数理解析研究所講究録
第 1557 巻 2007 年 211-214

Vilfredo Pareto and the Integrability Problem of Demand Functions

Shinichi Suda
Department of Economics
Keio University
2-15-45 Mita Minato-ku Tokyo 108-8345 Japan
E-mail: ssuda@econ.keio.ac.jp

Abstract

On the occasion of the centennial since the publication of Vilfredo Pareto's *Manuale di economia politica, con una introduzione alla scienza sociale*, Pareto's treatment of integrability problem is reexamined. This problem concerns whether one can "recover" a preference ordering that generates the given demand function. While the problem was mathematically solved by Antonelli in 1886, we found that Pareto used two different kinds of integrability conditions according to the measurability of utility. This finding will shed a new light on the discussion of the Pareto's attitude toward the measurability of utility. The present paper is based on the results obtained in Suda (2007).

Key words: Pareto, integrability problem, measurability of utility

1 Introduction

Today Vilfredo Pareto is recognized as one of the founders of general equilibrium theory, especially in the fields of the demand theory and the welfare economics. While the concept of utility underlies these theories, Pareto was well aware of the difficulty of its direct measurement whether it is cardinal or ordinal. In addition Pareto who started the career as an engineer could not possibly be satisfied on the theory based on unmeasurable objects. Therefore the derivation of an utility function from the observable demand function, i.e., the integrability problem, attracted his attension.

Despite the importance of the problem to Pareto's theory, his treatment has been evaluated relatively low. For example Chipman (1976) wrote: "Given the original and path-breaking nature of Pareto's approach to the theory of the consumer, it is all the more disappointing that his own solutions to the problems (of integrability) he posed were far from satisfactory, and contained a number of technical defects and confusions" (p. 81).

In this paper we will reexamine Pareto's discussion of the integrability problem, mainly in the series of papers he published in *Giornale degli economisti* during the years 1892-93 and try to identify what Pareto really meant by each of his integrability conditions. The paper will show that Pareto distinguished two different kinds of conditions, one is for the cardinal utility and the other is for the ordinal utility. Furthermore, it will be shown that the criticism toward Pareto's confusion originated partly from the unawareness of this distinction.

2 Problem of integrability of demand functions

Integrability problem concerns the recovery of a preference ordering that generates the given demand function. Putting one of the prices to unity (by the homogeneity) and assuming the invertibility of demand function, we get the inverse demand function which associates each consumption bundle to the marginal rate of substitution at that point. If the demand function is observable, the inverse demand function is also observable. Then the integrability problem becomes the problem of recovering a preference ordering (indifference map) from the information of the marginal rate of substitution at each point in consumption space.

In the case of two goods, the problem can be described by the following differential equation:

$$\frac{dx_2}{dx_1} = -r_1(x_1, x_2) \tag{1}$$

where $r_1(x_1, x_2)$ denotes the marginal rate of substitution at (x_1, x_2) [1]. If we assume the smoothness of the function $r_1(x_1, x_2)$, we can solve this differential equation and obtain the local solution $x_2(x_1) = x_2$ representing the indifference curve associated with the given inverse demand function.

Before moving to the general case, let us introduce another representation of the problem, namely the one using the total differential equation. Letting ϕ_1 and ϕ_2 be the marginal utilities of good 1 and 2 (so that $r_1(x) = \frac{\phi_1(x)}{\phi_2(x)}$), equation (1) can be rewritten by the following total differential equation:

$$\phi_1(x)dx_1 + \phi_2(x)dx_2 = 0.$$

Then it is easily seen that the general (n good) case can be represented by the following total differential equation:

$$\phi_1(x)dx_1 + \phi_2(x)dx_2 + \cdots + \phi_n(x)dx_n = 0. \tag{2}$$

Although in the two good case this differential equation always has a solution, in the case of three or more goods, the equation in general admits no solution. Moreover the following necessary and sufficient condition for the existence of solution is well known.

$$\phi_i\left(\frac{\partial\phi_j}{\partial x_k} - \frac{\partial\phi_k}{\partial x_j}\right) + \phi_j\left(\frac{\partial\phi_k}{\partial x_i} - \frac{\partial\phi_i}{\partial x_k}\right) + \phi_k\left(\frac{\partial\phi_i}{\partial x_j} - \frac{\partial\phi_j}{\partial x_i}\right) \neq 0,\ i \neq j \neq k,$$

[1] We consider only the local integrability problem.

which is usually called the "integrability condition" of the demand functions.

The problem of integrability in the consumer theory was firstly formulated by Antonelli (1886) and the integrability condition was clearly stated in the same monograph.

3 Pareto's treatment of the integrability problem

Pareto's first reference to the integrability problem appeared in Pareto (1892, Part 1 pp. 414-15). There he assumed that the consumers can absorb the idea of small variations in utility (marginal utility) and represented this variation by the following formula:

$$Qdx + Rdy \tag{3}$$

where Q and R are the marginal utilities for goods x and y. He then continued to say that if one can find a function P that satisfies the following condition:

$$\frac{\partial P}{\partial x} = Q, \ \frac{\partial P}{\partial y} = R, \tag{4}$$

then the total utility function exists. If we regard the equation (3) as the total differential equation:

$$Qdx + Rdy = 0,$$

then the condition (4) is unnecessary because, as we saw in Section 2, the two good case needs no integrability condition. This point was also indicated by Stigler (1950, p. 379, footnote 156), where he concluded Pareto made a mistake on the integrabililty condition. Chipman (1976) concluded the same way, which leads to his criticism. However there is another interpretation of this condition based on the cardinal utility measure.

In Pareto (1892-93), he clearly stated that he believed in the existence of marginal utility function. Therefore the natural procedure to obtain the total utility from the marginal utility is to integrate the latter, as Walras did in *Éléments d'économie politique pure.* Since, unlike the Walrasian marginal utility function, the Paretian marginal utility of a good depends on the quantity of other goods as well, the value of total utility depends on the path on which the integral operation is performed. Thus we can assume that Pareto considered the condition of the path independence of total utility, based on the following well known mathematical theorem.

Theorem

The following three propositions are equivalent:

(i) the equation (2) is an exact differential equation;

(ii) $\frac{\partial \phi_i}{\partial x_j} = \frac{\partial \phi_j}{\partial x_i}$, $i \neq j$;

(iii) the value of a line integral:

$$\int_0^1 \sum_{i=1}^{n} \phi_i(\zeta(t))d\zeta_i(t)$$

along a path $\zeta(t)$, $0 \leq t \leq 1$, connecting two points $\zeta(0) = a$, $\zeta(1) = b$ does not depend on the particular path ζ chosen.

Now we know the condition (4) is a necessary and sufficient condition for the path independence of the line integral, i.e., the determination of the total utility function. In this way we can consistently interpret the condition (4).

The above interpretation was further confirmed by the description in Part 5 of Pareto (1892-93), where he mentioned the ordinal utility function for the first time, correctly stating the fact that, for the two good case, the total differential equation could be solved without any condition (Pareto 1892-93, Part 5, pp. 299-300).

4 Concluding remarks

We showed that Pareto distinguished two different kinds of integrability conditions according to whether he assumed cardinal measure of utility or ordinal one. Some of the complaints against the Pareto's treatment on the integrability problem can be alleviated by realizing this distinction.

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