \frac{j}{6}\ (3 \leq j \leq 9)\Big\}.$$

(iii) If $\mathfrak{a} = (xy^5, x^3y^2, x^4y)$, then $S_Q^{[1]} = \{(2,4)\}$ for $\partial Q = \{(1,5), (3,2)\}$ with $L_Q(v_1, v_2) = (3v_1 + 2v_2)/13$, and

$$R_\mathfrak{a} = \Big\{\frac{i}{13}\ (5 \leq i \leq 17), \frac{j}{5}\ (2 \leq j \leq 6)\Big\}.$$

Here 19/13 is shifted to 6/13.

7.6. Comparison with exponents. If $n = 2$ and f has a nondegenerate Newton polygon with compact faces Q, then by Steenbrink [43]

$$E_f \cap (0,1] = \textstyle\bigcup_Q E_Q^{\leq 1} \quad \text{with} \quad E_Q^{\leq 1} = \{L_Q(u) \mid u \in \overline{\{0\} \cup Q} \cap \mathbf{Z}^2_{>0}\},$$

where $\overline{\{0\} \cup Q}$ is the convex hull of $\{0\} \cup Q$. Here we have the symmetry of E_f with center 1.

7.7. Another comparison. If $\mathfrak{a} = (x_1^{a_1}, \dots, x_n^{a_n})$, then

$$R_\mathfrak{a} = \big\{\textstyle\sum_i p_i/a_i \mid 1 \leq p_i \leq a_i\big\}.$$

On the other hand, if $f = \sum_i x_i^{a_i}$, then

$$\widetilde{R}_f = E_f = \big\{\textstyle\sum_i p_i/a_i \mid 1 \leq p_i \leq a_i - 1\big\}.$$

8. Hyperplane arrangements.

8.1. Let D be a central hyperplane arrangement in $X = \mathbf{C}^n$. Here, central means an affine cone of $Z \subset \mathbf{P}^{n-1}$. Let f be the reduced equation of D and $d := \deg f > n$. Assume D is not the pull-back of $D' \subset \mathbf{C}^{n'} (n' < n)$.

8.2. Theorem. (i) $\max R_f < 2 - \frac{1}{d}$. (ii) $m_1 = n$.

Proof of (i) uses a partial generalization of a solution of Aomoto's conjecture due to Esnault, Schechtman, Viehweg, Terao, Varchenko ([17], [40]) together with a generalization of Malgrange's formula (4.8) as below:

8.3. Theorem (Generalization of Malgrange's formula) [36]. *There exists a pole order filtration P on $H^{n-1}(F_0, \mathbf{C})_\lambda$ such that if $(\alpha + \mathbf{N}) \cap R'_f = \emptyset$, then*

$$\alpha \in R_f \Leftrightarrow \mathrm{Gr}_P^p H^{n-1}(F_0, \mathbf{C})_\lambda \neq 0, \tag{8.3.1}$$

with $p = [n - \alpha], \lambda = e^{-2\pi i\alpha}$, where $R'_f = \cup_{x\neq 0} R_{f,x}$.

This reduces the proof of (8.2)(i) to

$$P^i H^{n-1}(F_0, \mathbf{C})_\lambda = H^{n-1}(F_0, \mathbf{C})_\lambda, \tag{8.3.2}$$

for $i = n - 1$ if $\lambda = 1$ or $e^{2\pi i/d}$, and $i = n - 2$ otherwise.

8.4. Construction of the pole order filtration P. Let $U = \mathbf{P}^{n-1} \setminus Z$, and $F_0 = f^{-1}(0) \subset \mathbf{C}^n$. Then $F_0 = \widetilde{U}$ with $\pi : \widetilde{U} \to U$ a d-fold covering ramified over Z. Let $L^{(k)}$ be the local systems of rank 1 on U such that $\pi_*\mathbf{C} = \bigoplus_{0\le i<d} L^{(k)}$ and T acts on $L^{(k)}$ by $e^{-2\pi ik/d}$. Then

$$H^j(U, L^{(k)}) = H^j(F_0, \mathbf{C})_{\mathbf{e}(-k/d)}, \tag{8.4.1}$$

and P is induced by the pole order filtration on the meromorphic extension $\mathcal{L}^{(k)}$ of $L^{(k)} \otimes_{\mathbf{C}} \mathcal{O}_U$ over $\mathbf{P}^{n-1}$, see [15], [36], [37]. This is closely related to:

8.5. Solution of Aomoto's conjecture ([17], [40]). Let Z_i be the irreducible components of Z $(1 \le i \le d)$, g_i be the defining equation of Z_i on $\mathbf{P}^{n-1}\setminus Z_d$ $(i < d)$, and $\omega := \sum_{i<d} \alpha_i\omega_i$ with $\omega_i = dg_i/g_i$, $\alpha_i \in \mathbf{C}$. Let ∇ be the connection on $\mathcal{O}_U$ such that $\nabla u = du + \omega\wedge u$. Set $\alpha_d = -\sum_{i<d} \alpha_i$. Then $H^\bullet_{\mathrm{DR}}(U, (\mathcal{O}_U, \nabla))$ is calculated by

$$(\mathcal{A}^\bullet_\alpha, \omega\wedge) \quad \text{with} \quad \mathcal{A}^p_\alpha = \sum \mathbf{C}\omega_{i_1}\wedge\cdots\wedge\omega_{i_p},$$

if $\sum_{Z_i\supset L} \alpha_i \notin \mathbf{N} \setminus \{0\}$ for any *dense* edge $L \subset Z$ (see (8.7) below). Here an edge is an intersection of Z_i.

For the proof of (8.2)(ii) we have

8.6. Proposition. $N^{n-1}\psi_{f,\lambda}\mathbf{C} \neq 0$ *if* $\mathrm{Gr}^W_{2n-2} H^{n-1}(F_x, \mathbf{C})_\lambda \neq 0$.

(Indeed, $N^{n-1} : \mathrm{Gr}^W_{2n-2}\psi_{f,\lambda}\mathbf{C} \xrightarrow{\sim} \mathrm{Gr}^W_0\psi_{f,\lambda}\mathbf{C}$ by the definition of W, and the assumption of (8.6) implies $\mathrm{Gr}^W_{2n-2}\psi_{f,\lambda}\mathbf{C} \neq 0$.)

Then we get (8.2)(ii), since $\omega_{i_1}\wedge\cdots\wedge\omega_{i_{n-1}} \neq 0$ in $\mathrm{Gr}^W_{2n-2} H^{n-1}(\mathbf{P}^{n-1} \setminus Z, \mathbf{C}) = \mathrm{Gr}^W_{2n-2} H^{n-1}(F_x, \mathbf{C})_1$.

8.7. Dense edges. Let $D = \cup_i D_i$ be the irreducible decomposition. Then $L = \cap_{i\in I} D_i$ is called an edge of D $(I \neq \emptyset)$,

We say that an edge L is *dense* if $\{D_i/L \mid D_i \supset L\}$ is indecomposable. Here $\mathbf{C}^n \supset D$ is called decomposable if $\mathbf{C}^n = \mathbf{C}^{n'} \times \mathbf{C}^{n''}$ such that D is the union of the pull-backs from $\mathbf{C}^{n'}, \mathbf{C}^{n''}$ with $n', n'' \neq 0$.

Set $m_L = \#\{D_i \mid D_i \supset L\}$. For $\lambda \in \mathbf{C}$, define

$$\mathcal{DE}(D) = \{\text{dense edges of } D\}, \quad \mathcal{DE}(D, \lambda) = \{L \in \mathcal{DE}(D) \mid \lambda^{m_L} = 1\}.$$

We say that L, L' are *strongly adjacent* if $L \subset L'$ or $L \supset L'$ or $L \cap L'$ is non-dense. Let

$$m(\lambda) = \max\{|S| \mid S \subset \mathcal{DE}(D,\lambda) \text{ such that any } L, L' \in S \text{ are strongly adjacent}\}.$$

8.8. Theorem [37]. *$m_\alpha \le m(\lambda)$ with $\lambda = e^{-2\pi i\alpha}$.*

8.9. Corollary. *$R_f \subset \bigcup_{L\in\mathcal{DE}(D)} \mathbf{Z}m_L^{-1}$.*

8.10. Corollary. *If $\mathrm{GCD}(m_L, m_{L'}) = 1$ for any strongly adjacent $L, L' \in \mathcal{DE}(D)$, then $m_\alpha = 1$ for any $\alpha \in R_f \setminus \mathbf{Z}$.*

Theorem 2 follows from the canonical resolution of singularities $\pi : (\widetilde{X}, \widetilde{D}) \to (\mathbf{P}^{n-1}, D)$ due to [40], which is obtained by blowing up along the proper transforms of the dense edges. Indeed, mult $\widetilde{D}(\lambda)_{\mathrm{red}} \le m(\lambda)$, where $\widetilde{D}(\lambda)$ is the union of $\widetilde{D}_i$ such that $\lambda^{\widetilde{m}_i} = 1$ and $\widetilde{m}_i = \mathrm{mult}_{\widetilde{D}_i}\widetilde{D}$.

8.11. Theorem (Mustaţă [29]). For a central arrangement,

(8.11.1) $$\mathcal{J}(X, \alpha D) = I_0^k \text{ with } k = [d\alpha] - n + 1 \text{ if } \alpha < \alpha'_f,$$

where I_0 is the ideal of 0 and $\alpha'_f = \min_{x\ne 0}\{\alpha_{f,x}\}$.

(This holds for the affine cone of any divisor on $\mathbf{P}^{n-1}$, see [36].)

8.12. Corollary. *We have $\dim F^{n-1}H^{n-1}(F_0, \mathbf{C})_{\mathbf{e}(-k/d)} = \binom{k-1}{n-1}$ for $0 < \frac{k}{d} < \alpha'_f$, and the same holds with F replaced by P.*

8.13. Corollary. *$\alpha_f = \min(\alpha'_f, \frac{n}{d}) < 1$.*

(Note that α_f coincides with the minimal jumping number.)

8.14. Generic case. If D is a generic central hyperplane arrangement, then

(8.14.1) $$b_f(s) = (s+1)^{n-1}\textstyle\prod_{j=n}^{2d-2}(s + \frac{j}{d})$$

by U. Walther [46] (except for the multiplicity of -1). He uses a completely different method.

Note that Theorems (8.2) and (8.8) imply that the left-hand side divides the right-hand side of (8.14.1), and the equality follows using also (8.12).

8.15. Explicit calculation. Let $\alpha = k/d$, $\lambda = e^{-2\pi i\alpha}$ for $k \in \{1, \dots, d\}$. If $\alpha \ge \alpha'_f$, we assume there is $I \subset \{1, \dots, d-1\}$ such that $|I| = k-1$, and the condition of [40]

(8.15.1) $$\textstyle\sum_{Z_i \supset L} \alpha_i \notin \mathbf{N} \setminus \{0\} \text{ for any dense edge } L \subset Z,$$

is satisfied for

(8.15.2) $$\alpha_i = 1 - \alpha \text{ if } i \in I \cup \{d\}, \text{ and } -\alpha \text{ otherwise.}$$

Let $V(I)$ be the subspace of $H^{n-1}\mathcal{A}^\bullet_\alpha$ generated by

$$\omega_{i_1}\wedge\cdots\wedge\omega_{i_{n-1}} \quad \text{for} \quad \{i_1, \dots, i_{n-1}\} \subset I.$$

8.16. Theorem. *Let $\alpha = k/d$, $\lambda = e^{-2\pi i\alpha}$ for $k \in \{1, \dots, d\}$. Then*

(a) *If* $k = d-1$ *or* d, *then* $\alpha \in R_f$, $\alpha+1 \notin R_f$.
(b) *If* $\alpha < \alpha'_f$, *then* $\alpha \in R_f \Leftrightarrow k \geq d$.
(c) *If* $\binom{k-1}{n-1} < \dim H^{n-1}(F_0, \mathbf{C})_\lambda$, *then* $\alpha+1 \in R_f$.
(d) *If* $\alpha < \alpha'_f$, $\alpha \notin R'_f + \mathbf{Z}$ *and* $\binom{k-1}{n-1} = \chi(U)$, *then* $\alpha+1 \notin R_f$.
(e) *If* $\alpha \geq \alpha'_f$ *and* $V(I) \neq 0$, *then* $\alpha \in R_f$.
(f) *If* $\alpha \geq \alpha'_f$ *and* $V(I) = H^{n-1}\mathcal{A}^\bullet_\alpha$, *then* $\alpha+1 \notin R_f$.

8.17. Theorem [37]. *Assume* $n=3$, $\mathrm{mult}_z Z \leq 3$ *for any* $z \in Z \subset \mathbf{P}^2$, *and* $d \leq 7$. *Let* ν_3 *be the number of triple points of* Z, *and assume* $\nu_3 \neq 0$. *Then*

$$b_f(s) = (s+1)\prod_{i=2}^{4}(s+\tfrac{i}{3})\prod_{j=3}^{r}(s+\tfrac{j}{d}), \tag{8.17.1}$$

with $r = 2d-2$ *or* $2d-3$. *We have* $r = 2d-2$ *if* $\nu_3 < d-3$, *and the converse holds for* $d < 7$. *In case* $d = 7$, *we have* $r = 2d-3$ *for* $\nu_3 > 4$, *however, for* $\nu_3 = 4$, r *can be both* $2d-2$ *and* $2d-3$.

8.18. Remarks. (i) We have $\nu_3 < d-3$ if and only if

$$\chi(U) = \tfrac{(d-2)(d-3)}{2} - \nu_3 > \tfrac{(d-3)(d-4)}{2} = \binom{d-3}{2}. \tag{8.18.1}$$

(ii) By (8.4.1) we have $\chi(U) = h^2(F_0, \mathbf{C})_\lambda - h^1(F_0, \mathbf{C})_\lambda$ if $\lambda^d = 1$ and $\lambda \neq 0$.

(iii) Let ν'_i be the number of i-ple points of $Z' := Z \cap \mathbf{C}^2$. Then by [6]

$$b_0(U) = 1, \quad b_1(U) = d-1, \quad b_2(U) = \nu'_2 + 2\nu'_3, \tag{8.18.2}$$

8.19. Examples. (i) For $(x^2-1)(y^2-1) = 0$ in $\mathbf{C}^2$ with $d = 5$, (8.17.1) holds with $r = 7$, and $8/5 \notin R_f$. In this case we do not need to take I, because $(d-2)/d = 3/5 < \alpha'_f = 2/3$. We have $b_1(U) = b_2(U) = 4$ and $h^2(F_0, \mathbf{C})_\lambda = \chi(U) = 1$ if $\lambda^5 = 1$ and $\lambda \neq 1$. So $j/5 \in R_f$ for $3 \leq j \leq 7$ by (a), (b), (c), and $8/5 \notin R_f$ by (d).

(ii) For $(x^2-1)(y^2-1)(x+y) = 0$ in $\mathbf{C}^2$ with $d = 6$, (8.17.1) holds with $r = 9$, and $10/6 \notin R_f$. In this case we have $b_1(U) = 5, b_2(U) = 6, \chi(U) = 2, h^1(F_0, \mathbf{C})_\lambda = 1, h^2(F_0, \mathbf{C})_\lambda = 3$ for $\lambda = e^{\pm 2\pi i/3}$. Then $4/6 \in R_f$ by (e) and $10/6 \notin R_f$ by (f), where I^c corresponds to $(x+1)(y+1) = 0$. For other $j/6$, the argument is the same as in (i).

(iii) For $(x^2-y^2)(x^2-1)(y+2) = 0$ in $\mathbf{C}^2$ with $d = 6$, (8.17.1) holds with $r = 10$, and $10/6 \in R_f$. In this case we have $b_1(U) = 5, b_2(U) = 9, \chi(U) = 5, h^1(F_0, \mathbf{C})_\lambda = 0$, $h^2(F_0, \mathbf{C})_\lambda = 5$ for $\lambda = e^{\pm 2\pi i/3}$. Then $4/6 \in R_f$ by (e) and $10/6 \in R_f$ by (c), where I^c corresponds to $(x+1)(y+2) = 0$.

(iv) For $(x^2-y^2)(x^2-1)(y^2-1) = 0$ in $\mathbf{C}^2$ with $d = 7$, (8.17.1) holds with $r = 11$, and $12/7 \notin R_f$. In this case we have $b_1(U) = 6, b_2(U) = 9, \chi(U) = 4, h^2(F_0, \mathbf{C})_\lambda = 4$ if $\lambda^7 = 1$ and $\lambda \neq 1$. Then $5/7 \in R_f$ by (e) and $12/7 \notin R_f$ by (f), where I^c corresponds to $(x+1)(y+1) = 0$. Note that $5/7$ is not a jumping number.

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