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Micro-Macro Duality and Emergence of Macroscopic Levels

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Abstract

The mutual relation between quantum Micro and classical Macro is clarified by a unified formulation of instruments describing measurement processes and the associated amplification processes, from which some perspective towards a description of emergence processes of spacetime structure is suggested.

1 Sectors as Quantum-Classical Boundary

In the discussion on the mutual relations between micro- and macroscopic levels in the physical nature, it is important to identify the location of quantum-classical boundary. As is well known, however, the quantum theory lacks an intrinsic length scale to distinguish between quantum and classical levels, and hence, we need to find such an effective "boundary" in each problem according to its specific configuration of relevant scales. Once we succeed in it, the theoretical description of such a boundary can be given in terms of the notion of sectors. So, let me start from a brief account of the notions of *sectors*, *inter-sectorial structures*, *order parameters* to parametrize sectors and so on. According to these notions, we can formulate in a clear-cut manner the most important aspects of the mutual relations between the microscopic quantum world and the macroscopic classical levels, which is to be interpreted as a mathematical formulation [1] of the physically essential idea of "quantum-classical correspondence": the "boundary" and the gap between the former with non-commutative algebras of quantum physical variables and the latter with commutative algebras can be described by means of the notion of a (*superselection*) *sector structure* consisting of a family of *sectors* (or *pure phases*). To define it, we need to classify representations and states of a C*-algebra $\mathfrak{A}$ of quantum observables according to the *quasi-equivalence* $\pi_1 \approx \pi_2$ [2] defined by the unitary equivalence of representations π_1, π_2 *up to multiplicity*, which is equivalent to the isomorphism of representing von Neumann algebras $\pi_1(\mathfrak{A})'' \simeq \pi_2(\mathfrak{A})''$. A *sector*, or, a *pure phase* in the physical contexts, is then defined by a quasi-equivalence class of *factor* representations and states corresponding to a von Neumann algebra with a trivial centre which is a minimal unit among quasi-equivalence classes.

Representations belonging to different sectors π_a, π_b are mutually *disjoint* with no non-zero intertwiners, namely, any bounded operator T from the representation space $\mathfrak{H}_{\pi_a}$ of π_a to that $\mathfrak{H}_{\pi_b}$ of π_b vanishes, $T = 0$, if it satisfies $T\pi_a(A) = \pi_b(A)T$ for $\forall A \in \mathfrak{A}$. If π is not a factor representation belonging to a sector, it can be uniquely decomposed into the direct sum (or integral) of sectors, through the spectral decomposition of a non-trivial commutative algebra $\mathfrak{Z}(\pi(\mathfrak{A})'') = \pi(\mathfrak{A})'' \cap \pi(\mathfrak{A})' = \mathfrak{Z}_\pi(\mathfrak{A})$ as the centre of $\pi(\mathfrak{A})''$ admitting a "simultaneous diagonalization". Here each sector contained in π is faithfully parametrized by the Gel'fand spectrum $Spec(\mathfrak{Z}_\pi(\mathfrak{A}))$ of the centre. Thus, commutative classical observables belonging to the centre $\mathfrak{Z}_\pi(\mathfrak{A})$ physically plays the role of *macroscopic order parameters* and $Spec(\mathfrak{Z}_\pi(\mathfrak{A}))$ can be regarded as the *classifying space of sectors* to distinguish different sectors. In this way, we find in a *mixed phase* the coexistence of quantum(=intra-sectorial) and classical systems, which constitute an *inter-sectorial* structure concisely described by the centre $\mathfrak{Z}_\pi(\mathfrak{A})$ consisting of order parameters.

The traditional understanding of a sector is a "coherent subspace" where the "superposition principle" holds, but this "definition" applies only to sectors containing irreducible representations and pure states which are meaningful only in the contexts discussing the global aspects of quantum fields in the *vacuum* situation. Moreover, it leads to such a misleading interpretation of a "superselection rule" as an obstruction to the superposition of state vectors belonging to different sectors; actually the superposition of this sort is never "forbidden" but it simply reduces to statistical *mixtures* instead of superposed pure states, for lack of observables with non-vanishing off-diagonal terms connecting different sectors. In sharp contrast, the above general definition based on factoriality is applicable to any pure phases associated with reducible factor representations and mixed states which are common in the thermal and/or local aspects of quantum fields (latter even in the vacuum situations), owing to the inevitable relevance of *non-type I representations* (for which irreducible representations are almost useless).

2 Instruments for Intra-sectorial Searches

While the *inter-sectorial* structure can successfully be treated by means of the notions of sectors and of the macroscopic order parameters belonging to the centre, this is not sufficient for a satisfactory description of a given quantum system unless we combine it with the analysis of the intrinsic quantum structures *within* each sector, not only theoretically but also operationally (up to the resolution limits imposed by quantum theory itself). Since all the states in a sector share the same spectrum of the centre, however, the order parameters are of little use in the search of the *intra-sectorial* structures within a sector. For the purpose of detecting these invisible microscopic quantum structures we need a general scheme of quantum measurement which has been proposed in [3, 4] by extending the standard scheme [5] to systems with infinite degrees of freedom. This is based upon the notion of a maximal abelian subalgebra (MASA, for short) $\mathcal{A}$ of a factor von Neumann algebra $\mathcal{M} = \pi(\mathfrak{A})''$ describing a fixed sector, defined by the relation $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$; if we adopted the familiar condition $\mathcal{A} = \mathcal{A}'$ it would exclude the cases with $\mathcal{M}$ of non-type I common in quantum systems with infinite degrees of freedom. Given such a MASA $\mathcal{A} = \mathcal{A}' \cap \mathcal{M}$, the precise form of the *measurement coupling* can be specified between the observed system and the apparatus required for implementing a measurement, on the basis of which the central notion of *instrument* can concisely be formulated. The essence of the formulation can be summarized in terms of the following basic ingredients:

1. A (factor) von Neumann algebra $\mathcal{M}(:= \pi(\mathfrak{A})'')$ describing the observed system (in a fixed sector π) and its MASA $\mathcal{A} = \mathcal{M} \cap \mathcal{A}' = \mathcal{M}^{\mathcal{U}(\mathcal{A})}$ with the group $\mathcal{U}(\mathcal{A})$ of all unitaries in $\mathcal{A}$. Under the physically natural assumption that the representation Hilbert space $\mathfrak{H}_\pi$ of the present system $\mathcal{M}$ can be taken as separable, $\mathcal{A}$ as observables to be measured is generated by a locally compact abelian (Lie) group $\mathcal{U} \subset \mathcal{A} = \mathcal{U}''$ (with a Haar measure $d\mu$). Since the results of a measurement of $\mathcal{A}$ are given by the measured data belonging to *Spec*($\mathcal{A}$), the *algebra of the measuring system* can be identified with the subalgebra $\mathcal{A}$ itself of the observed system $\mathcal{M}$.

2. The *measurement coupling* between the observed and the measuring systems is specified by a representation W_U of the *Kac-Takesaki operator* (K-T operator, for short) W of the group $\mathcal{U}$ defined by $(W\eta)(u,v) := \eta(v^{-1}u, v)$ for $\eta \in L^2(\mathcal{U}\times\mathcal{U}, d\mu\otimes d\mu), u, v \in \mathcal{U}$ and characterized by the so-called pentagonal relation $W_{12}W_{23} = W_{23}W_{13}W_{12}$. When the action $\mathcal{M} \underset{\alpha}{\curvearrowleft} \mathcal{U}$ of the measuring system is implemented, $\alpha_u(M) = U_u M U_u^{-1}$ ($M \in \mathcal{M}$, $u \in \mathcal{U}$), by a unitary representation U of $\mathcal{U}$ on the (standard) representation Hilbert space $L^2(\mathcal{M})$ of $\mathcal{M}$, the representation W_U of W corresponding to $\alpha = AdU$ is defined by

$$(W_U\xi)(u) := U_u(\xi(u)) \quad \text{for } \xi \in L^2(\mathcal{M}) \otimes L^2(\mathcal{U}, d\mu),$$

satisfying the (modified) pentagonal relation

$$(W_U)_{12}W_{23} = W_{23}(W_U)_{13}(W_U)_{12},$$

and the intertwining relation $W_U(1 \otimes \lambda_u) = (U_u \otimes \lambda_u)W_U$. Here the suffices indicate the positions in the tensor product $L^2(\mathcal{M}) \otimes L^2(\mathcal{U}) \otimes$

$L^2(\mathcal{U})$ to which the operators act and λ_u is the regular representation of $\mathcal{U}$ defined by $(\lambda_u\eta)(v) := \eta(u^{-1}v)$ on $\eta \in L^2(\mathcal{U})$. The simplest standard choice of α common in the context of measurements is $\alpha_u(M) = uMu^{-1}$ (for $M \in \mathcal{M}$), $U_u = u$, which neglects the effect of the intrinsic dynamics of the observed system on the measurement process. In terms of the Lie generators X_a of the unitary representation U such that $U_u = \exp(\sum_a X_a\varphi^a(u))$, the coupling term can be written by $W_U = \exp(X_a \otimes \varphi^a(\hat{u}))$, where $\varphi^a(\hat{u})$ denotes an operator on $L^2(\mathcal{U})$ defined by $(\varphi^a(\hat{u})\eta)(u) = \varphi^a(u)\eta(u)$ for $\eta \in L^2(\mathcal{U}), u \in \mathcal{U}$ (similarly to the position operator $\hat{x}$ in quantum mechanics, where the displacement unitary $\lambda_x = \exp(-i\hat{p}x)$ corresponds to the unitary operator λ_u in the present context).

3. By restriction to $\mathcal{U}$ our measured data $\chi \in Spec(\mathcal{A})$ can be embedded as a group character $\chi \restriction_{\mathcal{U}}$ of $\mathcal{U}$ into the dual group $\widehat{\mathcal{U}}$ which is again a locally compact abelian group. By Fourier-transforming W_U to $\widetilde{W_U} := (id \otimes \mathcal{F})W_U(id \otimes \mathcal{F})^{-1}$ with $(\mathcal{F}\xi)(\gamma) := \int_{\mathcal{U}} \overline{\gamma(u)}\xi(u)d\mu(u)$ for $\xi \in L^2(\mathcal{U}, d\mu)$, we define an *instrument* $\mathcal{I}$ for measuring $\mathcal{A}$ by

$$\mathcal{I}(\Delta|\omega)(M) := (\omega \otimes m_{\mathcal{U}})\left(\widetilde{W_U}(M \otimes \chi_\Delta)\widetilde{W_U^*}\right),$$

for $M \in \mathcal{M}$, $\chi_\Delta \in \mathcal{A} = L^\infty(Spec(\mathcal{A}))$. While the identity element $\iota \in \widehat{\mathcal{U}}$ for a non-compact $\mathcal{U}$ is not represented by a normalized vector in $L^2(\mathcal{U})$, the above *invariant mean*[1] $m_{\mathcal{U}}$ over $\mathcal{U}$ physically plays the role of the *neutral position* ι of the measuring apparatus. All the ingredients relevant to a measurement process are incorporated in this instrument $\mathcal{I}$, such as the probability distribution $p(\Delta|\omega) = \mathcal{I}(\Delta|\omega)(1)$ of measured values of observables in $\mathcal{A}$ to be found in a Borel set $\Delta \subset Spec(\mathcal{A})$ and as the state change from an initial state ω to a final state $\mathcal{I}(\Delta|\omega)/p(\Delta|\omega)$ caused by the read-out of measured values $\in \Delta$ [5], according to which a process of the so-called "reduction of wave packets" is described.

4. Since $\mathcal{U}$ is abelian, we can consider the spectral decomposition, $U_u = \int_{\chi \in Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}} \overline{\chi(u)}dE(\chi)$ $(u \in \mathcal{U})$, of the unitary representation U (owing to the so-called SNAG theorem). Using this and the Fourier transform $V = (\mathcal{F} \otimes \mathcal{F})W^*(\mathcal{F} \otimes \mathcal{F})^{-1}$ of W as the K-T operator of the dual group $\widehat{\mathcal{U}}$ with the Plancherel measure $d\hat{\mu}$ satisfying the relation $(V\eta)(\gamma, \chi) = \eta(\gamma, \gamma^{-1}\chi)$ for $\eta \in L^2(\widehat{\mathcal{U}}, d\hat{\mu})$, we have a clearer picture of $\widetilde{W_U}$: $\widetilde{W_U} = \int_{\chi \in Spec(\mathcal{A})} dE(\chi) \otimes \lambda^*_\chi =: V^*_{\widehat{U}}$. In the Dirac notation

[1] To be precise, an invariant mean is a finitely additive measure as a state on a commutative C*-algebra, but, in general, it cannot be extended to a σ-additive measure. This point may play crucial roles in selecting which kind of quantities are readable with others non-readable.

(of non-normalizable generalized eigenvectors), the action of $V_{\widetilde{U}}$ on $L^2(\mathcal{M}) \otimes L^2(\widehat{\mathcal{U}})$ is given for $\gamma \in \widehat{\mathcal{U}}$, $\xi \in L^2(\mathcal{M})$ by $V_{\widetilde{U}}(\xi \otimes |\gamma\rangle) = \int_{\chi \in Spec(\mathcal{A})} dE(\chi)\xi \otimes |\chi\gamma\rangle$.

5. In terms of the above K-T operators, the crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$ is defined on $L^2(\mathcal{M}) \otimes L^2(\mathcal{U})$ as an important notion in the Fourier-Galois duality in the following two equivalent ways: either as a von Neumann algebra $\lambda^{\mathcal{M}}(L^1(\mathcal{U}, \mathcal{M}))''$ generated by the Fourier transform $\lambda^{\mathcal{M}}(\hat{F}) := \int_{\mathcal{U}} \hat{F}(u)U(u)d\mu(u)$ of $\mathcal{M}$-valued L^1-functions $\hat{F} \in L^1(\mathcal{U}, \mathcal{M})$ with the convolution product, $(\hat{F}_1 * \hat{F}_2)(u) = \int_{\mathcal{U}} \hat{F}_1(v)\alpha_v(\hat{F}_2(v^{-1}u)d\mu(v)$, mapped by $\lambda^{\mathcal{M}}$ into $\lambda^{\mathcal{M}}(\hat{F}_1 * \hat{F}_2) = \lambda^{\mathcal{M}}(\hat{F}_1)\lambda^{\mathcal{M}}(\hat{F}_2)$, or, as a von Neumann algebra $\pi_\alpha(\mathcal{M}) \vee (1 \otimes \lambda(\mathcal{U}))$ generated by $1 \otimes \lambda(\mathcal{U})$ and by

$$\pi_\alpha(\mathcal{M}) := \{\pi_\alpha(M) := Ad(W_U^*)(M \otimes 1);\ M \in \mathcal{M}\}.$$

These two versions are related by the mapping $\alpha(W) := Ad(W_U)$,

$$\lambda^{\mathcal{M}}(L^1(\mathcal{U}, \mathcal{M}))'' = (\mathcal{M} \otimes 1) \vee \{U_u \otimes \lambda_u; u \in \mathcal{U}\} \underset{\alpha(W)}{\overset{\alpha(W)^{-1}}{\rightleftarrows}} \pi_\alpha(\mathcal{M}) \vee (1 \otimes \lambda(\mathcal{U}),$$

which can be understood as the Schrödinger and Heisenberg pictures: the former $(\mathcal{M} \otimes 1) \vee \{U_u \otimes \lambda_u; u \in \mathcal{U}\}$ is in the Schrödinger picture with *unchanged* microscopic observables $\mathcal{M} \otimes 1$ and with the *coupling* $U_u \otimes \lambda_u$ *to change macroscopic states*, while, in the latter, all the coupling effects are concentrated in the observables $\pi_\alpha(\mathcal{M})$ in contrast to the *kinematical changes* of macroscopic *states* caused by $\lambda(\mathcal{U})$.

In the case of the instrument $\mathcal{I}$, the algebra to be observed is the tensor algebra $\mathcal{M} \otimes \mathcal{A} = \mathcal{M} \otimes L^\infty(Spec\mathcal{A})$ realized in the *initial* and *final* stages, respectively, before and after the measuring processes according to the switching-on and -off of the coupling α: $\mathcal{M} \otimes \mathcal{A} = \mathcal{M} \rtimes_{\alpha = id_{\mathcal{M}}} \mathcal{U} \to \mathcal{M} \rtimes_\alpha \mathcal{U} \to \mathcal{M} \otimes \mathcal{A}$, similarly to the scattering processes. All the effects of the measurement coupling $\widetilde{W_U}$ are encoded in the form of *macroscopic state changes* recorded in the spectrum of the non-trivial centre $\mathfrak{Z}(\mathcal{M} \otimes \mathcal{A}) = \mathcal{A} = L^\infty(Spec\mathcal{A})$ of $\mathcal{M} \otimes \mathcal{A}$, playing the same roles as the order parameters to specify sectors in the inter-sectorial context. For these reasons, the most natural physical essence of the formalism based on an instrument $\mathcal{I}$ should be found in the *interaction picture*, whose *coupling* term $\widetilde{W_U} = (id \otimes \mathcal{F})W_U(id \otimes \mathcal{F})^{-1}$ is responsible for deforming the decoupled algebra $\mathcal{M} \otimes \mathcal{A}$ into the above crossed product $\mathcal{M} \rtimes_\alpha \mathcal{U}$.

3 Amplification in Intra-sectorial Measurements

While the notion of an instrument provides a sufficient tool for the operational description of a measurement, the above state changes describe only *microscopic* changes of quantum states $\xi \otimes |\iota\rangle \longrightarrow \xi_\gamma \otimes |\gamma\rangle$ of the composite system of the observed system and the *probe system* taking place at their microscopic contact point. The question remains untouched as to how the invisible microscopic changes in the quantum states are *transformed into visible macroscopic changes* of the measuring pointer, without which measured values $\in Spec(\mathcal{A}) \subset \widehat{\mathcal{U}}$ cannot be read out or registered. To answer this question we need a mathematical formulation of the process of *amplification* from the microscopic state changes in the probe system caused by the measurement coupling into the macroscopic changes in the spatial positions of the measurement pointer. While I am not aware of known results of this sort, this kind of amplifying mechanism seems to be universally relevant to any bridges between micro-quantum systems and macro-classical world.

In the present approach, the mathematical essence of the amplification processes can be seen in the following simple form [6] based upon the quasi-equivalence $\lambda^{\otimes m} \approx \lambda^{\otimes n}$ $(\forall m, n \in \mathbb{N})$ among arbitrary tensor powers $\lambda^{\otimes n} = \lambda \otimes \cdots \otimes \lambda$ of the regular representation of a locally compact group $\widehat{\mathcal{U}}$ via the K-T operator V related closely to the measurement coupling. When V is applied arbitrarily many times to an initial state $\xi \otimes |\iota\rangle \otimes |\iota\rangle \cdots \otimes |\iota\rangle$ of the composite system where $\xi = \sum_{\gamma \in \widehat{\mathcal{U}}} c_\gamma \xi_\gamma$ is an initial state of the observed system, the resulting state becomes:

$$\begin{aligned}
&V_{N,N+1} \cdots V_{23} (V_{\widetilde{\mathcal{U}}})_{12} (\xi \otimes \underbrace{|\iota\rangle \otimes |\iota\rangle \cdots \otimes |\iota\rangle}_{N}) \\
&= \sum_{\gamma \in \widehat{\mathcal{U}}} c_\gamma V_{N,N+1} \cdots V_{34} V_{23} (\xi_\gamma \otimes |\gamma\rangle \otimes |\iota\rangle \cdots \otimes |\iota\rangle) \\
&= \sum_{\gamma \in \widehat{\mathcal{U}}} c_\gamma V_{N,N+1} \cdots V_{34} (\xi_\gamma \otimes |\gamma\rangle \otimes |\gamma\rangle \cdots \otimes |\iota\rangle) = \cdots \\
&= \sum_{\gamma \in \widehat{\mathcal{U}}} c_\gamma \xi_\gamma \otimes \underbrace{|\gamma\rangle \otimes |\gamma\rangle \cdots \otimes |\gamma\rangle}_{N} \underset{N \gg 1}{\longrightarrow} \sum_{\gamma \in \widehat{\mathcal{U}}} c_\gamma \xi_\gamma \otimes [|\gamma\rangle^{\otimes N}],
\end{aligned}$$

(whose validity is, to be precise, restricted to the case with $\widehat{\mathcal{U}}$ having a discrete spectrum). However, the corresponding formula in the Heisenberg picture given by

$$\begin{aligned}
&A \otimes f_2 \otimes \cdots \otimes f_{N+1} \\
&\longmapsto ((V_{\widetilde{\mathcal{U}}})^*_{12} V^*_{23} \cdots V^*_{N,N+1} (A \otimes f_2 \otimes \cdots \otimes f_{N+1}) V_{N,N+1} \cdots V_{23} (V_{\widetilde{\mathcal{U}}})_{12} \\
&= Ad((V_{\widetilde{\mathcal{U}}})^*_{12}) Ad(V^*_{23}) \cdots Ad(V^*_{V_{N,N+1}})(A \otimes f_2 \otimes \cdots \otimes f_{N+1}) \\
&= Ad(V^*_{\widetilde{\mathcal{U}}})(A \otimes Ad(V^*)(f_2 \otimes Ad(V^*)(\cdots \otimes Ad(V^*)(f_N \otimes f_{N+1}))) \cdots) \\
&\qquad\qquad\qquad\qquad \text{for } A \in \mathcal{M} \text{ and } f_i \in L^\infty(\widehat{\mathcal{U}}),
\end{aligned}$$

is similar to the one appearing in Accardi's formulation of quantum Markov chain [7] which is indepedent of the discreteness of the spectrum. According to the general basic idea of "quantum-classical correspondence", a classical macroscopic object can be identified with a condensed state of infinite number of quanta, as well exemplified by the macroscopic magnetization of Ising or Heisenberg ferromagnets described by the aligned states $|+\rangle^{\otimes\infty}$ of infinite number of microscopic spins. Therefore, the above state $|\gamma\rangle^{\otimes N}$ (with $N \gg 1$) can physically be interpreted as representing a macroscopic position of the measuring pointer, and hence, the above repeated action of the K-T operator V describes a cascade process or a domino effect of "decoherence" to amplify a state change at the microscopic end of the apparatus into the macroscopic classical motion $\iota \rightarrow \gamma$ of the measuring pointer. It is remarkable here that the quasi-equivalence of arbitrary tensor powers $\lambda^{\otimes n}$ of the regular representation λ guarantees the "unitarity" of the above amplification process, which provides the mathematical basis for not only the "repeatablity hypothesis" but also the possibility of the recurrent quantum interference even after the contact with the measuring apparatus under the situation that the number N of repetition need not be regarded as a real infinity (as the size of N depends on the length of the interval responsible for the amplification process between the microscopic and macroscopic ends of the measurement apparatus and also on the reaction rate of the flip from $|\iota\rangle$ to $|\gamma\rangle$). In this respect, the problem as to whether the situation is completely made classical or not depends highly on the relative configurations among many large or small numbers, which can consistently be described in the framework of the non-standard analysis (see, for instance, [8]). In relation to this, it is also interesting to note that the above amplification process is closely related to a Lévy process through its "infinite divisibility" as follows: similarly to the affine property $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$ $(\forall\lambda, \mu > 0)$ of a map f defined on a convex set which follows from the addivitiy $f(x+y) = f(x) + f(y)$, we can derive $\lambda \approx \lambda^{n/m}$ $(\forall m, n \in \mathbb{N})$ from $\lambda \approx \lambda^n$ $(\forall n \in \mathbb{N})$, which means the infinite divisibility $(AdV^*)^{t+s} \approx (AdV^*)^t(AdV^*)^s$ $(t, s > 0)$ of the process induced by the above transformation. In this way, we see that simple individual measurements with definite measured values are connected without gaps with discrete and/or continuous repetitions of measurements [9]. If this formulation exhausts the essence of the problem, the remaining tasks reduce to its physical and technological implementation through suitable choices of the media connecting the microscopic contact point between the system and the apparatus to the measuring pointer.

4 From Amplification to Emergence of Macro

In the mutual relations between invisible Micro and visible Macro, we find interesting recurrent patterns among dynamical systems, crossed products to formulate coupled systems and processes to amplify the results of state changing processes into readable data. The crucial roles are played here by the K-T operators and the Fourier duality to perform the spectral decomposition. To understand their natural operational meaning we compare the above scheme for an intra-sectorial search with the measurement of an inter-sectorial structure associated with an unbroken internal symmetry, whose basic ingredients are as follows:

1. A microscopic system described by a field algebra $\mathfrak{F}$ and a (compact) group G of internal symmetry constituting a dynamical system $\mathfrak{F} \underset{\alpha}{\curvearrowleft} G$.

2. The coupled system of observed and measuring systems is given by a *crossed product* $\mathfrak{F} \rtimes_\alpha G \simeq \mathfrak{F}^G \equiv \mathfrak{A}$ whose sector structure is parametrized by order parameters belonging to the set $\widehat{G}$ of equivalence classes of irreducible unitary representations of G.

3. Measured values (in a given representation π of $\mathfrak{F}$) are registered in $Spec(\mathfrak{Z}_\pi(\mathfrak{A})) = \widehat{G}$: note the *Fourier duality* between G *acting* on the system and its dual $\widehat{G}$ as sector indices to be *measured.*

4. The *K-T operator* relevant to measured data in $\widehat{G}$ is given in the form of $\hat{V} := \sigma V^* \sigma$ defined in $L^2(\hat{G}) = L^2(G)$ on the basis of the K-T operator V of G given by $(V\xi)(g_1, g_2) = \xi(g_1, g_1^{-1}g_2)$ for $g_1, g_2 \in G$ (where σ is the flip operator on the tensor product Hilbert space). For an abelian G, we have through the Fourier transform $(\hat{V}\eta)(\gamma_1, \gamma_2) = \eta(\gamma_1, \gamma_1^{-1}\gamma_2)$ for $\gamma_1, \gamma_2 \in \hat{G}$, which cannot, however, be literally reproduced for a *non-abelian* G owing to the relevance of *multi-dimensional vector spaces* to representations $\gamma \in \hat{G}$ of G.

In contrast, the problem of *parameter estimate* in covariant measurements is formulated as follows :

1. an algebra to be observed is $\mathfrak{A}$ or $\mathfrak{F} \rtimes_\alpha G = \mathfrak{A} \otimes \mathcal{K}(L^2(G))$.

2. The coupling between $\mathfrak{A}$ and $\hat{G}$ due to the *co-action* $\mathfrak{A} \curvearrowleft \hat{G}$ leads to a crossed product $\mathfrak{A} \rtimes \hat{G} \simeq \mathfrak{F}$ as a measurement is a process to *couple the system to the dual variables of what to be observed.*

3. What to be read out in this case as the outcome of the measurement is $g \in G$ whose non-commutativity requires an optimized choice of *positive operator-valued measures* (*POVM's*, for short) defined on G taking values in the representation space of $\mathfrak{F}$.

4. In the *Naimark extension* of a POVM, the augmented algebra $\widehat{\mathfrak{F}}$ of $\mathfrak{F}$ appears with a centre $\mathfrak{Z}(\widehat{\mathfrak{F}}) = L^\infty(G)$ whose spectrum is G (see [1]).

The duality of crossed products relevant to the above two cases can be summarized as follows:

$$\left[\begin{array}{lcl} \text{coupled system} & & \text{read-out } \in \hat{G} \\ \mathfrak{F} \rtimes_\alpha G \simeq \mathfrak{A} & \underset{V\otimes}{\overset{\text{amplify}}{\Longrightarrow}} & \leftrightarrow \text{sectors} \\ \Uparrow \qquad \circlearrowright \hat{G}\text{: coaction} & & \\ G \curvearrowright \qquad \Downarrow & & \\ \mathfrak{F} \simeq \mathfrak{A} \rtimes_{\hat{\alpha}} \hat{G} & & \end{array}\right]$$

$$\underset{\rightleftarrows}{\text{dual}} \left[\begin{array}{lcl} \text{coupled system} & & \text{read-out} \\ \mathfrak{A} \rtimes_{\hat{\alpha}} \hat{G} \simeq \mathfrak{F} & \overset{\text{amplify}}{\Longrightarrow} & \in G \\ \Uparrow \qquad \circlearrowright G\text{: action} & & \\ \hat{G} \curvearrowright \qquad \Downarrow & & \\ \mathfrak{A} \simeq \mathfrak{F} \rtimes_\alpha G & & \end{array}\right]$$

We encounter a physically natural context of this sort in the description of the sector structure associated with a *spontaneous symmetry breakdown (SSB)* of a bigger goup G into an unbroken symmetry with its closed subgroup H, as follows:

1. The inter-sectorial structure (I) consisting of degenerate "vacua" associated with SSB: the breakdown of an internal symmetry described by a group G is known to cause the violation of the Haag duality $\mathfrak{A}(\mathcal{O}') = \mathfrak{A}(\mathcal{O})'$ for the starting local net $\mathcal{O} \longmapsto \mathfrak{A}(\mathcal{O}) = \mathfrak{F}(\mathcal{O})^G$ of observable elements of quantum fields. Then it can be extended to the Haag dual net given by $\mathfrak{A}^d(\mathcal{O}) := \mathfrak{A}(\mathcal{O}')'$ to recover the Haag duality. Through the Doplicher-Roberts reconstruction [10] applied to $\mathfrak{A}^d$, we find a field algebra $\mathfrak{F} = \mathfrak{A}^d \rtimes \hat{H}$ with a compact Lie group H as a subgroup of G to describe an unbroken symmetry of $\mathfrak{F}$. Using the method developed in [1], we can construct an *augmented algebra* $\widehat{\mathfrak{F}} = \mathfrak{A}^d \rtimes \hat{G} = \mathfrak{F} \rtimes \widehat{(H\backslash G)}$ from the co-action of G on $\mathfrak{A}^d$ or equivalently from that of a homogeneous space $H\backslash G$ on $\mathfrak{F}$ such that its induced representation $\hat{\pi}$ from the vacuum representation of $\mathfrak{F}$ has automatically the unitary implementers of the broken G and that it has a non-trivial centre $L^\infty(G/H) = L^\infty(G)^H$ on which the action of G is ergodic. In this way, the degenerate "vacua" consisting of the base space G/H of the bundle of sectors can be detected as the spectrum of the order parameter $\mathfrak{Z}_{\hat{\pi}}(\widehat{\mathfrak{F}}) = L^\infty(G/H)$. The above second case of the parameter estimate of G in covariant measurements in the use of a POVM can be reproduced if we take $H = \{e\}$ here. Note the parallelism between the dynamical system $G \curvearrowright G/H$ and the *Galois group* G in *classical Galois theory* acting on the *space* G/H *of solutions.*

2. The inter-sectorial structure (II) concerning sectors arising from the unbroken symmetry H on one of degenerate "vacua": the above Haag

dual net algebra $\mathfrak{A}^d = \mathfrak{F}^H$ can be regarded as a coupled system $\mathfrak{F} \rtimes H \simeq \mathfrak{F}^H = \mathfrak{A}^d$ of the field algebra $\mathfrak{F}$ with its unbroken symmetry group H arising from the action of H: $\mathfrak{F} \curvearrowleft H \Longrightarrow \mathfrak{F} \rtimes H = \mathfrak{F}^H = \mathfrak{A}^d$ in the use of the Takesaki-Takai duality of crossed product. This coupled system is acted on by the group dual $\widehat{H}$, the latter of which can be measured to describe the sector structure of the unbroken H on a "vacuum" chosen among degenerate "vacua" (by means of, e.g., Casimir operators of $Lie(H)$). In this way, the sector structure due to a spontaneously broken symmetry constitutes a sector bundle $G \times_H \hat{H} \twoheadrightarrow G/H$ over the homogeneous space G/H with a standard fibre $\hat{H}$.

3. Intra-sectorial structure: detected by means of a suitable MASA (corresponding to a Cartan subalgebra of $Lie(H)$, for instance) of a factor algebra $\pi_\eta(\mathfrak{F}^H)'' = \pi_\eta(\mathfrak{A}^d)''$.

The relation above among a POVM of the space G/H, its Naimark extension and the augmented algebra $\widehat{\mathfrak{F}}$ with $\mathfrak{Z}_{\hat{\pi}}(\widehat{\mathfrak{F}}) = L^\infty(G/H) = \mathfrak{Z}_{\hat{\pi}}(\mathfrak{A}^d)$ can be naturally understood by means of the Stinespring theorem of dilations based upon the complete positivity of a POVM. Note here the mutual relations among *condensates, Goldstone modes and domain structures*: in SSB with G broken down to H, the *condensates* and *Goldstone modes* are both related to G/H but in quite a different manner. In the case with a Lie group G describing the spontaneously broken symmetry, the former corresponds to the base space G/H of the tangent bundle $T(G/H)$ and the latter to the fibre space $T_{\dot{g}}(G/H)$ at each point $\dot{g} \in G/H$ as follows:

1. Condensates (responsible for SSB): the list of all the possible condensates can be so parametrized by G/H that *each sector corresponds to a point* $\dot{g} \in G/H$. I.e., the relation of G/H to the condensates is that the set G/H exhausts all the possible choices of degenerate "vacua", among which only one point of G/H can be realized as a sector at each time.

2. Goldstone modes describe *virtual fluctuations around a fixed choice among the above condensates* without changing it.

3. In the case with *phase coexistence*, different choices of the condensates are realized in different regions of the real space through which a domain-structure is realized. "Real space" may be misunderstood as prior to the emergence of different phases, whereas such a "real space" may not be materialized without the coexistence of phases.

This last remark will play crucial roles in understanding classical geometrical structures visible at the macroscopic levels as something arising from the processes of *emergence* from the invisible microscopic worlds.

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