数理解析研究所講究録
第 1570 巻 2007 年 46-53

Preduals of Morrey-Campanato spaces

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1. INTRODUCTION

This is an announcement of my recent works.

Let $X=(X,d,\mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a nonnegative measure μ such that

$$d(x,y)\geq 0 \quad \text{and} \quad d(x,y)=0 \text{ if and only if } x=y,$$
$$d(x,y)=d(y,x),$$
$$d(x,y)\leq K_1\,(d(x,z)+d(z,y)), \tag{1.1}$$

the balls (d-balls) $B(x,r)=B^d(x,r)=\{y\in X: d(x,y)<r\}$, $r>0$, form a basis of neighborhoods of the point x, μ is defined on a σ-algebra of subsets of X which contains the balls, and

$$0<\mu(B(x,2r))\leq K_2\,\mu(B(x,r))<\infty, \tag{1.2}$$

where $K_i\geq 1$ $(i=1,2)$ are constants independent of $x,y,z\in X$ and $r>0$.

We note that every open subset of X is expressible as a countable union of balls (see [4], p.70), and so measurable.

If there are constants θ $(0<\theta\leq 1)$ and $K_3\geq 1$ such that

$$|d(x,z)-d(y,z)|\leq K_3\,(d(x,z)+d(y,z))^{1-\theta}d(x,y)^{\theta}, \qquad x,y,z\in X, \tag{1.3}$$

then the balls are open sets. Note that (1.1) for some $K_1\geq 1$ follows from (1.3) (Lemarié [12]). Conversely, from (1.1) it follows that there exist $\theta>0$, $K_3\geq 1$ and a quasi-distance which is equivalent to the original d such that (1.3) holds (Macías and Segovia [14]).

Using atoms, Coifman and Weiss [5] defined the Hardy space $H^p(X)$ as a subspace of the dual of $\mathrm{Lip}_\alpha(X)$ and they proved that $\mathrm{Lip}_\alpha(X)$ is the dual of $H^p(X)$. Their

2000 *Mathematics Subject Classification.* 42B30, 42B35, 46E15, 46E30.
Key words and phrases. Atom, Hardy space, BMO, Lipschitz space, Morrey space, Campanato space, space of homogeneous type, dual, predual .

results are generalization of the case $X = \mathbb{R}^n$. In [5] $\mathrm{Lip}_\alpha(X)$ was regarded the space of functions modulo constants. Therefore, we denote the fact above by

$$(H^p(X))^* = \mathrm{Lip}_\alpha(X)/\mathcal{C},$$

where $\mathcal{C}$ is the space of all constant functions. Let $\mathcal{L}_{p,\phi}(X)$ be the Campanato space which is a genaralization of $\mathrm{Lip}_\alpha(X)$. In this paper we define a generalized Hardy space $H_U^{[\phi,q]}(X)$ as a subspace of the dual of $\mathcal{L}_{q',\phi}(X)/\mathcal{C}$ and prove that $\mathcal{L}_{q',\phi}(X)/\mathcal{C}$ is the dual of $H_U^{[\phi,q]}(X)$, i.e.

$$\left(H_U^{[\phi,q]}(X)\right)^* = \mathcal{L}_{q',\phi}(X)/\mathcal{C},$$

where $1/q + 1/q' = 1$. The definition of $H^p(X)$ in [5] is a special case of ours. We note that the predual of $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ is not unique. Zorko [31] defined another predual of $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ in the case $X = \mathbb{R}^n$. Our definition is a generalization of both definitions.

We also define a space $B_U^{\Phi,q}(X)$ generated by blocks ("block" means an atom without the cancellation property), and prove that the dual of $B_U^{\Phi,q}(X)$ is a Morrey space $L_{p,\phi}(X)$. This is

$$\left(B_U^{\Phi,q}(X)\right)^* = L_{p,\phi}(X).$$

This result is a genaralization of Long [13] (1984).

It is known that $\mathcal{L}_{p,\phi}(X)/\mathcal{C} = L_{p,\phi}(X)$ under a certain condition (Nakai [24] (2006)). We show that $H_U^{\Phi,q}(X) = B_U^{\Phi,q}(X)$ under the correspondent condition.

2. Notations and definitions

Let (X, d, μ) be a space of homogeneous type satisfying (1.3).

Let $1 \le p < \infty$ and $\phi : X \times \mathbb{R}_+ \to \mathbb{R}_+$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L^1_{\mathrm{loc}}(X)$ and for a ball B, let $f_B = \mu(B)^{-1} \int_B f(x)\, d\mu(x)$. Then the Campanato spaces $\mathcal{L}_{p,\phi}(X)$, the Morrey spaces $L_{p,\phi}(X)$ and the Hölder spaces $\Lambda_\phi(X)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$, $\|f\|_{L_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\|f\|_{\mathcal{L}_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{L_{p,\phi}} = \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x)|^p \, d\mu(x) \right)^{1/p},$$

$$\|f\|_{\Lambda_\phi} = \sup_{x,y \in X,\, x \ne y} \frac{2|f(x) - f(y)|}{\phi(x, d(x,y)) + \phi(y, d(y,x))}.$$

Then $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$, $L_{p,\phi}(X)$ and $\Lambda_\phi(X)/\mathcal{C}$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$, $\|f\|_{L_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively.

If $\phi(x,r) = r^\alpha$ ($\alpha > 0$), $\Lambda_{r^\alpha}(X) = \mathrm{Lip}_\alpha(X)$. If $p = 1$, then $\mathcal{L}_{1,\phi}(X) = \mathrm{BMO}_\phi(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \mathrm{BMO}(X)$ and $\Lambda_\phi(X) = L^\infty(X)$. If $\phi(B) = \mu(B)^{-1/p}$, then $L_{p,\phi}(X) = L^p(X)$.

If $X = \mathbb{R}^n$, $d(x,y) = |x-y|$, μ is Lebesgue measure and $\phi(x,r) = r^\alpha$, then the following are known (Campanato, Mayers, Peetre, Spanne, Janson);

$$-n/p \le \alpha < 0 \Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n) \ (= L^p(\mathbb{R}^n) \text{ if } \alpha = -n/p),$$
$$\alpha = 0 \Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n) \supset L_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),$$
$$0 < \alpha \le 1 \Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n) = \mathrm{Lip}_\alpha(\mathbb{R}^n).$$

The relations above were generalized to spaces of homogeneous type by Macías and Segovia [14] (1979) and Nakai [24] (2006).

For functions $\tau, \kappa : (0,+\infty) \to (0,+\infty)$, we denote $\tau(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\tau(r) \le \kappa(r) \le C\tau(r) \quad \text{for} \quad r > 0.$$

A function $\tau : (0,+\infty) \to (0,+\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\tau(r) \le C\tau(s) \quad (\tau(r) \ge C\tau(s)) \quad \text{for} \quad r \le s.$$

A function $\tau : (0,+\infty) \to (0,+\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \le \frac{\tau(r)}{\tau(s)} \le C \quad \text{for} \quad \frac{1}{2} \le \frac{r}{s} \le 2.$$

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi : [0,+\infty) \to [0,+\infty)$. Then $\Phi(0) = 0$ and $\lim_{r\to+\infty} \Phi(r) = +\infty$ for $\Phi \in \mathcal{F}$.

Definition 2.1 ((Φ, q)-atom). Let $\Phi \in \mathcal{F}$ and $1 < q \le \infty$. A function a on X is called a (Φ, q)-atom if there exists a ball B such that

(i) $\operatorname{supp} a \subset \overline{B}$,
(ii) $\|a\|_q \le \mu(B)^{1/q}\Phi^{-1}(1/\mu(B))$,
(iii) $\displaystyle\int_X a(x)\,d\mu(x) = 0$,

where $\|a\|_q$ is the L^q norm of a, $\overline{B}$ is the closure of B. We denote by $A(\Phi, q)$ the set of all (Φ, q)-atoms.

Definition 2.2 (((Φ, q)-block). Let $\Phi \in \mathcal{F}$ and $1 < q \leq \infty$. A function a on X is called a (Φ, q)-block if there exists a ball B such that (i) and (ii) hold. We denote by $B(\Phi, q)$ the set of all (Φ, q)-blocks.

For $\Phi \in \mathcal{F}$ and for $B = B(x,r)$, let

$$\phi(x,r) = \phi(B) = \frac{1}{\mu(B)\Phi^{-1}(1/\mu(B))}. \tag{2.1}$$

If a is a (Φ, q)-atom, then, for a ball B satisfying (i)–(iii), we have

$$\begin{aligned} \left|\int_X a(x)g(x)\,d\mu(x)\right| &= \left|\int_B a(x)(g(x) - g_B)\,d\mu(x)\right| \\ &\leq \|a\|_q \left(\int_B |g(x) - g_B|\,d\mu(x)\right)^{1/q'} \\ &\leq \mu(B)\Phi^{-1}(1/\mu(B)) \left(\frac{1}{\mu(B)}\int_B |g(x) - g_B|\,d\mu(x)\right)^{1/q'} \leq \|g\|_{\mathcal{L}_{q',\phi}}. \end{aligned} \tag{2.2}$$

That is, the mapping $g \mapsto \int_X ag\,d\mu$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(X)/C$ with norm not exceeding 1.

Definition 2.3 ($H_U^{\Phi,q}(X)$). Let $\Phi, U \in \mathcal{F}$, U be concave, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and ϕ be as in (2.1). We define the space $H_U^{\Phi,q}(X) \subset (\mathcal{L}_{q',\phi}(X)/C)^*$ as follows:

$f \in H_U^{\Phi,q}(X)$ if and only if there exist sequences $\{a_j\} \subset A(\Phi, q)$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q',\phi}(X)/C)^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty. \tag{2.3}$$

In general, the expression (2.3) is not unique. We define

$$\|f\|_{H_U^{\Phi,q}} = \inf\left\{U^{-1}\left(\sum_j U(\lambda_j)\right)\right\},$$

where the infimum is taken over all expressions (2.3). We note that $\|f\|_{H_U^{\Phi,q}}$ is not a norm in general. Let $d(f,g) = U(\|f - g\|_{H_U^{\Phi,q}})$ for $f, g \in H_U^{\Phi,q}(X)$. Then $d(f,g)$ is a metric and $H_U^{\Phi,q}(X)$ is complete.

In the case $\Phi(r) = U(r) = r^p$, $p < 1$, then $H_U^{\Phi,q}(X) = H^p(X)$ defined by Coifman and Weiss [5] (1977). Let $I(r) = r$. Then $\|f\|_{H_I^{\Phi,q}}$ is a norm and $H_I^{\Phi,q}$ is a Banach space, which was defined by Zorko [31] (1986) in the case $X = \mathbb{R}^n$.

Definition 2.4 ($B_U^{\Phi,q}(X)$). Let $\Phi, U \in \mathcal{F}$, U be concave, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and ϕ be as in (2.1). Assume that $r\Phi^{-1}(1/r)$ is almost increasing. We define the space $B_U^{\Phi,q}(X) \subset (L_{q',\phi}(X))^*$ as follows:

$f \in B_U^{\Phi,q}(X)$ if and only if there exist sequences $\{a_j\} \subset B(\Phi, q)$ and positive numbers $\{\lambda_j\}$ such that

$$f = \sum_j \lambda_j a_j \text{ in } (L_{q',\phi}(X))^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty. \tag{2.4}$$

We define

$$\|f\|_{B_U^{\Phi,q}} = \inf\left\{U^{-1}\left(\sum_j U(\lambda_j)\right)\right\},$$

where the infimum is taken over all expressions (2.4).

Let $d(f,g) = U(\|f-g\|_{B_U^{\Phi,q}})$ for $f, g \in B_U^{\Phi,q}(X)$. Then $d(f,g)$ is a metric and $B_U^{\Phi,q}(X)$ is complete. Let $I(r) = r$. Then $\|f\|_{B_I^{\Phi,q}}$ is a norm and $B_I^{\Phi,q}$ is a Banach space.

If $X = \mathbb{R}^n$, $d(x,y) = |x-y|$, μ is Lebesgue measure, $\Phi(r) = r$ and $U(r) = r(1+\log^+(1/r))$, then $B_U^{\Phi,q}(X)$ is the space introduced by Taibleson and Weiss [29] (1983) and Lu, Taibleson and Weiss [10] (1982).

From the definition it follows that

- If $1 < q_1 < q_2 \le \infty$, then
$$H_U^{\Phi,q_2}(X) \subset H_U^{\Phi,q_1}(X), \quad B_U^{\Phi,q_2}(X) \subset B_U^{\Phi,q_1}(X).$$
- If $\Psi(r) \le \Phi(Cr)$ for all $r > 0$, then
$$H_U^{\Phi,q}(X) \subset H_U^{\Psi,q}(X), \quad B_U^{\Phi,q}(X) \subset B_U^{\Psi,q}(X).$$
- If $V(r) \le CU(r)$ for $0 \le r \le 1$, then
$$H_U^{\Phi,q}(X) \subset H_V^{\Phi,q}(X), \quad B_U^{\Phi,q}(X) \subset B_V^{\Phi,q}(X).$$
- For any concave function $U \in \mathcal{F}$,
$$H_U^{\Phi,q}(X) \subset H_I^{\Phi,q}(X), \quad B_U^{\Phi,q}(X) \subset B_I^{\Phi,q}(X).$$

In the above, the inclusion mapping are continuous.

3. Main results

Let $\left(H_U^{\Phi,q}(X)\right)^*$ and $\left(B_U^{\Phi,q}(X)\right)^*$ be the linear spaces of all continuous linear functionals on $H_U^{\Phi,q}(X)$ and on $B_U^{\Phi,q}(X)$, respectively.

Lemma 3.1. *Let Φ, q, U be as in Definition 2.3. If*

$$\sup_{0<s<1} \frac{U(rs)}{U(s)} \to 0 \quad (r \to 0), \tag{3.1}$$

then

$$\|\ell\|_{(H_U^{\Phi,q})^*} = \sup\left\{|\ell(f)| : \|f\|_{H_U^{\Phi,q}} \le 1\right\},$$
$$\|\ell\|_{(B_U^{\Phi,q})^*} = \sup\left\{|\ell(f)| : \|f\|_{B_U^{\Phi,q}} \le 1\right\}$$

are finite for all $\ell \in \left(H_U^{\Phi,q}(X)\right)^*$ *and for all* $\ell \in \left(B_U^{\Phi,q}(X)\right)^*$, *respectively.* $\|\ell\|_{(H_U^{\Phi,q})^*}$ *and* $\|\ell\|_{(B_U^{\Phi,q})^*}$ *are norms.*

Let $L^q_{\mathrm{comp}}(X)$ be the set of all L^q-functions with compact support, and let

$$L^{q,0}_{\mathrm{comp}}(X) = \left\{f \in L^q_{\mathrm{comp}}(X) : \int_X f\,d\mu = 0\right\}.$$

Then, for $1 < q \le \infty$, $L^q_{\mathrm{comp}}(X)$ and $L^{q,0}_{\mathrm{comp}}(X)$ are dense in $B_U^{\Phi,q}(X)$ and in $H_U^{\Phi,q}(X)$, respectively.

If $g \in \mathcal{L}_{q',\phi}(X)$ and $f \in L^{q,0}_{\mathrm{comp}}(X)$, then fg is integrable.

Theorem 3.2. *Let* Φ, q, U, ϕ *be as in Definition 2.3. If* U *satisfies* (3.1), *then*

$$\left(H_U^{\Phi,q}(X)\right)^* = \mathcal{L}_{q',\phi}(X)/\mathcal{C}.$$

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)$, *then the mapping* $\ell : f \mapsto \int_X f(g+C)\,d\mu$ $(f \in L^{q,0}_{\mathrm{comp}}(X))$ *can be extended to a continuous linear functional on* $H_U^{\Phi,q}(X)$. *Conversely, if* ℓ *is a continuous linear functional on* $H_U^{\Phi,q}(X)$, *then there exists* $g \in \mathcal{L}_{q',\phi}(X)$ *such that* $\ell(f) = \int_X f(g+C)\,d\mu$ *for* $f \in L^{q,0}_{\mathrm{comp}}(X)$. *The norm* $\|\ell\|$ *is equivalent to* $\|g\|_{\mathcal{L}_{q',\phi}}$.

Corollary 3.3. *Let* $\Phi(r) = r$. *Then, for* $1 < q \le \infty$ *and for any concave function* $U \in \mathcal{F}$ *with* (3.1),

$$\left(H_U^{\Phi,q}(X)\right)^* = \mathrm{BMO}(X)/\mathcal{C}.$$

Theorem 3.4. *Let* Φ, q, U, ϕ *be as in Definition 2.4. If* U *satisfies* (3.1), *then*

$$\left(B_U^{\Phi,q}(X)\right)^* = L_{q',\phi}(X).$$

More precisely, if $g \in L_{q',\phi}(X)$, *then the mapping* $\ell : f \mapsto \int_X fg\,d\mu$ $(f \in L^q_{\mathrm{comp}}(X))$ *can be extended to a continuous linear functional on* $B_U^{\Phi,q}(X)$. *Conversely, if* ℓ *is a continuous linear functional on* $B_U^{\Phi,q}(X)$, *then there exists* $g \in L_{q',\phi}(X)$ *such that* $\ell(f) = \int_X fg\,d\mu$ *for* $f \in L^q_{\mathrm{comp}}(X)$. *The norm* $\|\ell\|$ *is equivalent to* $\|g\|_{\mathcal{L}_{q',\phi}}$.

Theorem 3.5. *Assume that* $\mu(X) = \infty$ *and that there exists* $k > 1$ *s.t.*

$$\mu(B) \le \frac{1}{2}\mu(kB) \quad \textit{for all balls } B. \tag{3.2}$$

Let Φ, q, U, ϕ be as in Definition 2.4 and $U(rs) \le U(r)U(s)$ for $0 < r, s \le 1$. If there exists $C > 0$ such that

$$\int_r^\infty \frac{1}{t\Phi^{-1}(1/t)} \frac{dt}{t} \le C \frac{1}{r\Phi^{-1}(1/r)}, \quad 0 < r < \infty, \tag{3.3}$$

then $H_U^{\Phi,q}(X) = B_U^{\Phi,q}(X)$. More precisely, for $f \in B_U^{\Phi,q}(X)$, there exists a decomposition $f = \sum_j \lambda_j a_j$ with (Φ, q)-atoms such that

$$\langle f, g - c_g \rangle = \sum_j \lambda_j \int a_j g \quad \text{for all } g \in \mathcal{L}_{p,\phi}(X)/\mathcal{C},$$

where $c_g = \lim_{r\to\infty} g_{B(x_0,r)}$.

Remark 3.1. It is known that (3.3) is a necessary and sufficient condition for $\mathcal{L}_{q',\phi}(X)/\mathcal{C} = L_{q',\phi}(X)$ (Nakai [24] (2006)) with (2.1).

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