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Blaschke products with a critical point on the unit circle and rational functions with Siegel disks

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Abstract

We give a brief survey of results on Siegel disks of some rational functions with bounded type rotation number. A Siegel disk of some polynomial with bounded type rotation number has the quasicircle boundary containing its critical point. In order to construct such a Siegel disk not of a polynomial but of a rational function, we consider some Blaschke product and employ the quasiconformal surgery.

1 Results

Let $P_\alpha(z) = z^2 + e^{2\pi i\alpha}z$. Then the following theorem holds if α is of bounded type.

Theorem 1 (Ghys-Douady-Herman-Shishikura-Świątek, [6]). *If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of P_α centered at the origin is a quasicircle containing its critical point $-e^{2\pi i\alpha}/2$.*

Let $Q_{\alpha,m}(z) = e^{2\pi i\alpha}z(1+z/m)^m$. Geyer showed the following theorem which is extended to some polynomials. Note that P_α is conformally conjugate to $Q_{\alpha,1}$.

Theorem 2 (Geyer, [1]). *Let $m \geq 1$ be a positive integer. If an irrational number $\alpha \in [0,1]$ is of bounded type, then the boundary of the Siegel disk Δ of $Q_{\alpha,m}$ centered at the origin is a quasicircle containing its critical point $-m/(m+1)$.*

For complex numbers λ and μ with $\lambda\mu \neq 1$ and a positive integer m, let

$$F_{\lambda,\mu,m}(z) = z\left(\frac{z+\lambda}{\mu z+1}\right)^m.$$

The origin and the point at infinity are fixed points of $F_{\lambda,\mu,m}$ of multiplier λ^m and μ^m respectively. In the case that $\mu = 0$,

$$F_{\lambda,0,m}(z) = z(z+\lambda)^m .$$

Therefore the rational function $F_{\lambda,\mu,m}$ of degree $m+1$ is considered as a perturbation of the polynomial $F_{\lambda,0,m}$ of degree $m+1$. Note that $F_{\lambda,0,m}$ is conformally conjugate to $Q_{\alpha,m}$ if $\lambda^m = e^{2\pi i\alpha}$.

Main Theorem. *Let $m \geq 1$ be a positive integer and $\mu \in \overline{\mathbb{D}}$. If an irrational number $\alpha \in [0,1]$ is of bounded type and $e^{2\pi i\alpha}\mu^m \neq 1$, then there exist suitable pairs $\{(\lambda_j,\mu_j)\}_{j=1}^m$ with*

(i) $\lambda_j^m = e^{2\pi i\alpha}$, $\mu_j^m = \mu^m$ *and* $\lambda_j\mu_j \neq 1$ *for* $j \in \{1,\ldots,m\}$

(ii) $\lambda_j \neq \lambda_k$ *if* $j \neq k$

such that for each $j \in \{1,\ldots,m\}$, the boundary of the Siegel disk Δ_j of $F_{\lambda_j,\mu_j,m}$ centered at the origin is a quasicircle containing its critical point.

Main Theorem contains Theorems 1 and 2. Moreover we obtain the following corollary.

Corollary. *Let $m \geq 1$ be a positive integer, $\alpha \in [0,1]$ be an irrational number of bounded type, $\mu^m = e^{2\pi i\beta}$ with $e^{2\pi i\alpha}\mu^m \neq 1$ and $\{(\lambda_j,\mu_j)\}_{j=1}^m$ be as in Main Theorem. If $\beta \in [0,1]$ is an irrational number of bounded type, then the boundaries of Siegel disks Δ_j and Δ_j^∞ of $F_{\lambda_j,\mu_j,m}$ centered at the origin and the point at infinity respectively are quasicircles containing one critical point.*

2 Key Theorems

Let $m \geq 1$ be a positive integer. We consider the Blaschke product

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-bz}\right)^m$$

of degree $2m+1$ with $a\bar{b} \neq 1$ and $0 < |a| \leq |b| < \infty$. Let

$$x = \left\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\right\}^{-1} \times \left\{D_1 \cos 2\pi\varphi + D_2 \cos 2\pi(\varphi+\theta+\omega) + D_3 \cos 2\pi(3\varphi+\theta+\omega) + D_4 \cos 2\pi(3\varphi+2\theta+2\omega)\right\}$$

and

$$y = \left\{(m+1)^2 + (m-1)^2 r^2 + 2(m^2-1) r \cos 2\pi(2\varphi+\theta+\omega)\right\}^{-1} \times \left\{D_1 \sin 2\pi\varphi - D_2 \sin 2\pi(\varphi+\theta+\omega) + D_3 \sin 2\pi(3\varphi+\theta+\omega) - D_4 \sin 2\pi(3\varphi+2\theta+2\omega)\right\},$$

where

$$D_1 = (m+1)^2(2m+1) - 2m(m^2-1)r^2,$$
$$D_2 = 2m(m^2-1)r - (m-1)^2(2m-1)r^3,$$
$$D_3 = -(m+1)^2 r, \quad D_4 = -(m-1)^2 r^2.$$

Theorem A. *Let $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$ and let $a = a(\theta,\varphi)$ and $b = b(\theta,\varphi)$ with $|a| \leq |b|$ be complex numbers satisfying relations $a + b = x + iy$ and $ab = re^{-2\pi i(\theta+\omega)}$, that is, a and b are the solutions of the equation*

$$Z^2 - (x+iy)Z + re^{-2\pi i(\theta+\omega)} = 0, \qquad (\dagger)$$

where x and y are as above and $(\theta,\varphi) \in [0,1]^2$. Then the following holds:

(a) *In the case that $r = 0$, solutions of the equation $(\dagger)$ are $a = 0$ and $b = (2m+1)e^{2\pi i\varphi}$.*

(b) *In the case that $0 < r < 1$, the equation $(\dagger)$ does not have double roots. Moreover $0 < |a| < 1 < |b| < \infty$.*

(c) *In the case that $r = 1$ and $2\varphi + \theta + \omega \equiv 0 \pmod 1$, the equation $(\dagger)$ has double roots and $a = b = e^{2\pi i\varphi}$.*

(d) *In the case that $r = 1$ and $2\varphi + \theta + \omega \not\equiv 0 \pmod 1$, the equation $(\dagger)$ does not have double roots. Moreover $0 < |a| < 1 < |b| < \infty$.*

(e) *In the case* (a), (b) *or* (d),

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m$$

is a Blaschke product of degree $2m+1$ *and the point at infinity is a fixed point of* $B_{\theta,\varphi,m}$ *with multiplier* μ^m. *Moreover* $z = e^{2\pi i\varphi}$ *is a critical point of* $B_{\theta,\varphi,m}$ *and* $B_{\theta,\varphi,m}|_{\mathbb{T}} : T \to \mathbb{T}$ *is a homeomorphism, where* $\mathbb{T}$ *is the unit circle.*

Let $f : \mathbb{T} \to \mathbb{T}$ be an orientation preserving homeomorphism and denote by $\rho(f)$ the rotation number of f.

Theorem B. *Let* $\alpha \in [0, 1]$ *and let* $\mu = re^{2\pi i\omega} \in \overline{\mathbb{D}}$, $a = a(\theta, \varphi)$ *and* $b = b(\theta, \varphi)$ *be as in Theorem* A. *Then for the Blaschke product*

$$B_{\theta,\varphi,m}(z) = e^{2\pi i m\theta} z \left(\frac{z-a}{1-\bar{a}z}\right)^m \left(\frac{z-b}{1-\bar{b}z}\right)^m,$$

$B_{\theta,\varphi,m}|_{\mathbb{T}} : \mathbb{T} \to \mathbb{T}$ *is an orientation preserving homeomorphism. Moreover*

(a) *If* $0 \le r < 1$, *then there exists* $(\theta_0, \varphi_0) \in [0, 1]^2$ *such that* $\rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha$.

(b) *If* $r = 1$ *and* $\alpha + m\omega \not\equiv 0 \pmod 1$, *then there exists* $(\theta_0, \varphi_0) \in [0, 1]^2$ *such that* $\rho(B_{\theta_0,\varphi_0,m}|_{\mathbb{T}}) = \alpha$ *and* $2\varphi_0 + \theta_0 + \omega \not\equiv 0 \pmod 1$.

3 Proof

Proof of Main Theorem. By Theorem B, there exist $(\theta, \varphi) \in [0, 1]^2$ such that the degree of $B_{\theta,\varphi,m}$ is $2m + 1$ and $\rho(B_{\theta,\varphi,m}|_{\mathbb{T}}) = \alpha$. Then there exists a quasisymmetric homeomorphism $h : \mathbb{T} \to \mathbb{T}$ such that $h \circ B_{\theta,\varphi,m}|_{\mathbb{T}} \circ h^{-1}(z) = R_\alpha(z) = e^{2\pi i\alpha}z$ since α is of bounded type. By the theorem of Beurling and Ahlfors, h has a quasiconformal extension $H : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ with $H(0) = 0$. We define a new map $\mathfrak{B}_{\theta,\varphi,m}$ as

$$\mathfrak{B}_{\theta,\varphi,m} = \begin{cases} B_{\theta,\varphi,m} & \text{on } \hat{\mathbb{C}} \setminus \mathbb{D}, \\ H^{-1} \circ R_\alpha \circ H & \text{on } \mathbb{D}. \end{cases}$$

The map $\mathfrak{B}_{\theta,\varphi,m}$ is quasiregular on $\hat{\mathbb{C}}$ since $\mathbb{T}$ is an analytic curve. Moreover $\mathfrak{B}_{\theta,\varphi,m}$ is a degree $m+1$ branched covering of $\hat{\mathbb{C}}$. We define a conformal structure $\sigma_{\theta,\varphi,m}$ as

$$\sigma_{\theta,\varphi,m} = \begin{cases} H^*\sigma_0 & \text{on } \mathbb{D}, \\ \left(\mathfrak{B}^n_{\theta,\varphi,m}\right)^* \sigma_0 & \text{on } \mathfrak{B}^{-n}_{\theta,\varphi,m}(\mathbb{D}) \setminus \mathbb{D} \text{ for all } n \in \mathbb{N}, \\ \sigma_0 & \text{on } \hat{\mathbb{C}} \setminus \bigcup_{n=1}^{\infty} \mathfrak{B}^{-n}_{\theta,\varphi,m}(\mathbb{D}), \end{cases}$$

where σ_0 is the standard conformal structure on $\hat{\mathbb{C}}$. The conformal structure $\sigma_{\theta,\varphi,m}$ is invariant under $\mathfrak{B}_{\theta,\varphi,m}$ and its maximal dilatation is the dilatation of H since H is quasiconformal and $B_{\theta,\varphi,m}$ is holomorphic. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $\Psi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $\Psi^*\sigma_0 = \sigma_{\theta,\varphi,m}$. Therefore $\Psi \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi^{-1}$ is a rational map of degree $m+1$. We normalize $\Psi = \Psi_j$ by $\Psi_j(0) = 0$, $\Psi_j(b) = -\lambda_j$ and $\Psi_j(\infty) = \infty$, where $\lambda_j = e^{2\pi i(\alpha+j)/m}$ for $j \in \{1, \ldots, m\}$.

Lemma. *If $\mu \neq 0$, then there exists μ_j with $\mu_j^m = \mu^m$ such that*

$$F_{\lambda_j,\mu_j,m} = \Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}.$$

Proof of Lemma. Define ξ_j as $\xi_j = -\Psi_j(1/\overline{a})$. Note that $\lambda_j \neq \xi_j$ since such Ψ_j is unique. Since orders of zeros and poles are invariant under conjugation, we obtain that

$$\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}(z) = \eta_j z \left(\frac{z+\lambda_j}{z+\xi_j}\right)^m.$$

Since multipliers of fixed points are also invariant under conjugation, we obtain that

$$\left(\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}\right)'(0) = \frac{\eta_j \lambda_j^m}{\xi_j^m} = e^{2\pi i \alpha} \tag{1}$$

and

$$\frac{1}{\left(\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}\right)'(\infty)} = \frac{1}{\eta_j} = \mu^m. \tag{2}$$

By the equations (1) and (2), we obtain that $(\xi_j\mu)^m = 1$. Then there exists an m-th root of unity ν_j such that $\xi_j = \nu_j/\mu$. Therefore

$$\Psi_j \circ \mathfrak{B}_{\theta,\varphi,m} \circ \Psi_j^{-1}(z) = \frac{z}{\mu^m}\left(\frac{z+\lambda_j}{z+\nu_j/\mu}\right)^m = z\left(\frac{z+\lambda_j}{\mu z+\nu_j}\right)^m$$

$$= \frac{z}{\nu_j^m}\left(\frac{z+\lambda_j}{\left(\mu/\nu_j\right)z+1}\right)^m = z\left(\frac{z+\lambda_j}{\mu_j z+1}\right)^m = F_{\lambda_j,\mu_j,m}(z),$$

where $\mu_j = \mu/\nu_j$. □

Let $\mu_j = 0$ for all $j \in \{1, \dots, m\}$ if $\mu = 0$. It is easy to check that the pairs $\{(\lambda_j, \mu_j)\}_{j=1}^m$ satisfy (i) and (ii). The map $F_{\lambda_j,\mu_j,m}$ has a Siegel disk $\Delta = \Psi_j(\mathbb{D})$ with a critical point $\Psi_j(e^{2\pi i\varphi}) \in \partial\Delta$. Moreover $\partial\Delta = \Psi_j(\mathbb{T})$ is a quasicircle since Ψ_j is quasiconformal. □

Proof of Corollary. Let $\mathcal{I}(z) = 1/z$. Then $F_{\lambda_j,\mu_j,m} = \mathcal{I} \circ F_{\mu_j,\lambda_j,m} \circ \mathcal{I}$. Let Δ and Δ_∞ be Siegel disks of $F_{\lambda_j,\mu_j,m}$ centered at the origin and the point at infinity respectively. By Main Theorem, the boundary of Δ contains a critical point of $F_{\lambda_j,\mu_j,m}$. On the other hand, $\mathcal{I}(\Delta_\infty)$ is the Siegel disk of $F_{\mu_j,\lambda_j,m}$ centered at the origin. By Main Theorem, the boundary of $\mathcal{I}(\Delta_\infty)$ contains a critical point of $F_{\mu_j,\lambda_j,m}$. Therefore the boundary of Δ_∞ contains a critical point of $F_{\lambda_j,\mu_j,m}$. □

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