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THE SPECTRAL FUNCTION AT A MAXIMUM OF THE POTENTIAL

IVANA ALEXANDROVA, JEAN-FRANÇOIS BONY, AND THIERRY RAMOND

1. Introduction and Statement of Results

We study the structure of the spectral function of the Schrödinger operator with short range potential at an energy, which is a non-degenerate maximum of the potential. We prove that it is semi-classical Fourier integral operator quantizing the incoming and outgoing Lagrangian submanifolds associated to the fixed hyperbolic point. We then give the oscillatory integral representation of the spectral function implied by this result.

More precisely, we work in the following setting. We consider the operator

$$P(h)=-\frac{1}{2}h^2\Delta+V,\ 0<h\ll 1,$$

where $V\in C^\infty(\mathbb{R}^n;\mathbb{R})$, $n>1$, is a short range potential, i.e., for some $\rho>1$ and all $\alpha\in\mathbb{N}^n$

$$|\partial^\alpha V(x)|\leq C_\alpha(1+\|x\|)^{-\rho-|\alpha|},\ x\in\mathbb{R}^n. \tag{1}$$

Then $P(h)$ admits a unique self-adjoint realization on $L^2(\mathbb{R}^n)$ with domain $H_h^2(\mathbb{R}^n)$, the semi-classical Sobolev spaces of order 2 (see Appendix A). Denoting by $\{E_\lambda\}$ the spectral family of P, we shall use e_λ for the Schwartz kernel of E_λ for $\lambda>0$. The Limiting Absorption Principle states that in $\mathcal{B}\left(L^2_\alpha(\mathbb{R}^n),L^2_{-\alpha}(\mathbb{R}^n)\right)$, where $L^2_\alpha(\mathbb{R}^n)=\{f: f\langle\cdot\rangle^\alpha\in L^2(\mathbb{R}^n)\}$, $\alpha>\frac{1}{2}$, the limit $R(\lambda\pm i0,h)\overset{\text{def}}{=}\lim_{\epsilon\downarrow 0}\left(P(h)-(\lambda\pm i\epsilon)\right)^{-1}$ for $\lambda>0$ exists.

We let $p(x,\xi)=\frac{1}{2}\|\xi\|^2+V(x)$ denote the principal symbol of $P(h)$ and denote its Hamiltonian vector field by $H_p=\sum_{j=1}^n\left(\frac{\partial p}{\partial\xi_j}\frac{\partial}{\partial x_j}-\frac{\partial p}{\partial x_j}\frac{\partial}{\partial\xi_j}\right)$. An integral curve γ of H_p will be called a trajectory and will be denoted $\gamma(\cdot;x_0,\xi_0)$, if $(x_0,\xi_0)\in T^*\mathbb{R}^n$ are its initial conditions. We recall that

Definition 1. *The trajectory $\gamma(\cdot;x_0,\xi_0)$ is non-trapped if $\lim_{t\to\pm\infty}\|x(t;x_0,\xi_0)\|=\infty$. The energy $\lambda>0$ is non-trapping if for every $(x_0,\xi_0)\in T^*\mathbb{R}^n$ with $\frac{1}{2}\|\xi_0\|^2+V(x_0)=\lambda$ we have $\lim_{t\to\pm\infty}\|x(t;x_0,\xi_0)\|=\infty$.*

We refer to the Appendix for the relevant parts of semi-classical analysis used throughout this paper.

The structure of the spectral function for Schrödinger-like operators has been studied extensively. Popov and Shubin [14], Popov [13], and Vainberg [18] have established high energy asymptotics for the spectral function of second order elliptic operators under the non-trapping assumption.

Robert and Tamura [17] consider the spectral function for semi-classical Schrödinger operator with short range potentials and establish asymptotic expansions at fixed non-trapping and non-critical trapping energies in the sense of a distribution.

The microlocal structure of the spectral function has also been analyzed. In [19, Theorem XII.5] Vainberg establishes a high energy asymptotic expansion of the spectral function for compactly supported smooth perturbations of the Laplacian assuming that the energy 1 is non-trapping. This asymptotic expansion is expressed this in the form of a Maslov canonical operator $K_{\Lambda,\lambda}$ associated to a certain Lagrangian submanifold $\Lambda = \Lambda_y \subset T^*\mathbb{R}^n$ and acting on another asymptotic sum in λ. The Lagrangian submanifold Λ_y consists of the phase trajectories at energy 1 of the principal symbol of A passing through a fixed base point $x(0) = y$, while the terms of the asymptotic sum on which $K_{\Lambda,\lambda}$ acts solve a recurrent system of transport equations along the phase trajectories of the system.

Gerard and Martinez [10] prove that the spectral function for certain long-range Schrödinger operators at non-trapping energies λ is a semi-classical Fourier integral operator (h-FIO) associated to $\left(\cup_{t\in\mathbb{R}} \operatorname{graph} \exp(tH_p)|_{p^{-1}(\lambda)}\right)'$. Near the diagonal $\{(x,\xi;x,\xi) : p(x,\xi) = \lambda\}$ they also give the following oscillatory integral representation of the spectral function

$$e_\lambda(x,y,\lambda,h) \equiv \frac{1}{(2\pi h)^n} \int_{\mathbb{S}^{n-1}} e^{\frac{i}{h}\varphi(x,y,\omega,\lambda)} a(x,y,\omega,\lambda)\, d\omega,$$

where φ is such that $\left(\frac{\partial\varphi}{\partial x}\right)^2 + V(x) = \lambda$ and $\left.\frac{\partial\varphi}{\partial x}\right|_{\langle x-y,\omega\rangle=0} = \sqrt{\lambda - V(x)}\,\omega$, $\varphi|_{x=y} = 0$.

In [1] the first author has proven that the spectral function restricted away from the diagonal in $\mathbb{R}^n \times \mathbb{R}^n$ at non-trapping energies, and at trapping energies under the absence of resonances near the real axis, is an h-FIO associated to $\left(\cup_{t=0}^{T} \operatorname{graph} \exp(tH_p)|_{p^{-1}(\lambda)}\right)' \cup \left(\cup_{t=0}^{-T} \operatorname{graph} \exp(tH_p)|_{p^{-1}(\lambda)}\right)'$ for some $T > 0$ near a non-trapped trajectory. Under a certain geometric assumption [1] also gives an oscillatory integral representation of the spectral function of the form

$$e_\lambda(x,y,\lambda) \equiv \int e^{\frac{i}{h}S(x,y,t)}a(x,y,t)dt,$$

where $S(x,y,t) = \int_{l(t,x,y)} \frac{1}{2}\|\xi(t)\|^2 - V(x(t)) + \lambda dt$ is the action over the segment $l(t,x,y)$ of the trajectory which connects x with y at time t and $a \in S_{2n+1}^{\frac{n+3}{2}}(1)$.

Hassell and Wunsch [11] have studied the structure of the spectral function on compact manifolds with boundary equipped with scattering metrics. Their result roughly says that the spectral function is an intersecting Legendrian distribution.

Here we study the structure of the spectral function under the following additional assumptions:

(A1) V has a non-degenerate global maximum at $x = 0$, with $V(0) = E > 0$ and

$$V(x) = E - \sum_{j=1}^{n} \frac{\lambda_j^2}{2}x_j^2 + \mathcal{O}(x^3),\ x \to 0,$$

where $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

(A2) $\{(x,\xi) \in p^{-1}(E) : \exp(tH_p)(x,\xi) \nrightarrow \infty \text{ as } t \to \pm\infty\} \equiv \{(0,0)\}$

Then the linearized vector field of H_p at $(0,0)$ is

$$d_{(0,0)}H_p = \begin{pmatrix} 0 & I \\ \operatorname{diag}(\lambda_1^2,\ldots,\lambda_n^2) & 0 \end{pmatrix}.$$

Therefore, by the Stable Manifold Theorem, there exist Lagrangian submanifolds $\Lambda_\pm \subset T^*\mathbb{R}^n$ satisfying

$$\Lambda_\pm = \{(x,\xi) \in T^*\mathbb{R}^n : \exp tH_p(x,\xi) \to (0,0) \text{ as } t \to \mp\infty\}$$

(see Figure 1).

To state our main theorem, we further recall from [12] that if $\rho_\pm \in \Lambda_\pm$ and $\gamma_\pm(\cdot;\rho_\pm) = (x_\pm(\cdot;\rho_\pm), \xi_\pm(\cdot;\rho_\pm)) \stackrel{\text{def}}{=} \gamma(\cdot;\rho_\pm)$, then for some $g_\pm \in C^\infty(\mathbb{R}^{2n})$ and $\epsilon > 0$, $x_\pm(t;\rho_\pm) = g_\pm(\rho_\pm)e^{\pm\lambda_1 t} + \mathcal{O}(e^{\pm(\lambda_1+\epsilon)t})$ as $t \to \mp\infty$. We let $\tilde{\Lambda}_\pm = \{(x,\xi) \in \Lambda_\pm : g_-(x,\xi) = 0\}$ and recall from [7] that $\dim \tilde{\Lambda}_\pm = n - m$, where $m \stackrel{\text{def}}{=} \#\{j : \lambda_1 = \lambda_j\}$.

We also set $\tilde{\Lambda}_\pm(\rho_\mp) = \{\rho_\pm \in \Lambda_\pm : \langle g_\pm(\rho_\pm), g_\mp(\rho_\mp)\rangle = 0\}$.

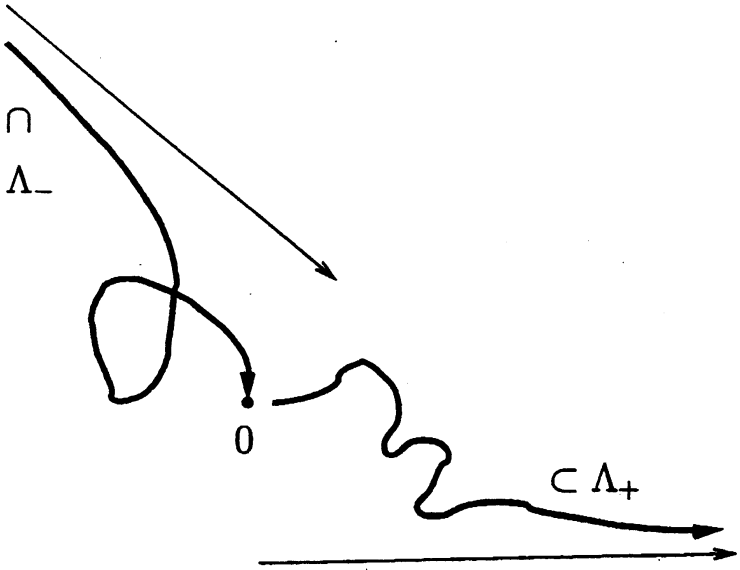

FIGURE 1. The incoming Λ_- and outgoing Λ_- Lagrangian submanifolds.

Our main result is the following

Theorem 1. *Microlocally near* $(\rho_+, \rho_-) \in \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-$, *the resolvent* $R(E+i0) \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \left(\mathbb{R}^{2n}, \Lambda_+ \times \Lambda_-\right)$.

Similarly, microlocally near $(\rho_-, \rho_+) \in \Lambda_- \setminus \tilde{\Lambda}_-(\rho_+) \times \Lambda_+ \setminus \tilde{\Lambda}_+$, *the resolvent* $R(E-i0) \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \left(\mathbb{R}^{2n}, \Lambda_- \times \Lambda_+\right)$.

Remark. If $\lambda_2 > \lambda_1$, then $\tilde{\Lambda}_+(\rho_-) = \tilde{\Lambda}_+$ and $R(E+i0) \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \left(\mathbb{R}^{2n}, \Lambda_+ \setminus \tilde{\Lambda}_+ \times \Lambda_- \setminus \tilde{\Lambda}_-\right)$.

The structure of the resolvent in various settings has been studied in [3], [4], and [11]. For compactly supported and short range potentials, the resolvent has been shown to be a h-FIO associated to the Hamiltonian flow relation of the principal symbol of P restricted to the energy surface in [3] and [4]. Hassell and Wunsch [11] identify the Schwartz kernel of the resolvent on a compact scattering manifold with a Legendrian distribution.

Using Stone's formula

$$\frac{dE_\lambda}{d\lambda}(E) = \frac{1}{2\pi i}\left(R(E+i0) - R(E-i0)\right),$$

we now easily obtain from Theorem 1 the following

Corollary 1. *Microlocally near* $(\rho_-, \rho_+) \in \Lambda_- \setminus \widetilde{\Lambda}_- \times \Lambda_+ \setminus \widetilde{\Lambda}_+(\rho_-)$, *the spectral function* $e_E \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}(\mathbb{R}^{2n}, \Lambda_+ \times \Lambda_-)$.

Microlocally near $(\rho_+, \rho_-) \in \Lambda_+ \setminus \widetilde{\Lambda}_+ \times \Lambda_- \setminus \widetilde{\Lambda}_-(\rho_+)$, *the spectral function* $e_E \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}}(\mathbb{R}^{2n}, \Lambda_- \times \Lambda_+)$.

We now introduce some of the notation we shall use below. For a sequentially continuous operator $W : C_c^\infty(\mathbb{R}^m) \to \mathcal{D}'(\mathbb{R}^n)$ we shall denote by K_W its Schwartz kernel. On any smooth manifold M we denote by σ the canonical symplectic form on T^*M and everywhere below we work with the canonical symplectic structure on T^*M. If $C \subset T^*M_1 \times T^*M_2$, where M_j, $j = 1, 2$, are smooth manifolds, we will use the notation $C' = \{(x, \xi; y, -\eta) : (x, \xi; y, \eta) \in C\}$. We also set $B(0, r) = \{x \in \mathbb{R}^n : \|x\| < r\}$.

We prove our main theorem in Section 2 and in Section 3 we give the microlocal representation of the spectral function implied by Theorem 1.

2. The Resolvent as a Semi-Classical Fourier Integral Operator

Here we prove Theorem 1.

The resolvent estimate from [5, Theorem 2.1], $\|R(E \pm i0)\|_{\mathcal{B}(L^2_\alpha(\mathbb{R}^n), L^2_{-\alpha}(\mathbb{R}^n))} = \mathcal{O}\left(\frac{|\log h|}{h}\right)$, for $\alpha > \frac{1}{2}$, and [4, Lemma 1] give that $K_{R(E \pm i0)} \in \mathcal{D}'_h(\mathbb{R}^{2n})$.

Let

$$\Gamma_\pm(R, d, \sigma) = \left\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \|x\| > R, \frac{1}{d} < \|\xi\| < d, \pm \cos(x, \xi) > \pm\sigma\right\}$$

with $R > 1$, $d > 1$, $\sigma \in (-1, 1)$, and $\cos(x, \xi) = \frac{\langle x, \xi\rangle}{\|x\|\,\|\xi\|}$, be the outgoing and incoming subsets of phase space, respectively. We choose $d > 0$ such that $\frac{1}{d} < E < d$.

Let $u_- \in \mathcal{D}'_h(\mathbb{R}^n)$ be such that $MS(u_-) \subset \Gamma_-(R, d, \sigma)$ is compact, where MS denotes its microsupport.

We shall prove that $u = R(E + i0)u_-$ solves the problem

$$\begin{cases} (P - E)u = 0 \text{ microlocally near } (0,0) \\ u = \frac{i}{h}\int_0^T e^{\frac{i}{h}tE} e^{-\frac{i}{h}tP} dt\, u_- \text{ near } \Gamma_-(R, d, \sigma) \end{cases}$$

for some $T > 0$ sufficiently large.

The first condition is clear.

For the second condition, let $w_- \in S^0_{2n}(1)$ have compact support and observe that for any $T>0$

$$w_-(x,hD_x)R(E+i0)u_- = w_-(x,hD_x)\frac{i}{h}\int_0^T e^{\frac{i}{h}tE}e^{-\frac{i}{h}tP}dt\, u_- + e^{\frac{i}{h}TE}w_-(x,hD_x)R(E+i0)e^{-\frac{i}{h}TP}u_-.$$

For the second term, observe that, by [5, Lemma 5.1] there exist $\sigma_+ \in (0,1)$ and $T_0>0$ such that for $T>T_0$

$$MS\left(e^{-\frac{iT}{h}P}u_-\right) \subset T^*B\left(0,\frac{R}{2}\right) \cup \Gamma_+\left(\frac{R}{2},d,\sigma_+\right).$$

Let, now, $w_+ \in S^0_{2n}(1)$ have compact support in $\Gamma_+\left(\frac{R}{3},d_1,\bar\sigma_+\right)$ for some $d_1>d$ and $\bar\sigma_+<\sigma_+$ with $w_+=1$ on $MS\left(e^{-\frac{iT}{h}P}u_-\right)\cap\Gamma_+\left(\frac{R}{2},d,\sigma_+\right)$ and let $\chi \in C_c^\infty(\mathbb{R}^n)$ be such that $\chi\equiv 1$ on $B\left(0,\frac{R}{2}\right)$. Then two consecutive applications of [16, Lemma 2.3] give

$$\begin{aligned}
&w_-(x,hD_x)R(E+i0)e^{-\frac{i}{h}TP}u_-\\
&= w_-(x,hD_x)R(E+i0)\chi e^{-\frac{i}{h}TP}u_- + w_-(x,hD_x)R(E+i0)w_+(x,hD_x)e^{-\frac{i}{h}TP}u_-\\
&\quad + \mathcal{O}(h^\infty)\\
&= \mathcal{O}(h^\infty).
\end{aligned}$$

The same proof as of [5, Lemma 5.1] now gives that for $R>0$ sufficiently large, we have that $\Lambda_\pm \cap T^*\left(\mathbb{R}^n\backslash B\left(0,\frac{R}{2}\right)\right) \subset \Gamma_\pm\left(\frac{R}{2},d,\sigma_\pm\right)$. Therefore, by [7, Theorem 2.6] and [7, Remark 2.7], if $Op_h(a_\pm)$ have compact wavefront sets in $\Gamma_\pm\left(\frac{R}{2},d,\sigma_\pm\right)$ near $\rho_\pm$, respectively, then microlocally near $(\rho_+,\rho_-) \in \Lambda_+\setminus\tilde\Lambda_+(\rho_-)\times\Lambda_-\setminus\tilde\Lambda_-$,

(2)

$$Op_h(a_+)R(E+i0)Op_h(a_-) \equiv Op_h(a_+)\mathcal{J}(E)\frac{i}{h}\int_o^T e^{\frac{i}{h}tE}e^{-\frac{i}{h}tP}dt\, Op_h(a_-),$$

if $\operatorname{supp} a_+$ is close to $(0,0)$

$$Op_h(a_+)R(E+i0)Op_h(a_-) \equiv e^{-\frac{i}{h}s(P-E)}Op_h\left(a_{+,s}\right)\mathcal{J}(E)\frac{i}{h}\int_o^T e^{\frac{i}{h}tE}e^{-\frac{i}{h}tP}dt\, e^{\frac{i}{h}s(P-E)}Op_h(a_-),$$

if $\operatorname{supp} a_+$ is far from $(0,0)$, $s>0$ is large enough,

and $\text{ess-supp}_h\, a_{+,s} \subset \exp(-sH_p)\,\text{ess-supp}_h\, a_+$

where microlocally near $(\rho_+,\rho_-) \in \Lambda_+\backslash\tilde\Lambda_+(\rho_-)\times\Lambda_-\backslash\tilde\Lambda_-$, $\mathcal{J}(E) \in \mathcal{I}_h^{-\frac{\sum_{j=1}^n\lambda_j}{2\lambda_1}}\left(\mathbb{R}^{2n},\Lambda_+\times\Lambda_-\right)$.

We now have the following

Lemma 1. $\frac{i}{h}\int_0^T e^{\frac{i}{h}tE}e^{-\frac{i}{h}tP}dt \in \mathcal{I}_h^{\frac{1}{2}}\left(\mathbb{R}^{2n}, \Lambda_E(R)\right)$, *where*
$\Lambda_E(R) = \left(\cup_{t>0}\,\mathrm{graph}\exp(tH_p)|_{p^{-1}(E)}\right)'$.

Proof. We recall the well known fact that $e^{-\frac{i}{h}sP} \in \mathcal{I}_h^0\left(\mathbb{R}^{2n}, (\mathrm{graph}\exp(sH_p))'\right)$ for $s \in \mathbb{R}$. For t sufficiently small we further have from [15, Proposition IV-14]

$$K_{e^{-\frac{i}{h}t(P-E)}} = \frac{i}{(2\pi h)^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(\varphi(t,x,\theta)-y\cdot\theta+tE)}a(x,y,\theta)d\theta, \tag{3}$$

where $\varphi \in C^\infty(\mathbb{R}^{2n+1})$ satisfies $\varphi_t'+p\,(x,\varphi_x')=0$ and $(x,\nabla_x\varphi(t,x,\theta)) = \exp(tH_p)\,(\nabla_\theta\varphi(t,x,\theta),\theta)$, and $a \in S_{3n}^0(1)$.

We now use the following result, the proof of which we postpone until later.

Lemma 2. *Let* $\chi \in C_c^\infty(\mathbb{R})$. *Then*

$$\frac{i}{h}\int_0^\infty \chi(t)e^{-\frac{i}{h}t(P-E)}dt \in \mathcal{I}_h^{\frac{1}{2}}\left(\mathbb{R}^{2n}, \Lambda_E(R)\right).$$

Let, now, $\chi \in C_c^\infty(\mathbb{R}^n)$ have support near 0 and satisfy $\sum_{l\in\epsilon\mathbb{Z}}\chi(t-l) = 1$ for $t \in \mathbb{R}$ and some $\epsilon > 0$ sufficiently small. Then

$$\frac{i}{h}\int_0^T e^{-\frac{i}{h}t(P-E)}dt = \frac{i}{h}\int_0^T \sum_{l\in\epsilon\mathbb{Z}}\chi(t-l)e^{-\frac{i}{h}t(P-E)}dt = \frac{i}{h}\sum_{l\in\epsilon\mathbb{Z}}\int_l^{T+l}\chi(s)e^{-\frac{i}{h}s(P-E)}ds\; e^{-\frac{i}{h}l(P-E)}.$$

It is now easy to see that the manifolds

$$\Lambda_E(R)' \times \mathrm{graph}\exp\left(tH_p\right)$$

and

$$T^*\mathbb{R}^n \times \mathrm{diag}\left(T^*\mathbb{R}^n \times T^*\mathbb{R}^n\right) \times T^*\mathbb{R}^n$$

intersect transversely and therefore

$$\frac{i}{h}\int_0^T e^{-\frac{i}{h}t(P-E)}dt \in \mathcal{I}_h^{\frac{1}{2}}\left(\mathbb{R}^{2n}, \Lambda_E(R)\right).$$

□

We now return to the analysis of (2), It is easy to see that the manifolds

$$\Lambda_+ \times \Lambda_-' \times \Lambda_E(R)'$$

and

$$T^*\mathbb{R}^n \times \mathrm{diag}\left(T^*\mathbb{R}^n \times T^*\mathbb{R}^n\right) \times T^*\mathbb{R}^n$$

intersect cleanly with excess 1 and from (2) and [9] we then have that microlocally near $(\rho_+,\rho_-) \in \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-$, $R(E+i0) \in \mathcal{I}_h^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}} \left(\mathbb{R}^{2n}, \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-\right)$.

The second part of the theorem is proven analogously.

Proof of Lemma 2. As in (3) we have

$$\frac{i}{h}\int \chi(t) e^{-\frac{i}{h}t(P-E)}dt = \frac{i}{(2\pi)^n h^{n+1}}\int_0^\infty \int_{\mathbb{R}^n} \chi(t) e^{-\frac{i}{h}(\varphi(t,x,\theta)-y\cdot\theta+tE)} a(t,x,y,\theta)d\theta dt.$$

We shall prove that $\Phi(x,y;t,\theta) \stackrel{\text{def}}{=} \varphi(t,x,\theta) - y\theta + tE$ is a non-degenerate phase function. Let

$$C_\Phi \stackrel{\text{def}}{=} \{(x,y,t,\theta) \in \mathbb{R}^{3n+1} : \nabla_{t,\theta}\Phi(x,y;t,\theta) = 0\}$$
$$= \{(x,y,t,\theta) \in \mathbb{R}^{3n+1} : \varphi_t'(t,x,\theta) = -E, \nabla_\theta \varphi(t,x,\theta) = y\}$$

and for $(x,y,t,\theta) \in C_\Phi$ consider

$$\begin{bmatrix} d\Phi_t' \\ d\Phi_\theta' \end{bmatrix}(x,y;t,\theta) = \begin{bmatrix} \Phi_{tx}'' & \Phi_{ty}'' & \Phi_{tt}'' & \Phi_{t\theta}'' \\ \Phi_{\theta x}'' & \Phi_{\theta y}'' & \Phi_{\theta t}'' & \Phi_{\theta\theta}'' \end{bmatrix}(x,y;t,\theta) = \begin{bmatrix} \varphi_{tx}'' & 0 & \varphi_{tt}'' & \varphi_{t\theta}'' \\ \varphi_{\theta x}'' & I & \varphi_{\theta t}'' & \varphi_{\theta\theta}'' \end{bmatrix}(x,y;t,\theta)$$

The bottom n rows in the above matrix are clearly linearly independent. The last row is never 0 for (x,y,t,θ) such that $\varphi_t(t,x,\theta) = -E = -p(x,\varphi_x(t,x,\theta))$ because from Assumption 2 it follows that $dp \neq 0$ on $\{p=E\}\setminus\{(0,0)\}$. Therefore $d\Phi|_{C_\Phi}$ has maximum rank and Φ is a non-degenerate phase function. This implies that $\frac{i}{h}\int_0^T e^{-\frac{i}{h}t(P-E)}dt$ is an h-FIO associated to

$$\Lambda_\Phi \stackrel{\text{def}}{=} \{(x, \nabla_x\Phi(x,y;t,\theta); y, \nabla_y\Phi(x,y;t,\theta)) : (x,y;t,\theta) \in C_\Phi\}$$
$$= \{(x,\nabla_x\varphi(t,x,\theta); y, -\theta) : (x,y;t,\theta) \in C_\Phi\} = \Lambda_E(R).$$

From [2, Theorem 2] we obtain that the order of this h-FIO is $\frac{1}{2}$.

3. Microlocal Representation of the Spectral Function

Here we present the representation of the spectral function as an oscillatory integral operator near Microlocally near $(\rho_+,\rho_-) \in \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-$. The oscillatory integral representation near $(\rho_-,\rho_+) \in \Lambda_- \setminus \tilde{\Lambda}_-(\rho_+) \times \Lambda_+ \setminus \tilde{\Lambda}_+$ is analogous.

Theorem 2. *Let $(\rho_+,\rho_-) \in \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-$.*

Then there exists a non-degenerate phase function $\Psi \in C^\infty(\mathbb{R}^{2n+m})$ and a symbol $b \in S_{2n+m}^{1-\frac{\sum_{j=1}^n\lambda_j}{2\lambda_1}+\frac{m}{2}+\frac{n}{2}}(1)$ such that microlocally near (ρ_+,ρ_-)

$$e_E \equiv \int_{\mathbb{R}^m} e^{\frac{i}{h}\Psi(x,y,\tau)} b(x,y,\tau)d\tau.$$

Proof. The assertion of the theorem follows from [2, Theorem 1] and Theorem 1. $\square$

Remark. If $(\rho_+, \rho_-) \in \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-$ are such that the projection from $T^*\mathbb{R}^n$ to the base $\mathbb{R}^n$ restricted to $\Lambda_\pm$ is a diffeomorphism in some neighborhoods of $\rho_\pm$, we have from [6, Theorem 46.D] that near $\rho_\pm$, $\Lambda_\pm = \{(x, d_x S_+(x); y, d_y S_-(y))\}$, where $S_\pm = \int_{\gamma_\pm(\rho_\pm)} \frac{1}{2}\|\xi_\pm(t)\|^2 - V(x_\pm(t))dt$ are the actions over the half-trajectories $\gamma_\pm(\rho_\pm) = (x_\pm, \xi_\pm) \subset \tilde{\Lambda}_\pm$ which start at $\rho_\pm$ and approach $(0,0)$ as $t \to \mp\infty$. Therefore, from [2, Theorem 1] we have that there exist $b \in S_{2n}^{1-\frac{\sum_{j=1}^n \lambda_j}{2\lambda_1}+\frac{n}{2}}(1)$ such that

$$e_E \equiv e^{\frac{i}{h}(S_+ + S_-)}b \text{ microlocally near } (\rho_+, \rho_-) \in \Lambda_+ \setminus \tilde{\Lambda}_+(\rho_-) \times \Lambda_- \setminus \tilde{\Lambda}_-.$$

Appendix A. Elements of Semi-Classical Analysis

In this section we recall some of the elements of semi-classical analysis which we use in this paper. First we recall the definitions of the following two classes of symbols

$$S_{2n}^m(1) = \left\{a \in C^\infty\left(\mathbb{R}^{2n} \times (0, h_0]\right) : \forall \alpha, \beta \in \mathbb{N}^n, \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)\right| \le C_{\alpha,\beta} h^{-m}\right\}$$

and

$$S^{m,k}(T^*\mathbb{R}^n) = \left\{a \in C^\infty\left(T^*\mathbb{R}^n \times (0, h_0]\right) : \forall \alpha, \beta \in \mathbb{N}^n, \left|\partial_x^\alpha \partial_\xi^\beta a(x, \xi; h)\right| \le C_{\alpha,\beta} h^{-m} \langle \xi \rangle^{k-|\beta|}\right\},$$

where $h_0 \in (0, 1]$ and $m, k \in \mathbb{R}$. For $a \in S_{2n}^m(1)$ or $a \in S^{m,k}(T^*\mathbb{R}^n)$ we define the corresponding semi-classical pseudodifferential operator of class $\Psi_h^m(1, \mathbb{R}^n)$ or $\Psi_h^{m,k}(\mathbb{R}^n)$, respectively, by setting

$$Op_h(a)u(x) = \frac{1}{(2\pi h)^n} \int\int e^{\frac{i\langle x-y, \xi\rangle}{h}} a(x, \xi; h) u(y)\, dy d\xi,\ u \in \mathcal{S}(\mathbb{R}^n),$$

and extending the definition to $\mathcal{S}'(\mathbb{R}^n)$ by duality (see [8]). Here we work only with symbols which admit asymptotic expansions in h and with pseudodifferential operators which are quantizations of such symbols. For $A \in \Psi_h^k(1, \mathbb{R}^n)$ or $A \in \Psi_h^{m,k}(\mathbb{R}^n)$, we shall use $\sigma_0(A)$ and $\sigma(A)$ to denote its principal symbol and its complete symbol, respectively. A semi-classical pseudodifferential operator is said to be of principal type if its principal symbol a_0 satisfies

$$a_0 = 0 \implies da_0 \neq 0.$$

For $a \in S^{m,k}(T^*\mathbb{R}^n)$ or $a \in S^m_{2n}(1)$ we define

$$
\begin{aligned}
&\text{ess-supp}_h\, a \\
&= \left\{ (x,\xi) \in T^*\mathbb{R}^n \mid \exists\, \epsilon > 0\ \partial_x^\alpha \partial_\xi^\beta a(x',\xi') = \mathcal{O}_{C(B((x,\xi),\epsilon))}(h^\infty),\ \forall \alpha,\beta \in \mathbb{N}^n \right\}^c \\
&\cup \Bigg(\Bigg\{ (x,\xi) \in T^*\mathbb{R}^n \backslash \{0\} \mid \exists\, \epsilon > 0\ \partial_x^\alpha \partial_\xi^\beta a(x',\xi') = \mathcal{O}\left(h^\infty \langle\xi\rangle^{-\infty}\right), \\
&\quad \text{uniformly in } (x',\xi') \text{ such that } \|x - x'\| + \frac{1}{\|\xi'\|} + \left\| \frac{\xi}{\|\xi\|} - \frac{\xi'}{\|\xi'\|} \right\| < \epsilon \Bigg\} / \mathbb{R}_+ \Bigg)^c \\
&\subset T^*\mathbb{R}^n \sqcup S^*\mathbb{R}^n,
\end{aligned}
$$

where we define $S^*\mathbb{R}^n = (T^*\mathbb{R}^n \backslash \{0\})/\mathbb{R}_+$ and denote by $\bullet^c$ the complement of the set $\bullet$. For $A \in \Psi_h^{m,k}(\mathbb{R}^n)$, we then define

$$WF_h(A) = \text{ess-supp}_h\, a,\ A = Op_h(a).$$

We also define the class of semi-classical distributions $\mathcal{D}'_h(\mathbb{R}^n)$ with which we will work here

$$
\begin{aligned}
\mathcal{D}'_h(\mathbb{R}^n) = &\{ u \in C_h^\infty((0,1]; \mathcal{D}'(\mathbb{R}^n)) : \forall \chi \in C_c^\infty(\mathbb{R}^n)\ \exists\, N \in \mathbb{N} \text{ and } C_N > 0 : \\
&|\mathcal{F}_h(\chi u)(\xi)| \le C_N h^{-N} \langle\xi\rangle^N \}
\end{aligned}
$$

where

$$\mathcal{F}_h(\chi u)(\xi) = \left\langle e^{-\frac{i}{h}\langle\cdot,\xi\rangle}, \chi u \right\rangle,$$

and $\langle\cdot,\cdot\rangle$ denotes the distribution pairing. We also extend this definition in the obvious way to $\mathcal{E}'_h(\mathbb{R}^n)$.

The L^2-based semi-classical Sobolev spaces $H_h^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, which consist of the distributions $u \in \mathcal{E}'_h(\mathbb{R}^n)$ such that $\|u\|^2_{H_h^s(\mathbb{R}^n)} \overset{\text{def}}{=} \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\mathcal{F}_h(u)(\xi)|^2\, d\xi < \infty$.

For $u \in \mathcal{D}'_h(\mathbb{R}^n)$ we also define its finite semi-classical wavefront set as follows.

Definition 2. *Let $u \in \mathcal{D}'_h(\mathbb{R}^n)$ and let $(x_0,\xi_0) \in T^*(\mathbb{R}^n)$. Then the point (x_0,ξ_0) does not belong to $WF_h^f(u)$ if there exist $\chi \in C_c^\infty(\mathbb{R}^n)$ with $\chi(x_0) \neq 0$ and an open neighborhood U of ξ_0, such that $\forall N \in \mathbb{N}$, $\forall \xi \in U$, $|\mathcal{F}(\chi u)(\xi)| \le C_N h^N$.*

We say that $u = v$ *microlocally* (or $u \equiv v$) near an open set $U \subset T^*\mathbb{R}^n$, if $P(u-v) = \mathcal{O}(h^\infty)$ in $C_c^\infty(\mathbb{R}^n)$ for every $P \in \Psi_h^0(1,\mathbb{R}^n)$ such that

$$WF_h(P) \subset \tilde{U},\ \bar{U} \Subset \tilde{U} \Subset T^*\mathbb{R}^n,\ \tilde{U} \text{ open}.$$

We also say that u satisfies a property $\mathcal{P}$ *microlocally* near an open set $U \subset T^*\mathbb{R}^n$ if there exists $v \in \mathcal{D}'_h(\mathbb{R}^n)$ such that $u = v$ microlocally near U and v satisfies property $\mathcal{P}$.

We extend these notions to compact manifolds through the following definition of semi-classical pseudodifferential operators on compact manifolds. Let M be a smooth compact manifold and $\kappa_j : M_j \to X_j$, $j = 1, \dots, N$, a set of local charts. A linear continuous operator $A : C^\infty(M) \to \mathcal{D}'_h(M)$ belongs to $\Psi_h^m(1, M)$ or $\Psi_h^{m,k}(T^*M)$ if for all $j \in \{1, \dots, N\}$ and $u \in C_c^\infty(M_j)$ we have $Au \circ \kappa_j^{-1} = A_j\left(u \circ \kappa_j^{-1}\right)$ with $A_j \in \Psi_h^m(1, X_j)$ or $A_j \in \Psi_h^{m,k}(X_j)$, respectively, and $\chi_1 A \chi_2 : \mathcal{D}'_h(M) \to h^\infty C^\infty(M)$ if $\operatorname{supp}\chi_1 \cap \operatorname{supp}\chi_2 = \emptyset$.

We now define global semi-classical Fourier integral operators.

Definition 3. *Let M be a smooth k-dimensional manifold and let $\Lambda \subset T^*M$ be a smooth closed Lagrangian submanifold with respect to the canonical symplectic structure on T^*M. Let $r \in \mathbb{R}$. Then the space $I_h^r(M, \Lambda)$ of semi-classical Fourier integral distributions of order r associated to Λ is defined as the set of all $u \in \mathcal{E}'_h(M)$ such that*

$$\left(\prod_{j=0}^{N} A_j\right)(u) = \mathcal{O}_{L^2(M)}\left(h^{N-r-\frac{k}{4}}\right), h \to 0,$$

for all $N \in \mathbb{N}_0$ and for all $A_j \in \Psi_h^0(1, M)$, $j = 0, \dots, N-1$, with compact wavefront sets and principal symbols vanishing on Λ, and any $A_N \in \Psi_h^0(1, M)$ with compact wavefront set.

A continuous linear operator $C_c^\infty(M_1) \to \mathcal{D}'_h(M_2)$, where M_1, M_2 are smooth manifolds, whose Schwartz kernel is an element of $I_h^r(M_1 \times M_2, \Lambda)$ for some Lagrangian submanifold $\Lambda \subset T^*M_1 \times T^*M_2$ and some $r \in \mathbb{R}$ will be called a global semi-classical Fourier integral operator of order r associated to Λ. We denote the space of these operators by $\mathcal{I}_h^r(M_1 \times M_2, \Lambda)$.

Lastly, we define the microlocal equivalence of two semi-classical Fourier integral operators.

Definition 4. *Let M_j, $j = 1, 2$, be smooth manifolds, $\Lambda \subset T^*M_1 \times T^*M_2$— a Lagrangian submanifold, and $W, W' \in \mathcal{I}_h^r(M_1 \times M_2, \Lambda)$ for some $r \in \mathbb{R}$. For open or closed sets $U \subset T^*M_1$ and $V \subset T^*M_2$ the operators W and W' are said to be microlocally equivalent near $U \times V$ if there exist open sets $\tilde{U} \Subset T^*M_1$ and $\tilde{V} \Subset T^*M_2$ with $\bar{U} \Subset \tilde{U}$ and $\bar{V} \Subset \tilde{V}$ such that for any $A \in \Psi_h^0(1, M_1)$ and $B \in \Psi_h^0(1, M_2)$ with $WF_h(A) \subset \tilde{U}$ and $WF_h(B) \subset \tilde{V}$ we have that*

$$B\left(W - W'\right)A = \mathcal{O}\left(h^\infty\right) : \mathcal{D}'_h(M_1) \to C^\infty(M_2).$$

If $X \subset M_1 \times M_2$ is an open set, we shall also write $W \in \mathcal{I}_h^r(X, \Lambda)$ to indicate that $K_W|_X \in I_h^r(X, \Lambda)$, where $\Lambda \subset T^*X$ is a Lagrangian submanifold. We shall also write $W \equiv W'$ near $V \times U$.

References

[1] Alexandrova, Ivana. Semi-Classical Behavior of the Spectral Function. Proc. AMS. **2006**, *134* (8), 2295 – 2302.

[2] Alexandrova, Ivana. Semi-Classical Wavefront Set and Fourier Integral Operators. To appear in Canadian Journal of Mathematics.

[3] Alexandrova, Ivana. Structure of the Semi-Classical Amplitude for General Scattering Relations. Communications in Partial Differential Equations **2005**, *30* (10-12), 1505–1535.

[4] Alexandrova, Ivana. Structure of the Short Range Amplitude for General Scattering Relations. Asymptotic Analysis **2006**, *50*, 13-30.

[5] Alexandrova, Ivana. Bony, Jean-François; Ramond, Thierry. Semiclassical Scattering Amplitude at the Maximum Point of the Potential. Preprint.

[6] Arnold, Vladimir. *Mathematical Methods of Classical Mechanics*; Springer-Verlag: New York, 1980.

[7] Bony, Jean-François; Fujie, Setsuro; Ramond, Thierry; Zerzeri, Maher. Microlocal Kernel of Pseudodifferential Operators at a Hyperbolic Fixed Point. Preprint.

[8] Dimassi, Mouez; Sjöstrand, Johannes. *Spectral Asymptotics in the Semi-Classical Limit*; Cambridge University Press: Cambridge, 1999.

[9] Sandrine, Dozias. Opérateurs h-pseudodifférentiels à flot périodique. Ph. D. Thesis. Université Paris Nord. 1994.

[10] Gérard, Christian and Martinez, Andre. Semiclassical Asymptotics for the Spectral Function of Long-Range Schrödinger Operators. Journal of Functional Analysis. **1989**, *184* (1), 226 – 254.

[11] Hassell, Andrew and Wunsch, Jared. The Semiclassical Resolvent and the Propagator for Nontrapping Scattering Metrics. Preprint.

[12] Helffer, Bernard, and Sjöstrand, Johannes. Multiple Wells in the Semiclassical Limit. III. Interaction through nonresonant Wells. Math. Nachr. **1985**, *124*, 263–313.

[13] Popov, Georgi. Spectral Asymptotics for Elliptic Second Order Differential Operators. J. Math. Kyoto Univ. **1985**, *25* (4), 659–681.

[14] Popov, Georgi and Shubin, Mikhail. Asymptotic Expansion of the Spectral Function for Second Order Elliptic Operators in $\mathbf{R}^n$. Funct. Anal. Appl. **1983**, *17*.

[15] Robert, Didier. *Autour de l'Approximation Semi-Classique*; Birkhäuser, Boston, 1987.

[16] Robert, Didier and Tamura, Hideo. Asymptotic Behavior of Scattering Amplitudes in Semi-Classical and Low Energy Limits. Annales de l'Institut Fourier **1989**, *39* (1), 155–192.

[17] Robert, Didier and Tamura, Hideo. Semi-Classical Asymptotics for Local Spectral Densities and Time Delay Problems in Scattering Processes. Journal of Functional Analysis **1988**, *80*, 124–147.

[18] Vainberg, Boris. Complete Asymptotic Expansion of the Spectral Function of Second-Order Elliptic Operators in $\mathbb{R}^n$. Matematicheskii Sbornik **1984**, *123 (165)*, (2), 195–211.

[19] Vainberg, Boris. *Asymptotic Methods in Equations of Mathematical Physics.* Gordon and Breach Science Publishers. New York. **1989**.

Ivana Alexandrova, Department of Mathematics, East Carolina University, Greenville, NC 27858, USA

E-mail address: `alexandrovai@ecu.edu`

Jean-François Bony, Laboratoire MAB, CNRS, Université de Bordeaux I, Bordeaux, France

E-mail address: `bony@math.u-bordeaux1.fr`

Thierry Ramond, Mathématiques, Université Paris Sud, (UMR CNRS 8628), France

E-mail address: `thierry.ramond@math.u-psud.fr`