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A NOTE ON GAMMA FUNCTIONS

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1. Introduction

Let X be the Gamma distributed random variable,

$$P\{X \leq x\} = \int_0^x \frac{\beta^\xi}{\Gamma(\xi)} t^{\xi-1} e^{-\beta t} dt,$$

for $x > 0$, where $\beta > 0$, $\xi > 0$. Let us denote $Y = \alpha \log X$ for $\alpha \neq 0$. We obtain

$$P\{Y \leq x\} = \int_{-\infty}^x \frac{\beta^\xi}{|\alpha|\,\Gamma(\xi)} e^{(\xi/\alpha)\,t} e^{-\beta e^{t/\alpha}} dt, \ x \in \mathrm{R}, \tag{1}$$

and then the characteristic function of distribution function of the random variable Y is

$$Ee^{izY} = \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}, \ z \in \mathrm{R}. \tag{2}$$

The author will discuss about the Lèvy representation,

$$\begin{aligned}
&\frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \\
&= \exp\Big[iz\Big\{\alpha \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha \log \beta + \alpha^3 \int_{-\infty}^{-0} \frac{x^2}{1+\alpha^2 x^2} \frac{e^{\xi x}}{(1-e^x)} dx\Big\} \\
&+ \int_{-\infty}^{-0} \Big(e^{izx} - 1 - \frac{izx}{1+x^2}\Big) \frac{e^{(\xi/\alpha)\,x}}{(1-e^{x/\alpha})|x|} dx\Big],
\end{aligned} \tag{3}$$

for $\alpha > 0$ and also discuss on Thorin's representation. The Lévy representation and Thorin's representation were found by B. Grigelionis in the paper [1]. In this note, as an application of the Lévy representation it is shown that Gauss's multiplication formula or the duplication formula of Legendre can be obtained from a property of Lévy measure.

2. On an infinitely divisible characteristic function

We first show (1). Suppose that α is a positive constant. We see by change of variable, $u = e^{t/\alpha}$ that

$$
\begin{aligned}
P\{Y \le x\} &= P\{\log X \le \frac{x}{\alpha}\} \\
&= P\{X \le \exp(\frac{x}{\alpha})\} \\
&= \int_0^{\exp(\frac{x}{\alpha})} \frac{\beta^\xi}{\Gamma(\xi)} u^{\xi-1} \exp(-\beta u) du \\
&= \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^{x} \exp\Big\{\frac{\xi}{\alpha} t - \beta e^{\frac{t}{\alpha}}\Big\} \frac{dt}{\alpha}. \qquad (4)
\end{aligned}
$$

Next, suppose that α is a negative constant. Let us set $\alpha' = -\alpha$. We see by change of variable, $u = e^{-t/\alpha'}$ that

$$
\begin{aligned}
P\{Y \le x\} &= P\{\log X \ge -\frac{x}{\alpha'}\} = P\{X \ge \exp(-\frac{x}{\alpha'})\} \\
&= 1 - P\{X < \exp(-\frac{x}{\alpha'})\} \\
&= 1 - \int_0^{\exp(-\frac{x}{\alpha'})} \frac{\beta^\xi}{\Gamma(\xi)} u^{\xi-1} \exp(-\beta u) du \\
&= 1 - \frac{\beta^\xi}{\Gamma(\xi)} \int_{\infty}^{x} \Big(e^{-\frac{t}{\alpha'}}\Big)^\xi \exp\Big(-\beta e^{-\frac{t}{\alpha'}}\Big)\Big(-\frac{dt}{\alpha'}\Big) \\
&= 1 - \frac{\beta^\xi}{\Gamma(\xi)} \int_{x}^{\infty} \exp\Big\{-\frac{\xi}{\alpha'} t - \beta e^{-\frac{t}{\alpha'}}\Big\} \frac{dt}{\alpha'} \\
&= 1 + \frac{\beta^\xi}{\Gamma(\xi)} \int_{x}^{\infty} \exp\Big\{\frac{\xi}{\alpha} t - \beta e^{\frac{t}{\alpha}}\Big\} \frac{dt}{\alpha}. \qquad (5)
\end{aligned}
$$

Suppose that α is a positive constant. Next, we will get a characteristic function of distribution fuction of Y. We see that

$$
\begin{aligned}
Ee^{izY} &= \int_{-\infty}^{\infty} e^{izy} \frac{\beta^\xi}{\Gamma(\xi)} e^{\frac{\xi}{\alpha} y} e^{-\beta e^{\frac{y}{\alpha}}} \frac{dy}{\alpha} \\
&= \frac{\beta^\xi}{\Gamma(\xi)} \int_{-\infty}^{\infty} e^{(iz+\frac{\xi}{\alpha})y - \beta e^{\frac{y}{\alpha}}} \frac{dy}{\alpha} \\
&= \frac{\beta^\xi}{\Gamma(\xi)} \int_{+0}^{\infty} x^{(\xi+i\alpha z)} e^{-\beta x} \frac{dx}{x} \\
&= \frac{\beta^\xi}{\Gamma(\xi)} \int_{+0}^{\infty} v^{(\xi+i\alpha z-1)} e^{-v} dv \frac{1}{\beta^{\xi+i\alpha z}}
\end{aligned}
$$

$$= \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}. \tag{6}$$

Next, suppose that α is a negative constant. Let us set $\alpha' = -\alpha$. We see that

$$\begin{aligned} Ee^{izY} &= \int_{-\infty}^{\infty} e^{izy} \frac{\beta^{\xi}}{\Gamma(\xi)} e^{-\frac{\xi}{\alpha'} y} e^{-\beta e^{-\frac{y}{\alpha'}}} \frac{dy}{\alpha'} \\ &= \frac{\beta^{\xi}}{\Gamma(\xi)} \int_{-\infty}^{\infty} e^{i(z - \frac{\xi}{\alpha'})y - \beta e^{-\frac{y}{\alpha'}}} \frac{dy}{\alpha'} \\ &= \frac{\beta^{\xi}}{\Gamma(\xi)} \int_{\infty}^{+0} x^{(\xi - i\alpha' z)} e^{-\beta x} (-1) \frac{dx}{x} \\ &= \frac{\beta^{\xi}}{\Gamma(\xi)} \int_{+0}^{\infty} v^{(\xi - i\alpha' z - 1)} e^{-v} dv \frac{1}{\beta^{\xi - i\alpha' z - 1}\beta} \\ &= \frac{\Gamma(\xi - i\alpha' z)}{\Gamma(\xi)\beta^{-i\alpha' z}} = \frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}. \end{aligned} \tag{7}$$

Let $z = -iu$ and u real in an interval of the real line which includes the origin such that $\xi + \alpha u > 0$ if ξ is a positive constant. Let us take the principal logarithm such that

$$\log\left\{\frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}\right\} = 0$$

for $z = -iu = 0$ and let us denote

$$\Psi(u) = \log\Gamma(\xi + \alpha u) - \log\Gamma(\xi)\beta^{\alpha u}.$$

Theorem 1. *The characteristic function of the distribution function (4) or (5) is given in the following form.*

$$\begin{aligned} &\frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}} \\ &= \exp\Big[iz\Big\{\alpha \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha\log\beta + \alpha^3 \int_{-\infty}^{-0} \frac{x^2}{1 + \alpha^2 x^2} \frac{e^{\xi x}}{(1 - e^{x})} dx\Big\} \\ &+ \int_{-\infty}^{-0} \Big(e^{izx} - 1 - \frac{izx}{1 + x^2}\Big) \frac{e^{(\xi/\alpha)\, x}}{(1 - e^{x/\alpha})|x|} dx\Big] \end{aligned} \tag{8}$$

Proof. By the result in [2] we see that

$$\begin{aligned} \frac{d\Psi(u)}{du} &= \Big(\log\Gamma(\xi + \alpha u) - \log\Gamma(\xi)\beta^{\alpha u}\Big)' \\ &= \frac{\Gamma(\xi + \alpha u)'}{\Gamma(\xi + \alpha u)}\alpha - \alpha\log\beta \\ &= \alpha \int_{+0}^{\infty} \Big\{\frac{e^{-x}}{x} - \frac{e^{-(\xi + \alpha u)x}}{1 - e^{-x}}\Big\} dx - \alpha\log\beta. \end{aligned} \tag{9}$$

Consider the case that α is positive. Integrating from 0 to u, we have

$$\Psi(u)-\Psi(0)=\alpha\int_0^u\Big(\int_{+0}^{\infty}\Big\{\frac{e^{-x}}{x}-\frac{e^{-(\xi+\alpha t)x}}{1-e^{-x}}\Big\}dx\Big)dt-\ \alpha u\log\beta \tag{10}$$

and so we obtain

$$\begin{aligned}\Psi(u)\quad &=\alpha\int_{+0}^{\infty}\Big(\int_{+0}^{u}\Big\{\frac{e^{-x}}{x}-\frac{e^{-(\xi+\alpha t)x}}{1-e^{-x}}\Big\}dt\Big)dx-\alpha u\log\beta\\ &=\alpha\int_{+0}^{\infty}\Big(u\frac{e^{-x}}{x}-\frac{e^{-\xi x}}{1-e^{-x}}\frac{e^{-\alpha ux}-1}{-\alpha x}\Big)dx-\alpha u\log\beta\\ &=\alpha u\int_{+0}^{\infty}\Big(\frac{e^{-x}}{x}-\frac{e^{-\xi x}}{1-e^{-x}}\Big)dx\\ &+\alpha u\int_{+0}^{\infty}\Big(1-\frac{1}{1+\alpha^2x^2}\Big)\frac{e^{-\xi x}}{1-e^{-x}}dx\\ &+\int_{+0}^{\infty}(e^{-\alpha ux}-1+\frac{\alpha ux}{1+\alpha^2x^2})\frac{e^{-\xi x}}{(1-e^{-x})x}dx-\alpha u\log\beta. \end{aligned}\tag{11}$$

and by the facts that

$$\frac{\Gamma'(\xi)}{\Gamma(\xi)}=\int_{+0}^{\infty}\Big(\frac{e^{-x}}{x}-\frac{e^{-\xi x}}{1-e^{-x}}\Big)dx$$

and

$$\begin{aligned}&\int_{+0}^{\infty}\Big((e^{-\alpha ux}-1+\frac{\alpha ux}{1+\alpha^2x^2}\Big)\frac{e^{-\xi x}}{(1-e^{-x})x}dx\\ &=\int_{+0}^{\infty}\Big(e^{-ut}-1+\frac{ut}{1+t^2}\Big)\frac{e^{-(\xi/\alpha)\,t}}{(1-e^{-t/\alpha})t}dt\\ &=\int_{-\infty}^{-0}\Big(e^{uy}-1-\frac{uy}{1+y^2}\Big)\frac{e^{(\xi/\alpha)\,y}}{(1-e^{y/\alpha})|y|}dy\end{aligned}\tag{12}$$

we obtain

$$\begin{aligned}\Psi(u)\quad &=\alpha u\frac{\Gamma'(\xi)}{\Gamma(\xi)}+\alpha^3u\int_{+0}^{\infty}\frac{x^2}{1+\alpha^2x^2}\frac{e^{-\xi x}}{1-e^{-x}}dx-\alpha u\log\beta\\ &+\int_{-\infty}^{-0}\Big(e^{uy}-1-\frac{uy}{1+y^2}\Big)\frac{e^{(\xi/\alpha)\,y}}{(1-e^{y/\alpha})|y|}dy.\end{aligned}\tag{13}$$

Next, suppose that α is negative. In the same way as the case that α is positive, we obtain

$$\begin{aligned}\Psi(u)\quad &=\alpha u\frac{\Gamma'(\xi)}{\Gamma(\xi)}+\alpha^3u\int_{+0}^{\infty}\frac{x^2}{1+\alpha^2x^2}\frac{e^{-\xi x}}{1-e^{-x}}dx-\alpha u\log\beta\\ &+\int_{+0}^{\infty}\Big(e^{uy}-1-\frac{uy}{1+y^2}\Big)\frac{e^{(\xi/\alpha)\,y}}{(1-e^{y/\alpha})y}dy\end{aligned}\tag{14}$$

and hence the Lévy representation (8). q.e.d

3. On Thorin's representation of characteristic function

We will show Thorin's representation of the characteristic function (2).

Theorem 2. *We obtain Thorin's representation in the following form.*

$$\frac{\Gamma(\xi+i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}$$
$$= \exp\Big[i\alpha z\Big(\frac{\Gamma'(\xi)}{\Gamma(\xi)} - \log\beta + \sum_{k=0}^{\infty}\frac{\alpha^2}{(\alpha^2+(\xi+k)^2)(\xi+k)}\Big)$$
$$+\sum_{k=0}^{\infty}\Big\{-\log\frac{i\alpha z+\xi+k}{\xi+k} + iz\frac{(\xi+k)\alpha}{\alpha^2+(\xi+k)^2}\Big\}\Big] \tag{15}$$

Proof. Suppose that α is positive. We see that

$$\int_{-\infty}^{-0}\Big(e^{ut}-1-\frac{ut}{1+t^2}\Big)\frac{e^{(\xi/\alpha)\,t}}{(1-e^{t/\alpha})|t|}dt$$
$$=\int_{-\infty}^{-0}\Big(\frac{e^{ut}-1}{-t}+\frac{u}{1+t^2}\Big)\frac{e^{(\xi/\alpha)\,t}}{(1-e^{t/\alpha})}dt$$
$$=-\int_{-\infty}^{-0}\Big(\int_{+0}^{u}e^{ty}dy-\frac{u}{1+t^2}\Big)e^{(\xi/\alpha)\,t}\sum_{k=0}^{\infty}e^{k/\alpha\,t}dt$$
$$=-\sum_{k=0}^{\infty}\Big\{\int_{+0}^{u}\Big(\int_{-\infty}^{-0}(e^{(y+(\xi+k)/\alpha)t}dt\Big)dy$$
$$-u\int_{-\infty}^{-0}\frac{1}{1+t^2}e^{(\xi+k)/\alpha\,t}dt\Big\}$$
$$=-\sum_{k=0}^{\infty}\Big\{\int_{+0}^{u}\frac{dy}{y+(\xi+k)/\alpha}-u\frac{(\xi+k)/\alpha}{1+(\xi+k)^2/\alpha^2}\Big\}$$
$$-u\sum_{k=0}^{\infty}\Big\{\frac{(\xi+k)/\alpha}{1+(\xi+k)^2/\alpha^2}-\int_{-\infty}^{-0}\frac{1}{1+t^2}e^{(\xi+k)/\alpha\,t}dt\Big\}. \tag{16}$$

From the above (13) we see that

$$\Psi(u) = \alpha u\frac{\Gamma'(\xi)}{\Gamma(\xi)}+\sum_{k=0}^{\infty}\alpha^3u\int_{+0}^{\infty}\frac{x^2}{1+\alpha^2x^2}e^{-(\xi+k)x}dx-\alpha u\log\beta$$
$$-\sum_{k=0}^{\infty}\Big\{\int_{+0}^{u}\frac{dy}{y+(\xi+k)/\alpha}dy-u\frac{(\xi+k)/\alpha}{1+(\xi+k)^2/\alpha^2}\Big\}$$
$$-u\sum_{k=0}^{\infty}\Big\{\frac{(\xi+k)/\alpha}{1+(\xi+k)^2/\alpha^2}-\int_{-\infty}^{-0}\frac{1}{1+t^2}e^{(\xi+k)/\alpha\,t}dt\Big\}$$

$$= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha u \log \beta$$
$$- \sum_{k=0}^{\infty} \Big\{ \int_{+0}^{u} \frac{dy}{y + (\xi + k)/\alpha} dy - u \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} \Big\}$$
$$-u \sum_{k=0}^{\infty} \Big\{ \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} - \alpha \int_{-\infty}^{-0} e^{(\xi+k)\,t} dt \Big\}$$
$$= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha u \log \beta$$
$$+ \sum_{k=0}^{\infty} \Big\{ -\log \frac{u + (\xi + k)/\alpha}{(\xi + k)/\alpha} + u \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} \Big\}$$
$$-\alpha u \sum_{k=0}^{\infty} \Big\{ \frac{\xi + k}{\alpha^2 + (\xi + k)^2} - \frac{1}{\xi + k} \Big\}$$
$$= \alpha u \frac{\Gamma'(\xi)}{\Gamma(\xi)} - \alpha u \log \beta + \alpha u \sum_{k=0}^{\infty} \frac{\alpha^2}{(\alpha^2 + (\xi + k)^2)(\xi + k)}$$
$$+ \sum_{k=0}^{\infty} \Big\{ -\log \frac{u + (\xi + k)/\alpha}{(\xi + k)/\alpha} + u \frac{(\xi + k)/\alpha}{1 + (\xi + k)^2/\alpha^2} \Big\}. \quad (17)$$

Therefore we obtain Thorin's representation

$$\frac{\Gamma(\xi + i\alpha z)}{\Gamma(\xi)\beta^{i\alpha z}}$$
$$= \exp\Big[i\alpha z \Big(\frac{\Gamma'(\xi)}{\Gamma(\xi)} - \log \beta + \sum_{k=0}^{\infty} \frac{\alpha^2}{(\alpha^2 + (\xi + k)^2)(\xi + k)} \Big)$$
$$+ \sum_{k=0}^{\infty} \Big\{ -\log \frac{i\alpha z + \xi + k}{\xi + k} + iz \frac{(\xi + k)\alpha}{\alpha^2 + (\xi + k)^2} \Big\} \Big]. \quad (18)$$

For the case that α is negative, in the same way as the above we obtain the same expression as the above formula. q.e.d

4. The duplication formula of Legendre and Gauss's multiplication formula

Let $\alpha = 1$, $z = \xi + i\eta$ and $m = 2, 3, \ldots$. The duplication formula of Legendre is

$$\Gamma(z)\Gamma(z + \frac{1}{2}) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)$$

and Gauss's multiplication formula is

$$\Pi_{k=0}^{m-1}\Gamma(z+\frac{k}{m}) = m^{1/2-mz}(2\pi)^{1/2(m-1)}\Gamma(mz). \tag{19}$$

In what follows, we show that Gauss' multiplication formula can be deduced from a property of the Lévy measure in the Lévy representation (3). We see that the left hand side of (19) can be written in the following form:

$$\Pi_{k=0}^{m-1}\Gamma(\xi+i\eta+\frac{k}{m}) = \Gamma(\xi)\beta^{i\eta}\Gamma(\xi+\frac{1}{m})\beta^{i\eta}\cdots\Gamma(\xi+\frac{m-1}{m})\beta^{i\eta}$$
$$\cdot\exp\Big[i\eta\Big\{\frac{\Gamma'(\xi)}{\Gamma(\xi)}-\log\beta+\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{\xi x}}{(1-e^x)}dx\Big\}$$
$$+\int_{-\infty}^{-0}\Big(e^{i\eta x}-1-\frac{i\eta x}{1+x^2}\Big)\frac{e^{\xi\, x}}{(1-e^x)|x|}dx\Big]$$
$$\exp\Big[i\eta\Big\{\frac{\Gamma'(\xi+1/m)}{\Gamma(\xi+1/m)}-\log\beta+\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{(\xi+1/m)x}}{(1-e^x)}dx\Big\}$$
$$+\int_{-\infty}^{-0}\Big(e^{i\eta x}-1-\frac{i\eta x}{1+x^2}\Big)\frac{e^{(\xi+1/m)\,x}}{(1-e^x)|x|}dx\Big]$$
$$\cdots\cdot\exp\Big[i\eta\Big\{\frac{\Gamma'(\xi+(m-1)/m)}{\Gamma(\xi+(m-1)/m)}-\log\beta$$
$$+\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{(\xi+(m-1)/m)x}}{(1-e^x)}dx\Big\}$$
$$+\int_{-\infty}^{-0}\Big(e^{i\eta x}-1-\frac{i\eta x}{1+x^2}\Big)\frac{e^{(\xi+(m-1)/m)\,x}}{(1-e^x)|x|}dx\Big]$$
$$=\Gamma(\xi)\Gamma(\xi+\frac{1}{m})\cdots\Gamma(\xi+\frac{m-1}{m})\beta^{im\eta}$$
$$\cdot\cdot\exp\Big[i\eta\Big\{\frac{\Gamma'(\xi)}{\Gamma(\xi)}+\frac{\Gamma'(\xi+1/m)}{\Gamma(\xi+1/m)}+\cdots+\frac{\Gamma'(\xi+(m-1)/m)}{\Gamma(\xi+(m-1)/m)}\Big\}$$
$$-i\eta m\log\beta+i\eta\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{\xi x}}{(1-e^{x/m})}dx\Big\}$$
$$+\int_{-\infty}^{-0}\Big(e^{i\eta x}-1-\frac{i\eta x}{1+x^2}\Big)\frac{e^{\xi\, x}}{(1-e^{x/m})|x|}dx\Big].$$

By change of variable we see that

$$\int_{-\infty}^{-0}\Big(e^{i\eta x}-1-\frac{i\eta x}{1+x^2}\Big)\frac{e^{\xi\, x}}{(1-e^{x/m})|x|}dx$$
$$=\int_{-\infty}^{-0}\Big(e^{im\eta x}-1-\frac{im\eta x}{1+x^2}\Big)\frac{e^{m\xi\, x}}{(1-e^x)|x|}dx$$

$$+\int_{-\infty}^{-0}\Big(\frac{im\eta x}{1+x^2}-\frac{im\eta x}{1+m^2x^2}\Big)\frac{e^{m\xi x}}{(1-e^x)|x|}dx$$

and

$$\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{\xi x}}{(1-e^{x/m})}dx=\int_{-\infty}^{-0}\frac{m^3x^2}{1+m^2x^2}\Big)\frac{e^{m\xi x}}{(1-e^x)}dx.$$

Therefore we obtain

$$\int_{-\infty}^{-0}\Big(\frac{im\eta x}{1+x^2}-\frac{im\eta x}{1+m^2x^2}\Big)\frac{e^{m\xi x}}{(1-e^x)|x|}dx$$
$$+i\eta\int_{-\infty}^{-0}\frac{m^3x^2}{1+m^2x^2}\frac{e^{m\xi x}}{(1-e^x)}dx=im\eta\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{m\xi x}}{(1-e^x)}dx$$

and so we obtain

$$\Pi_{k=0}^{m-1}\Gamma(\xi+i\eta+\frac{k}{m})=\Gamma(\xi)\Gamma(\xi+\frac{1}{m})\cdots\Gamma(\xi+\frac{m-1}{m})\beta^{im\eta}$$
$$\cdot\exp\Big[i\eta\Big\{\frac{\Gamma'(\xi)}{\Gamma(\xi)}+\frac{\Gamma'(\xi+1/m)}{\Gamma(\xi+1/m)}+\cdots+\frac{\Gamma'(\xi+(m-1)/m)}{\Gamma(\xi+(m-1)/m)}\Big\}$$
$$-i\eta m\log\beta+im\eta\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{m\xi x}}{(1-e^x)}dx\Big\}$$
$$+\int_{-\infty}^{-0}\Big(e^{im\eta x}-1-\frac{im\eta x}{1+x^2}\Big)\frac{e^{m\xi x}}{(1-e^x)|x|}dx\Big]. \qquad (20)$$

We obtain

$$\frac{m\Gamma'(m\xi)}{\Gamma(m\xi)}-\frac{\Gamma'(\xi)}{\Gamma(\xi)}+\frac{\Gamma'(\xi+1/m)}{\Gamma(\xi+1/m)}+\cdots+\frac{\Gamma'(\xi+(m-1)/m)}{\Gamma(\xi+(m-1)/m)}$$
$$=m\int_{+0}^{\infty}\frac{e^{-t}-e^{-mt}}{t}dt$$
$$=m\log m \qquad (21)$$

and

$$\frac{\Gamma(\xi)\Gamma(\xi+1/m)\cdots\Gamma(\xi+(m-1)/m)}{\Gamma(m\xi)}$$
$$=\frac{\Gamma(1/m)\Gamma(2/m)\cdots\Gamma((m-1)/m)}{m^{m\xi-1}}=(2\pi)^{(m-1)/2}m^{1/2-m\xi}. \qquad (22)$$

From the above results we see that

$$\Pi_{k=0}^{m-1}\Gamma(\xi+i\eta+\frac{k}{m}) = (2\pi)^{(m-1)/2}m^{1/2-m(\xi+i\eta)}\Gamma(m\xi)\beta^{im\eta}$$
$$\cdot\exp\Big[im\eta\Big\{\frac{\Gamma'(m\xi)}{\Gamma(m\xi)}-\log\beta+\int_{-\infty}^{-0}\frac{x^2}{1+x^2}\frac{e^{m\xi x}}{(1-e^x)}dx\Big\}$$
$$+\int_{-\infty}^{-0}\Big(e^{im\eta x}-1-\frac{im\eta x}{1+x^2}\Big)\frac{e^{m\xi x}}{(1-e^x)|x|}dx\Big]$$
$$= (2\pi)^{(m-1)/2}m^{1/2-m(\xi+i\eta)}\Gamma(m(\xi+i\eta)) \quad (23)$$

for a positive number ξ. By analytic continuation we obtain Gauss's formula (19).

References

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