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On Convergence of the dqds and mdLVs Algorithms for Computing Matrix Singular Values

東京大学大学院 数理情報学専攻 相島 健助 (Kensuke Aishima)
松尾 宇泰 (Takayasu Matsuo)
室田 一雄 (Kazuo Murota)
杉原 正顯 (Masaaki Sugihara)
Graduate School of Information Science and Technology, University of Tokyo

Abstract

Convergence theorems are established with mathematical rigour for two algorithms for the computation of singular values of bidiogonal matrices: the differential quotient difference with shift (dqds) and the modified discrete Lotka-Volterra with shift (mdLVs). For the dqds algorithm, global convergence is guaranteed under a fairly general assumption on the shift, and the asymptotic rate of convergence is 1.5 for the Johnson bound shift. Also for the mdLVs algorithm, global convergence is guaranteed in a realistic assumption, a substantial improvement of the convergence analysis by Iwasaki and Nakamura. The asymptotic rate of convergence of the mdLVs algorithm is 1.5 when the Johnson bound shift is employed. Numerical examples support these theoretical results.

1 Introduction

Every $n \times m$ real matrix A (with $\mathrm{rank}(A) = r$) can be decomposed into

$$A = U\Sigma V^{\mathrm{T}}$$

by suitable orthogonal matrices $U \in \mathrm{R}^{n\times n}$ and $V \in \mathrm{R}^{m\times m}$, where

$$\Sigma = \begin{pmatrix} D & O_{r,m-r} \\ O_{n-r,r} & O_{n-r,m-r} \end{pmatrix}, \qquad D = \mathrm{diag}(\sigma_1, \ldots, \sigma_r).$$

The values $\sigma_1 \geq \cdots \geq \sigma_r > 0$ are the singular values of A.

In the computation of matrix singular values, a matrix is often transformed first to an upper bidiagonal matrix by appropriate orthogonal matrices, and then its singular values are calculated by some iterative algorithm. A common iterative algorithm for bidiagonal matrices is the differential quotient difference with shift (dqds) algorithm [7]. The dqds is now implemented as DLASQ in LAPACK [3, 10, 13] and widely used by many practitioners because of its high accuracy, speed, and numerical stability. The dqds is integrated into Multiple Relatively Robust Representations (MR^3) algorithm [4, 5, 6]. On the other hand, quite recently a new iterative algorithm, called the modified discrete Lotka-Volterra with shift (mdLVs) algorithm, was proposed [11], and has been rapidly expanding its influence due to its high efficiency comparable to the dqds.

The aim of this report is to investigate the theoretical aspects of the two iterative algorithms. So far, the convergence for the dqds has been proved only under the condition that the shift is off. Independently of that, the rate of convergence has been shown to be locally quadratic or cubic when the shift satisfies some stringent assumptions [7]. In this report, we first prove that the dqds always converges as far as the shift satisfies a certain natural condition. Then we show that, if the shift is determined by the Johnson bound [9], the asymptotic rate of convergence is 1.5. For the mdLVs, a convergence theorem is known under a certain condition on the shift, and the local rate of convergence has been shown to be quadratic or cubic under certain conditions. In this report we establish a stronger convergence theorem for a wider class of shift choices, and also show that, with the shift by the Johnson bound, the asymptotic rate of convergence is also 1.5.

2 Problem setting

We assume that the given real matrix A has already been transformed to a bidiagonal matrix

$$B = \begin{pmatrix} b_1 & b_2 & & \\ & b_3 & \ddots & \\ & & \ddots & b_{2m-2} \\ & & & b_{2m-1} \end{pmatrix}, \tag{1}$$

to which the dqds or the mdLVs algorithm is applied.

Following [7], we assume

Assumption (A) The bidiagonal elements of B are positive, i.e., $b_k > 0$ for $k = 1, 2, \dots, 2m-1$.

This assumption guarantees (see [12]) that the singular values of B are all distinct: $\sigma_1 > \cdots > \sigma_m > 0$.

Assumption (A) is not restrictive, in theory or in practice. In fact, if a subdiagonal element is zero, i.e., $b_{2k} = 0$ for some k, then the problem reduces to two independent problems on matrices of smaller sizes, $k \times k$ and $(m-k) \times (m-k)$. If there is a zero element on the diagonal, several iterations of the dqd algorithm (i.e., the dqds algorithm without shifts) suffice to remove the diagonal zero, and the problem is again separated into a set of smaller problems (see [7] for details). Finally it is easy to see that the singular values are invariant if b_k is replaced by $|b_k|$.

In our problem setting we have assumed real matrices, whereas the singular value decomposition is also defined for complex matrices. Our restriction to real matrices is justified by the fact that any complex matrix can be transformed to a real bidiagonal matrix by, say, (complex) Householder transformations, while keeping its singular values [7].

3 Convergence of the dqds Algorithm

In this section, convergence results for the dqds algorithm are established with mathematical rigour. See [1] for the proofs.

3.1 The dqds algorithm

The dqds algorithm can be described in computer program form as follows.

Algorithm 3.1 The dqds algorithm

Initialization: $q_k^{(0)} = (b_{2k-1})^2 \ (k = 1, 2, \ldots, m); \quad e_k^{(0)} = (b_{2k})^2 \ (k = 1, 2, \ldots, m-1)$

for $n := 0, 1, \cdots$ **do**
  choose shift $s^{(n)} (\geq 0)$
  $d_1^{(n+1)} := q_1^{(n)} - s^{(n)}$
  for $k := 1, \cdots, m-1$ **do**
    $q_k^{(n+1)} := d_k^{(n+1)} + e_k^{(n)}$
    $e_k^{(n+1)} := e_k^{(n)} q_{k+1}^{(n)} / q_k^{(n+1)}$
    $d_{k+1}^{(n+1)} := d_k^{(n+1)} q_{k+1}^{(n)} / q_k^{(n+1)} - s^{(n)}$
  end for
  $q_m^{(n+1)} := d_m^{(n+1)}$
end for

The outermost loop is terminated when some suitable convergence criterion, say, $\|e_{m-1}^{(n)}\| \leq \epsilon$ for some prescribed constant $\epsilon > 0$, is satisfied. At the termination we have

$$\sigma_m{}^2 \approx q_m^{(n)} + \sum_{l=0}^{n-1} s^{(l)} \tag{2}$$

and hence σ_m can be approximated by $\sqrt{q_m^{(n)} + \sum_{l=0}^{n-1} s^{(l)}}$. Then by the deflation process the problem is shrunk to an $(m-1) \times (m-1)$ problem, and the same procedure is repeated until $\sigma_{m-1}, \ldots, \sigma_1$ are obtained in turn.

It turns out to be convenient to introduce additional notations $e_0^{(n)}$ and $e_m^{(n)}$ with "boundary conditions":

$$e_0^{(n)} = 0, \quad e_m^{(n)} = 0 \quad (n = 0, 1, \ldots)$$

to simplify the expression of the algorithm. Put

$$B^{(n)} = \begin{pmatrix} b_1^{(n)} & b_2^{(n)} & & & \\ & b_3^{(n)} & \ddots & & \\ & & \ddots & b_{2m-2}^{(n)} & \\ & & & b_{2m-1}^{(n)} & \end{pmatrix}, \tag{3}$$

$b_k^{(0)} = b_k$ $(k = 1, 2, \ldots, 2m-1)$, and

$$q_k^{(n)} = (b_{2k-1}^{(n)})^2 \quad (k = 1, 2, \ldots, m;\ n = 0, 1, \ldots), \tag{4}$$
$$e_k^{(n)} = (b_{2k}^{(n)})^2 \quad (k = 1, 2, \ldots, m-1;\ n = 0, 1, \ldots). \tag{5}$$

Then Algorithm 3.1 can be rewritten in terms of the Cholesky decomposition (with shifts):

$$(B^{(n+1)})^{\mathrm{T}} B^{(n+1)} = B^{(n)} (B^{(n)})^{\mathrm{T}} - s^{(n)} I, \tag{6}$$

where $B^{(0)} = B$. From (6) it follows that

$$(B^{(n)})^{\mathrm{T}} B^{(n)} = W^{(n)} \left((B^{(0)})^{\mathrm{T}} B^{(0)} - \sum_{l=0}^{n-1} s^{(l)} I \right) (W^{(n)})^{-1}, \tag{7}$$

where $W^{(n)} = (B^{(n-1)} \cdots B^{(0)})^{-\mathrm{T}}$ is a nonsingular matrix. Therefore the eigenvalues of $(B^{(n)})^{\mathrm{T}} B^{(n)}$ are the same as those of $(B^{(0)})^{\mathrm{T}} B^{(0)} - \sum_{l=0}^{n-1} s^{(l)} I$.

If $s^{(n)} < (\sigma_{\min}^{(n)})^2$ in each iteration n, where $\sigma_{\min}^{(n)}$ is the smallest singular value of $B^{(n)}$, the variables in the dqds algorithm are always positive so that the algorithm does not break down as follows.

Lemma 3.1 (Positivity of the variables in the dqds algorithm). *Suppose the dqds algorithm is applied to the matrix B satisfying Assumption* (A). *If $s^{(n)} < (\sigma_{\min}^{(n)})^2$ $(n = 0, 1, 2, \ldots)$, then $(B^{(n)})^{\mathrm{T}} B^{(n)}$ are positive definite, and hence $q_k^{(n)} > 0$ $(k = 1, \ldots, m)$, $e_k^{(n)} > 0$ $(k = 1, \ldots, m-1)$, and $d_k^{(n)} > 0$ $(k = 1, \ldots, m)$ for $n = 0, 1, 2, \ldots$.* ■

3.2 Global convergence of the dqds algorithm

The next theorem establishes the convergence of the dqds algorithm. Moreover, the theorem states that the variables $q_k^{(n)}$ converge to the square of the singular values minus the sum of the shifts, and that they are placed in the descending order.

Theorem 3.1 (Convergence of the dqds algorithm). *Suppose the matrix B satisfies Assumption* (A), *and the shift in the dqds algorithm is taken so that $0 \le s^{(n)} < (\sigma_{\min}^{(n)})^2$ holds for all n. Then*

$$\sum_{n=0}^{\infty} s^{(n)} \le {\sigma_m}^2. \tag{8}$$

Moreover,

$$\lim_{n\to\infty} e_k^{(n)} = 0 \quad (k = 1, 2, \ldots, m-1), \tag{9}$$
$$\lim_{n\to\infty} q_k^{(n)} = {\sigma_k}^2 - \sum_{n=0}^{\infty} s^{(n)} \quad (k = 1, 2, \ldots, m). \tag{10}$$

■

The next theorem states the asymptotic rate of convergence of the dqds algorithm. Let us define

$$\rho_k = \frac{{\sigma_{k+1}}^2 - \sum_{n=0}^{\infty} s^{(n)}}{{\sigma_k}^2 - \sum_{n=0}^{\infty} s^{(n)}} \qquad (k = 1, \ldots, m-1), \tag{11}$$

$$r_k^{(n)} = (q_k^{(n)} + \sum_{l=0}^{n-1} s^{(l)}) - {\sigma_k}^2 \qquad (k = 1, \ldots, m). \tag{12}$$

In view of (2), $r_k^{(n)}$ is the error in the approximated eigenvalue of $B^{\mathrm{T}}B$. Note that $0 < \rho_k < 1$ $(k = 1, \ldots, m-2)$, and $0 < \rho_{m-1} < 1$ if ${\sigma_m}^2 - \sum_{n=0}^{\infty} s^{(n)} > 0$ and $\rho_{m-1} = 0$ if ${\sigma_m}^2 - \sum_{n=0}^{\infty} s^{(n)} = 0$.

Theorem 3.2 (Rate of convergence of the dqds algorithm). *Under the same assumption as in Theorem* 3.1, *we have*

$$\lim_{n\to\infty} \frac{e_k^{(n+1)}}{e_k^{(n)}} = \rho_k \qquad (k = 1, \ldots, m-1), \tag{13}$$

$$\lim_{n\to\infty} \frac{r_1^{(n+1)}}{r_1^{(n)}} = \rho_1, \tag{14}$$

$$\lim_{n\to\infty} \frac{r_m^{(n+1)}}{r_m^{(n)}} = \rho_{m-1}. \tag{15}$$

Furthermore, if $\rho_{k-1} \neq \rho_k$ $(k = 2, \ldots, m-1)$, *then*

$$\lim_{n\to\infty} \frac{r_k^{(n+1)}}{r_k^{(n)}} = \max\{\rho_{k-1}, \rho_k\} \qquad (k = 2, \ldots, m-1). \tag{16}$$

Therefore, $e_k^{(n)}$ $(k = 1, \ldots, m-2)$ *and* $r_k^{(n)}$ $(k = 1, \ldots, m-1)$ *are of linear convergence as* $n \to \infty$. *The bottommost elements* $e_{m-1}^{(n)}$ *and* $r_m^{(n)}$ *are also of linear convergence when* $\rho_{m-1} > 0$, *i.e.,* $\sigma_m^2 - \sum_{n=0}^{\infty} s^{(n)} > 0$, *and of superlinear convergence when* $\rho_{m-1} = 0$, *i.e.,* $\sigma_m^2 - \sum_{n=0}^{\infty} s^{(n)} = 0$. ■

Remark 3.1. When $\rho_{k-1} = \rho_k$, we have a weaker claim that

$$r_k^{(n)} = \frac{\sum_{l=n+1}^{\infty} e_{k-1}^{(l)}}{e_{k-1}^{(n+1)}} \cdot e_{k-1}^{(n+1)} - \frac{\sum_{l=n}^{\infty} e_k^{(l)}}{e_k^{(n)}} \cdot e_k^{(n)},$$

which implies that, for any small $\epsilon > 0$,

$$|r_k^{(n)}| \le O((\rho_k + \epsilon)^n) \qquad (k = 2, \ldots, m-1).$$

That is, the convergence is at least linear, and can sometimes be better. ■

3.3 Convergence rate of the dqds with the Johnson bound

In this section, we show that the asymptotic rate of convergence of the dqds algorithm is 1.5 if the shift is determined by the Johnson bound [9]. Though the Johnson bound is valid for a general matrix, we present here its version for a bidiagonal matrix B.

Lemma 3.2 (Johnson bound [9]). *For a matrix B of the form* (1), *define*

$$\lambda = \min_{k=1,\ldots,m} \left\{ \mid b_{2k-1} \mid - \frac{\mid b_{2k-2} \mid + \mid b_{2k} \mid}{2} \right\},$$

where $b_0 = b_{2m} = 0$ and let σ_m denote the smallest singular value of B. Then $\sigma_m \geq \lambda$. Moreover, if the subdiagonal elements $(b_2, b_4, \ldots, b_{2m-2})$ are nonzero, then $\sigma_m > \lambda$. ■

With reference to (3), (4) and (5) we define the shift by the Johnson bound as follows:

$$\lambda^{(n)} = \min_{k=1,\ldots,m} \left\{ \sqrt{q_k^{(n)}} - \frac{1}{2}\left(\sqrt{e_{k-1}^{(n)}} + \sqrt{e_k^{(n)}}\right) \right\}, \tag{17}$$

$$s^{(n)} = \left(\max\{\lambda^{(n)},\ 0\}\right)^2. \tag{18}$$

This choice of the shift guarantees the condition $0 \leq s^{(n)} < (\sigma_{\min}^{(n)})^2$ in each iteration n, and hence the dqds is convergent by Theorem 3.1. The precise rate of convergence can be revealed through a scrutiny of the shift.

The following theorem shows that the rate of convergence of the dqds is 1.5. The theorem refers only to the lower right two elements of $B^{(n)}$, and the error in the approximation of the smallest eigenvalue of $B^{\mathrm{T}}B$. This is sufficient from the practical point of view since whenever the lower right elements converge to zero, the deflation is applied to reduce the matrix size.

Theorem 3.3 (Rate of convergence of the dqds). *Suppose the dqds algorithm with the Johnson bound is applied to a matrix B that satisfies Assumption* (A). *Then we have*

$$\lim_{n\to\infty} \frac{e_{m-1}^{(n+1)}}{(e_{m-1}^{(n)})^{3/2}} = \frac{1}{\sqrt{{\sigma_{m-1}}^2 - {\sigma_m}^2}}, \tag{19}$$

$$\lim_{n\to\infty} \frac{q_m^{(n+1)}}{(q_m^{(n)})^{3/2}} = \frac{1}{\sqrt{{\sigma_{m-1}}^2 - {\sigma_m}^2}}, \tag{20}$$

$$\lim_{n\to\infty} \frac{r_m^{(n+1)}}{(r_m^{(n)})^{3/2}} = \frac{1}{\sqrt{{\sigma_{m-1}}^2 - {\sigma_m}^2}}. \tag{21}$$

That is, the rate of convergence is 1.5. ■

4 Convergence of the mdLVs Algorithm

In this section, convergence results for the mdLVs algorithm are established with mathematical rigour. See [2] for the proofs.

4.1 The mdLVs algorithm

The mdLVs algorithm [8, 11] can be described as follows.

Algorithm 4.1 mdLVs algorithm

Initialization: $w_0^{(0)} = 0;\quad w_{2m}^{(0)} = 0;\quad w_k^{(0)} = (b_k)^2\quad (k = 1, 2, \ldots, 2m-1)$

for $n := 0, 1, \cdots$ **do**
  choose shift $s^{(n)} (\geq 0)$ and parameter $\delta^{(n)} (> 0)$ based on $w_k^{(n)}$
  $u_0^{(n)} := 0;\quad u_{2m}^{(n)} := 0$
  for $k := 1, \cdots, 2m-1$ **do**
    $u_k^{(n)} := w_k^{(n)} / (1 + \delta^{(n)} u_{k-1}^{(n)})$
  end for
  $v_0^{(n)} := 0;\quad v_{2m}^{(n)} := 0$
  for $k := 1, \cdots, 2m-1$ **do**
    $v_k^{(n)} := u_k^{(n)} (1 + \delta^{(n)} u_{k+1}^{(n)})$
  end for
  $w_0^{(n+1)} := 0;\quad w_{2m}^{(n+1)} := 0$
  if $s^{(n)} > 0$ **then**
    for $k := 1, \cdots, m-1$ **do**
      $w_{2k-1}^{(n+1)} := v_{2k-1}^{(n)} + v_{2k-2}^{(n)} - w_{2k-2}^{(n+1)} - s^{(n)}$
      $w_{2k}^{(n+1)} := v_{2k-1}^{(n)} v_{2k}^{(n)} / w_{2k-1}^{(n+1)}$
    end for
    $w_{2m-1}^{(n+1)} := v_{2m-1}^{(n)} + v_{2m-2}^{(n)} - w_{2m-2}^{(n+1)} - s^{(n)}$
  else
    for $k := 1, \cdots, 2m-1$ **do**
      $w_k^{(n+1)} := v_k^{(n)}$
    end for
  end if
end for

The outermost loop is terminated when some suitable convergence criterion, say, $\|w_{2m-2}^{(n)}\| \leq \epsilon$ for some prescribed constant $\epsilon > 0$, is satisfied. At the termination we have

$$\sigma_m{}^2 \approx w_{2m-1}^{(n)} + \sum_{l=0}^{n-1} s^{(l)} \tag{22}$$

and hence σ_m can be approximated by $\sqrt{w_{2m-1}^{(n)} + \sum_{l=0}^{n-1} s^{(l)}}$. Then by the deflation process the problem is shrunk to an $(m-1) \times (m-1)$ problem, and the same procedure is repeated until $\sigma_{m-1}, \ldots, \sigma_1$ are obtained in turn. In Algorithm 4.1, $\delta^{(n)} > 0$ is a free parameter.

It turns out to be convenient to introduce

$$w_0^{(n)} = w_{2m}^{(n)} = 0, \quad u_0^{(n)} = u_{2m}^{(n)} = 0, \quad v_0^{(n)} = v_{2m}^{(n)} = 0 \tag{23}$$

for $n = 0, 1, 2, \ldots$ as boundary conditions.

Similarly to the dqds, we use the notation (3) with $b_k^{(0)} = b_k$ $(k = 1, 2, \ldots, 2m-1)$, and put

$$\begin{aligned} w_{2k-1}^{(n)} &= (b_{2k-1}^{(n)})^2 \quad (k = 1, 2, \ldots, m;\ n = 0, 1, \ldots), \\ w_{2k}^{(n)} &= (b_{2k}^{(n)})^2 \quad (k = 1, 2, \ldots, m-1;\ n = 0, 1, \ldots). \end{aligned}$$

Again we denote by $\sigma_{\min}^{(n)}$ the smallest singular value of $B^{(n)}$. The next lemma states that the algorithm does not break down if $s^{(n)} < (\sigma_{\min}^{(n)})^2$ in each iteration n.

Lemma 4.1 (Positivity of the variables in the mdLVs algorithm). *Suppose the mdLVs algorithm is applied to the matrix B satisfying Assumption* (A). *If $s^{(n)} < (\sigma_{\min}^{(n)})^2$ $(n = 0, 1, 2, \ldots)$, then $(B^{(n)})^{\mathrm{T}} B^{(n)}$ are positive definite, and hence $u_k^{(n)} > 0$, $v_k^{(n)} > 0$, and $w_k^{(n)} > 0$ $(k = 1, \ldots, 2m-1)$ for $n = 0, 1, 2, \ldots$.* ■

4.2 Global convergence of the mdLVs

A global convergence theorem for the mdLVs algorithm is available in [11]. The theorem, however, assumes not only $0 \le s^{(n)} < (\sigma_m^{(n)})^2$ but also $\sum_{n=0}^{\infty} s^{(n)} < {\sigma_m}^2$ for the chosen shift. It can easily be suspected from the convergence analysis for the dqds algorithm that the latter assumption is not met when superlinear convergence is realized. As we see later, this is in fact the case with the mdLVs algorithm with the Johnson bound shift. Thus we are motivated to establish a stronger convergence theorem that works also in the case of $\sum_{n=0}^{\infty} s^{(n)} = {\sigma_m}^2$.

The next theorem establishes the convergence of the mdLVs. Moreover, the theorem states that the variables $w_{2k-1}^{(n)}$ converge to the square of the singular values minus the sum of the shifts, and that they are placed in the descending order.

Theorem 4.1 (Convergence of the mdLVs algorithm). *Suppose the matrix B satisfies Assumption* (A), *and the shift in the mdLVs algorithm is taken so that $0 \le s^{(n)} < (\sigma_{\min}^{(n)})^2$ holds for all n, and the parameter is taken so that*

$$\lim_{n\to\infty} \frac{1}{\delta^{(n)}} = D_0 \tag{24}$$

for some nonnegative constant $D_0 \ge 0$. Then we have

$$\sum_{n=0}^{\infty} s^{(n)} \le {\sigma_m}^2. \tag{25}$$

Moreover,

$$\begin{aligned} \lim_{n\to\infty} w_{2k}^{(n)} &= 0 \qquad (k = 1, 2, \ldots, m-1), &(26)\\ \lim_{n\to\infty} w_{2k-1}^{(n)} &= {\sigma_k}^2 - \sum_{n=0}^{\infty} s^{(n)} \qquad (k = 1, 2, \ldots, m). &(27) \end{aligned}$$

■

The next theorem states the asymptotic rate of convergence of the dqds algorithm.

Theorem 4.2 (Rate of convergence of the mdLVs algorithm). *Under the same assumption as in Theorem* 4.1, *we have*

$$\lim_{n\to\infty} \frac{e_k^{(n+1)}}{e_k^{(n)}} = \frac{{\sigma_{k+1}}^2 + D_0 - \sum_{n=0}^{\infty} s^{(n)}}{{\sigma_k}^2 + D_0 - \sum_{n=0}^{\infty} s^{(n)}} < 1 \qquad (k = 1, \ldots, m-1). \tag{28}$$

Therefore $e_k^{(n)}$ $(k = 1, \ldots, m-2)$ are of linear convergence as $n \to \infty$. The bottommost element $e_{m-1}^{(n)}$ is also of linear convergence when ${\sigma_m}^2 + D_0 - \sum_{n=0}^{\infty} s^{(n)} > 0$, and of superlinear convergence when ${\sigma_m}^2 + D_0 - \sum_{n=0}^{\infty} s^{(n)} = 0$. ■

4.3 Convergence rate of the mdLVs with the Johnson bound

For the mdLVs algorithm it is proposed in [8, 11] to use the shift $s^{(n)}$ determined from the Johnson bound as follows:

$$\lambda^{(n)} = \min_{k=1,\ldots,m} \left\{ \sqrt{w_{2k-1}^{(n)}} - \frac{1}{2}\left(\sqrt{w_{2k-2}^{(n)}} + \sqrt{w_{2k}^{(n)}} \right) \right\}, \tag{29}$$

$$s^{(n)} = \left(\max\{\lambda^{(n)},\ 0\} \right)^2. \tag{30}$$

The following theorem shows that the rate of convergence of the dqds is 1.5 if the parameter $\delta^{(n)}$ satisfies

$$\lim_{n\to\infty} \delta^{(n)} = +\infty \tag{31}$$

as well as another natural condition. Note that the condition (31) is a special case of (24) with $D_0 = 0$. The theorem refers only to the lower right two elements of $B^{(n)}$. This is sufficient from the practical point of view since whenever the lower right elements converge to zero, the deflation is applied to reduce the matrix size.

Theorem 4.3 (Rate of convergence of the mdLVs). *Suppose the mdLVs algorithm with the Johnson bound is applied to a matrix B that satisfies Assumption* (A). *If the parameter $\delta^{(n)}$ satisfies* (31) *and*

$$D_1 \le \delta^{(n)} w_{2m-2}^{(n)} \qquad (n = 1, 2, \ldots) \tag{32}$$

for some positive constant D_1, we have

$$\lim_{n\to\infty} \frac{w_{2m-2}^{(n+1)}}{(w_{2m-2}^{(n)})^{3/2}} = \frac{1}{\sqrt{{\sigma_{m-1}}^2 - {\sigma_m}^2}},$$

$$\lim_{n\to\infty} \frac{w_{2m-1}^{(n+1)}}{(w_{2m-1}^{(n)})^{3/2}} = \frac{1}{\sqrt{{\sigma_{m-1}}^2 - {\sigma_m}^2}}.$$

That is, the rate of convergence is 1.5. ■

5 Numerical experiments

In this section, simple numerical results are presented to illustrate the theory. We consider an $m \times m$ symmetric tridiagonal matrix

$$T = \begin{pmatrix} a & b & & 0 \\ b & a & \ddots & \\ & \ddots & \ddots & b \\ 0 & & b & a \end{pmatrix}, \tag{33}$$

the eigenvalues of which are

$$a + 2b\cos\left(\frac{\pi k}{m+1}\right) \quad (k = 1, \dots, m).$$

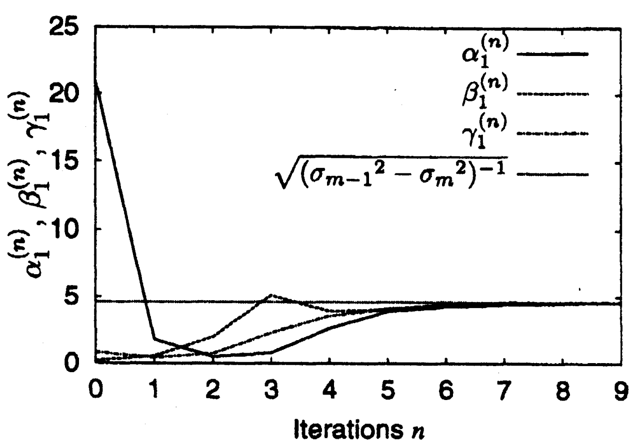
As a test matrix, the bidiagonal matrix B is obtained from the Cholesky decomposition of T. The parameters are taken as $m = 10$, $a = 1.0$ and $b = 0.2$.

First we show the result with the dqds algorithm. In view of Theorem 3.3, we define

$$\alpha_1^{(n)} = \frac{e_{m-1}^{(n+1)}}{(e_{m-1}^{(n)})^{3/2}}, \qquad \beta_1^{(n)} = \frac{q_m^{(n+1)}}{(q_m^{(n)})^{3/2}}, \qquad \gamma_1^{(n)} = \frac{r_m^{(n+1)}}{(r_m^{(n)})^{3/2}},$$

which should converge to the constant $1/\sqrt{({\sigma_{m-1}}^2 - {\sigma_m}^2)}$ according to the theory. The result is shown in Figure 1. The solid line (——) shows $\alpha_1^{(n)}$, the chained line (-------) shows $\beta_1^{(n)}$ and the dashed-dotted line (-·-·-) shows $\gamma_1^{(n)}$. The dotted line (·········) shows $1/\sqrt{({\sigma_{m-1}}^2 - {\sigma_m}^2)} = 4.60$. The solid line, the chained line and the dashed-dotted line all approach the dotted line in Figure 1.

In Figure 2, $e_{m-1}^{(n)}$, $q_m^{(n)}$ and $r_m^{(n)}$ are plotted in the single logarithmic graph. The solid line shows $e_{m-1}^{(n)}$, the chained line shows $q_m^{(n)}$ the dashed-dotted line shows $r_m^{(n)}$. The variables $e_{m-1}^{(n)}$, $q_m^{(n)}$ and $r_m^{(n)}$ converge to zero. By Figure 1 and Figure 2 we can say that the rate of convergence is 1.5.



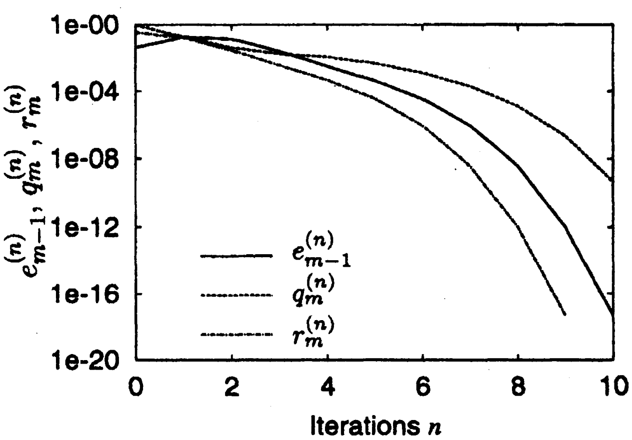


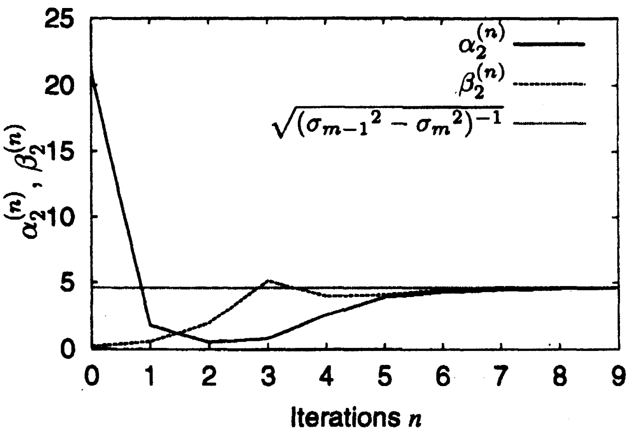
Figure 1: dqds algorithm: $\alpha^{(n)}$, $\beta^{(n)}$ and $\gamma^{(n)}$. Figure 2: dqds algorithm: $e_{m-1}^{(n)}$, $q_m^{(n)}$ and $r_m^{(n)}$.

Second we show the result with the mdLVs algorithm. In view of Theorem 4.3, we define

$$\alpha_2^{(n)} = \frac{w_{2m-2}^{(n+1)}}{(w_{2m-2}^{(n)})^{3/2}}, \qquad \beta_2^{(n)} = \frac{w_{2m-1}^{(n+1)}}{(w_{2m-1}^{(n)})^{3/2}},$$

which should converge to the constant $1/\sqrt{(\sigma_{m-1}{}^2 - \sigma_m{}^2)}$ according to the theory. We chose the parameter $D_1 = 100$ in Theorem 4.3. The result is shown in Figure 3. The solid line (——) shows $\alpha_2^{(n)}$ and the chained line (-------) shows $\beta_2^{(n)}$. The dotted line (———) shows $1/\sqrt{(\sigma_{m-1}{}^2 - \sigma_m{}^2)} = 4.60$. The solid line and the chained line approach the dotted line in Figure 3.

In Figure 4, $w_{2m-2}^{(n)}$ and $w_{2m-1}^{(n)}$ are plotted in the single logarithmic graph. The solid line shows $w_{2m-2}^{(n)}$ and the chained line shows $w_{2m-1}^{(n)}$. The variables $w_{2m-2}^{(n)}$ and $w_{2m-1}^{(n)}$ converge to zero. By Figure 3 and Figure 4 we can say that the rate of convergence is 1.5.



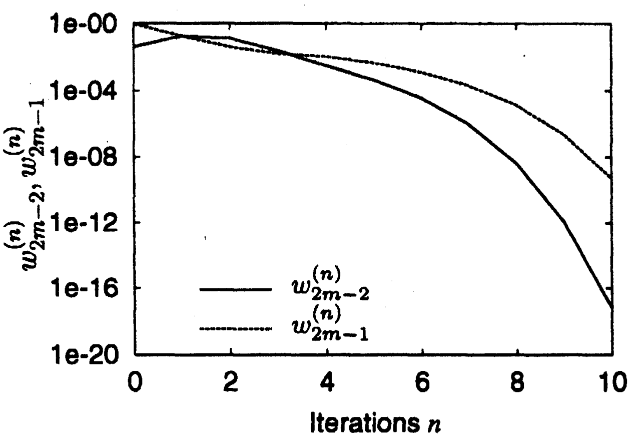


Figure 3: mdLVs algorithm: $a = 1$, $b = 0.2$, $D_1 = 100$

Figure 4: mdLVs algorithm: $a = 1$, $b = 0.2$, $D_1 = 100$

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