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Equivalence relations on measure spaces and von Neumann algebras

— One-cocycles and coactions —

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1 Introduction

This note is a brief survey on the several topics treated in the articles [2] and [22], and is prepared for the report of the workshop「作用素環論の新展開」held at RIMS in September, 2007. What is discussed here is a very intriguing relation between 1-cocycles in ergodic theory and a certain type of coactions in von Neumann algebra theory. This relation illustrates how every 1-cocycle on a principal measured groupoid or an ergodic $\mathbf{R}$-space gives rise to a special kind of coaction on the von Neumann algebra associated with such a dynamical system and vice versa. We shall thus see that the study of 1-cocycles described above is roughly "equivalent" to that of coactions with a specific property. I hope that the results presented in this note would provide us with a new interesting approach to analyze both 1-cocycles in ergodic theory and coactions on von Neumann algebras.

2 1-cocycles on measured equivalence relations and coactions associated with them

Throughout the rest of this section, we fix a principal measured groupoid (i.e., a measured equivalence relation) $(\mathcal{R}, \{\lambda^x\}_{x\in X}, \mu)$ on a standard Borel probability space (X, μ). Associated with such a system and a normalized 2-cocycle σ on $\mathcal{R}$ is a von Neumann algebra $W^*(\mathcal{R},\sigma)$ acting on the Hiilbert space $L^2(\mathcal{R},\nu)$, called the (twisted) goupoid von Neumann algebra, obtained by the method indicated in [11], [16], [19], [8], ... Here ν is the measure on $\mathcal{R}$ defined by integrating the Haar system $\{\lambda^x\}_{x\in X}$ with respect to the quasi-invariant measure μ: $\nu := \int_X \lambda^x d\mu(x)$. Inside $W^*(\mathcal{R},\sigma)$, there is a $*$-isomorphic image D of $L^\infty(X,\mu)$, which we call the diagonal algebra of $W^*(\mathcal{R},\sigma)$. In the case of $\mathcal{R}$ being discrete, D is usually called a Cartan subalgebra.

Suppose now that we are given a Borel 1-cocycle c from $\mathcal{R}$ into a (second countable) locally compact group K. We denote the set of such 1-cocycles by $Z^1(\mathcal{R},K)$. Then we define a unitary U_c on $L^2(K)\otimes L^2(\mathcal{R},\nu)$ by

$$\{U_c\xi\}(k,(x,y)) := \xi(c(x,y)^{-1}k,(x,y)) \qquad (\xi\in L^2(K)\otimes L^2(\mathcal{R},\nu), k\in K, (x,y)\in\mathcal{R})$$

Then it is easy to check that the map $\alpha_c : W^*(\mathcal{R},\sigma)\to B(L^2(K)\otimes L^2(\mathcal{R},\nu))$

$$\alpha_c(a) := U_c(1\otimes a)U_c^* \qquad (a\in W^*(\mathcal{R},\sigma))$$

defines a coaction of K on $W^*(\mathcal{R},\sigma)$, that is to say, it is a unital injective normal $*$-homomorphism from $W^*(\mathcal{R},\sigma)$ into $W^*(K)\otimes W^*(\mathcal{R},\sigma)$ satisfying

$$(\Delta_K\otimes id)\circ\alpha_c=(id\otimes\alpha_c)\circ\alpha_c,$$

where $W^*(K)$ is the group von Neumann algebra of K and Δ_K stands for the usual coproduct of Δ_K. One of the characteristics of α_c is that it fixes the diagonal algebra D pointwise. In general, for a coaction β on a von Neumann algebra A, an element $a\in A$ is said to be fixed by β if it satisfies $\beta(a)=1\otimes a$. The von Neumann subalgebra consisting of all elements fixed by β is called the fixed-point algebra of β and is denoted by A^β. As we see below, the above property of α_c completely characterizes the coactions on $W^*(\mathcal{R},\sigma)$ which arise from 1-cocycles on $\mathcal{R}$.

Theorem 2.1. Let α be a coaction of a locally compact group K on $W^*(\mathcal{R},\sigma)$ that fixes D pointwise. Then there exists a cocycle $c\in Z^1(\mathcal{R},K)$ such that $\alpha=\alpha_c$.

Before we give an outline of a proof to this theorem, let us treat the case of $\mathcal{R}$ being discrete, because this case is fairly easy to deal with.

First choose nonsingular partial Borel automorhisms $\{\phi_i\}_{i\in I}$ on X such that (i) $\Gamma(\phi_i)$ (:=the graph of ϕ_i) $\subseteq\mathcal{R}$; (ii) $\mathcal{R}=\bigcup_{i\in I}\Gamma(\phi_i)$ (disjoint). Let v_i be the partial isometry in $W^*(\mathcal{R},\sigma)$ corresponding to ϕ_i — it satisfies $v_i^*v_i, v_iv_i^*\in D$, $v_vDv_i^*=v_iv_i^*D$. Then check that $w_i:=\alpha(v_i^*)(1\otimes v_i)\in W^*(K)\otimes D=L^\infty(X,W^*(K))$ and that it satisfies $(\Delta_K\otimes id)(w_i)=(w_i)_{12}(w_i)_{13}$. This means that, as a bounded $W^*(K)$-valued Borel function, w_i satisfies $\Delta_K(w_i(x))=w_i(x)\otimes w_i(x)$. Since

$$\{u\in W^*(K)\setminus\{0\}:\Delta_K(u)=u\otimes u\}=\lambda_K(K),$$

where λ_K is the left regular representation of K, there exists, for each $i\in I$, a Borel function $k_i:X\to K$ such that $\alpha(v_i^*)(1\otimes v_i)(x)=\lambda_K(k_i(x))$. Define $c:\mathcal{R}\to K$ by

$$c(x,y):=k_i(x)\quad\text{if }(x,y)\in\Gamma(\phi_i).$$

The map c is the desired 1-cocycle.

Proof of Theorem 2.1. (Outline) Let U be the canonical implementation of α on $L^2(\mathcal{R},\nu)$ in the sense of [20]. Thanks to the identity $(\Delta_K\otimes id)(U)=U_{23}U_{13}$, the equation

$$\Pi(\omega):=(\omega\otimes id)(U^*)\qquad(\omega\in W^*(K)_*)$$

defines a $*$-representation Π of the abelian Banach $*$-algebra $W^*(K)_*$ on $L^2(\mathcal{R},\nu)$. Note that the Gelfand spectrum of $W^*(K)_*$ is (homeomorphic to) K ([5]). So, by the spectral theorem of $*$-representations of abelian Banach $*$-algebras (see [6]), there exists an $L^2(\mathcal{R},\nu)$-projection-valued measure P on K such that $\Pi(\omega)=\int_K\hat{\omega}\,dP$ for any $\omega\in W^*(K)_*$, where $\hat{\omega}$ is the Gelfand transform of ω. Observe that the image of Π is contained in $(D\cup JDJ)''$. Here J denotes the modular conjugation of $\mathcal{R}$. Since the groupoid $\mathcal{R}$ is principal, we have $(D\cup JDJ)''=L^\infty(\mathcal{R},\nu)$. It follows that there exists a Borel map $c:\mathcal{R}\to K$ such that $P(B)=\chi_{c^{-1}(B^{-1})}$ for any Borel subset B of K. By replacing c by a suitable Borel map almost everywhere equal to c if necessary, we get the desired 1-cocycle. □

Remark. Theorem 2.1 can be strengthened to the case of a locally compact quantum group action as follows. Let $\mathbb{G} = (M, \Delta)$ be a locally compact quantum group in the sense of Kustermans and Vaes ([17]). Suppose that α is a faithful action of $\mathbb{G}$ on $W^*(\mathcal{R}, \sigma)$ ([20]). If D is regular in $W^*(\mathcal{R}, \sigma)$ and if the fixed-point algebra $W^*(\mathcal{R}, \sigma)^\alpha$ of α contains the diagonal algebra D, then $\mathbb{G}$ must be cocommutative. Therefore, α is a coaction of some locally compact group K.

Definition 2.2. Let α be a coaction of K on a von Neumann algebra A. A unitary V in $W^*(K) \otimes A$ is called an α-1-cocycle if V satisfies the following:

$$(\Delta_K \otimes id_A)(V) = V_{23}(id_{W^*(K)} \otimes \alpha)(V).$$

For each α-1-cocycle V, $\operatorname{Ad} V \circ \alpha$ is also a coaction of K on A. A coaction α' of K on A is said to be cocycle conjugate to α if there exist an α-1-cocycle V and a $*$-automorphism θ of A such that $(id_{W^*(K)} \otimes \theta) \circ \alpha' \circ \theta^{-1} = \operatorname{Ad} V \circ \alpha$.

Proposition 2.3. Let K be a locally compact group and $c, c' \in Z^1(\mathcal{R}, K)$.

(1) If c is cohomologous to c', i.e., if there is a Borel map $\phi : X \to K$ such that $c'(x, y) = \phi(x)c(x, y)\phi(y)^{-1}$ a.e. $(x, y) \in \mathcal{R}$, then there exists an α_c-1-cocycle V in $W^*(K) \otimes D$ so that $\alpha_{c'} = \operatorname{Ad} V \circ \alpha_c$. Hence α_c is cocycle conjugate to $\alpha_{c'}$.

(2) If there exist an α_c-1-cocycle V and a $*$-automorphism θ of $W^*(\mathcal{R}, \sigma)$ such that $\theta(D) = D$ and $(id_{W^*(K)} \otimes \theta) \circ \alpha_{c'} \circ \theta^{-1} = \operatorname{Ad} V \circ \alpha_c$, then c is weakly equivalent to c' in the sense that there exists a measure-class preserving Borel groupoid automorphism ρ of $\mathcal{R}$ such that $c' \circ \rho^{-1}$ is cohomologous to c.

From this proposition, we see that the assignment $c \in Z^1(\mathcal{R}, K) \longmapsto \alpha_c$ passes to the map from $Z^1(\mathcal{R}, K)$ modulo the cohomologous equivalence to the set of coactions α of K on $W^*(\mathcal{R}, \sigma)$ with the property $D \subseteq W^*(\mathcal{R}, \sigma)^\alpha$, modulo the special cocycle conjugacy described in Proposition 2.3 (2). Let us remark that the converse of Proposition 2.3 (2) is also true if $\mathcal{R}$ is discrete and $\sigma = 1$.

Thus we may say that study of 1-cocycles on measured equivalence relations is almost equivalent to that of coactions on $W^*(\mathcal{R}, \sigma)$ fixing D pointwise.

To add yet another evidence to support our statement above, we consider the asymptotic range of a Borel 1-cocycle on a measured equivalence relation (cf. [7], [19]).

Definition 2.4. Let $c \in Z^1(\mathcal{R}, K)$. The essential range $\sigma(c)$ of c is by definition $\sigma(c) := \bigcap\{\overline{c(B)} : m(B^c) = 0\}$. Then the asymptotic range $r^*(c)$ of c is defined to be $r^*(c) := \bigcap\{\sigma(c_B) : \mu(B) > 0\}$, where $c_B := c|_{\mathcal{R} \cap (B \times B)}$. It is known (see [7], [19] for example) that the asymptotic range is a closed subgroup of K.

With the notion of an asymptotic range on one hand, we have the notion of a Connes spectrum $\Gamma(\alpha)$ of a coaction α on the other. Refer to [18] for the definition of $\Gamma(\alpha)$. The next theorem states that the assignment $c \in Z^1(\mathcal{R}, K) \mapsto \alpha_c$ behaves nicely in the level of the invariants $r^*(c)$ and $\Gamma(\alpha)$ introduced above.

Theorem 2.5. For any $c \in Z^1(\mathcal{R}, K)$, we have $r^*(c) = \Gamma(\alpha_c)$.

By combining this theorem with one of the results of Golodets and Sinel'shchikov in [10], we can extend the main result of Kawahigashi in [15] concerning the classification, up to cocycle conjugacy, of actions of locally compact abelian groups on the AFD type II factors which fix Cartan subalgebras.

Theorem 2.6. Let A be an AFD type II factor. Suppose that α and α' are coactions of a locally compact group K on A such that each of A^α and $A^{\alpha'}$ contains a Cartan subalgebra of A. If $\Gamma(\alpha) = \Gamma(\alpha') = K$, then α is cocycle conjugate to α'

Sketch of proof. Suppose that A^α (resp. $A^{\alpha'}$) contains a Cartan subalgebra D_1 (resp. D_2) of A. By [3], there exists a $*$-automorphism θ of A such that $\theta(D_1) = D_2$. Set $\alpha_\theta := (id_{W^*(K)} \otimes \theta^{-1}) \circ \alpha \circ \theta$. Then we have $A^{\alpha_\theta} = \theta(A^\alpha)$. So $D_2 = \theta(D_1) \subsetneq \theta(A^\alpha) = A^{\alpha_\theta}$. Clearly, α_θ is cocycle conjugate to α. Hence it suffices to assume from the outset that $D_1 = D_2 =: D$.

We may assume that the inclusion $(D \subsetneq A)$ is of the form $(L^\infty(X) \subsetneq W^*(\mathcal{R}))$ for an amenable ergodic type II discrete measured equivalence relation $\mathcal{R}$ on a standard Borel space $(X, \mathfrak{B}, \mu)$ with an invariant measure μ. By Theorem 2.1, there exist $c, c' \in Z^1(\mathcal{R}, K)$ such that $\alpha = \alpha_c$ and $\alpha' = \alpha_{c'}$. Due to Theorem 2.5, we have $r^*(c) = r^*(c') = K$. Then, from [10], two cocycles c, c' are weakly equivalent. By the remark made right after Proposition 2.3, α is cocycle conjugate to α'. □

3 1-cocycles on an ergodic R-space and extended modular coactions

Let us fix a (separable) type III factor N and a dominant weight ϕ on N for the time being. Then consider the so-called continuous decomposition $\{N_\phi, \mathbf{R}, \theta\}$ of N with respect to ϕ ([4]). Hence, in particular, the crossed product $\mathbf{R}\,_\theta\!\ltimes N_\phi$ is $*$-isomorphic to N, and we denote this isomorphism by Ψ. The commutative dynamical system $\{Z(N_\phi), \mathbf{R}, \theta|_{Z(N_\phi)}\}$ is usually called the smooth flow of weights on N ([4]). But, in this note, instead of looking at the algebra itself, we mainly focus on a measure-theoretical realization of this system as follows:

$$Z(N_\phi) = L^\infty(X_N, \mu), \qquad \theta_t(f)(x) = f(F^N_{-t}x) \qquad (f \in L^\infty(X_N, \mu),\ x \in X_N,\ t \in \mathbf{R}).$$

It is known that $\{F^N_t\}$ is an ergodic flow.

Let K be a locally compact group. A Borel map $c : \mathbf{R} \times X_N \to K$ is said to be a (K-valued) 1-cocycle on the ergodic $\mathbf{R}$-space $\{X_N, F^N\}$ if it satisfies

$$c(s+t, x) = c(s, F^N_t x)c(t, x) \qquad (s, t \in \mathbf{R}, x \in X_N).$$

The set of all (K-valued) Borel 1-cocycles on $\{X_N, F^N\}$ is denoted by $Z^1(F^N, K)$. Our first main theorem of this section is the following:

Theorem 3.1. Let $c \in Z^1(F^N, K)$. Then there exists a coaction β_c^ϕ of K on N satisfying the identities

$$\beta_c^\phi(a) = 1 \otimes a \qquad (\forall a \in N_\phi); \tag{3.1}$$

$$\beta_c^\phi(u(s)) = Q_s(1 \otimes u(s)) \qquad (\forall s \in \mathbf{R}), \tag{3.2}$$

where $u(s) := \Psi(\lambda_{\mathbf{R}}(s) \otimes 1) \in N$ and $Q_s \in W^*(K) \otimes Z(N_\phi) = L^\infty(X_N, W^*(K))$ is a $W^*(K)$-valued Borel function on X_N defined by $Q_s(x) := \lambda_K(c(-s,x))^*$. If ψ is another dominant weight on N, then β_c^ψ is conjugate to β_c^ϕ. Therefore, the conjugacy class of β_c^ϕ is independent of choice of a dominant weight on N.

Definition 3.2. The coaction β_c^ϕ is called the extended modular coaction associated with ϕ and c.

Remark. Let us examine the extended modular coaction β_c^ϕ in the case where K is the one-dimensional torus $\mathbf{T}$. In this case, by using the Fourier transform, β_c^ϕ corresponds to a usual automorphic action of the dual group $\mathbf{Z} = \widehat{\mathbf{T}}$ on N. Thus β_c^ϕ determines a single automorphism of N, still denoted by β_c^ϕ. To clarify what this automorphism is, we define a function $a : \mathbf{R} \to Z(N_\phi) = L^\infty(X_N)$ by

$$a_t(x) := c(t,x) \in \mathbf{T} \qquad (t \in \mathbf{R},\ x \in X_N).$$

Thus a_t is a unitary element in $Z(N_\phi) = L^\infty(X_N)$ for any $t \in \mathbf{R}$. From the cocycle identity of c, it easily follows that the function a satisfies $a_{s+t} = a_s\theta_s(a_t)$ for all $s, t \in \mathbf{R}$. In other words, a is a unitary θ-1-cocycle in $Z(N_\phi)$. Hence it induces a $*$-automorphism σ_a^ϕ of N, called the extended modular automorphism ([4]). By using the identities (3.1), (3.2), it is easy to check that β_c^ϕ actually equals σ_a^ϕ. This fact justifies our terminology of "extended modular coaction."

Proposition 3.3. Let $c, c' \in Z^1(F^N, K)$. Then the following are equivalent:

(1) The cocycles c and c' are cohomologous, i.e., there is a Borel map $q : X \to K$ such that, for each $t \in \mathbf{R}$, $c'(t,x) = q(F_t^N x)^{-1} c(t,x) q(x)$ for a.e. $x \in X_N$.

(2) There exists a β_c^ϕ-1-cocycle R such that $\beta_{c'}^\phi = \operatorname{Ad} R \circ \beta_c^\phi$ (in particular, the coactions are cocycle conjugate).

Sketch of proof. (1) $\Rightarrow$ (2): Suppose that c and c' are cohomologous. Let $q : X \to K$ be a Borel map as above. This in turn induces the Borel map $V_q : x \in X_N \mapsto \lambda_K(q(x))^* \in W^*(K)$. Then V_q is a unitary in $L^\infty(X_N, W^*(K)) = W^*(K) \otimes Z(N_\phi)$. By (3.1), $(\Delta_K \otimes id)(V_q) = (V_q)_{23}(id \otimes \beta_c^\phi)(V_q)$, i.e., V_q is a β_c^ϕ-1-cocycle. Take V_q for the desired unitary R.

(2) $\Rightarrow$ (1): Let R be as in (2). By (3.1) and the Connes-Takesaki Relative Commutant Theorem ([4]), $R \in W^*(K) \otimes Z(N_\phi)$. Thus R can be viewed as a $W^*(K)$-valued Borel function on X_N. Since $(\Delta_K \otimes id)(R) = R_{23}(id \otimes \beta_c^\phi)(R) = R_{23}R_{13}$, it follows that $\Delta_K(R(x)) = R(x) \otimes R(x)$. Hence there is a Borel function $p : X_N \to K$

such that $R(x) = \lambda_K(p(x))^*$. By (3.2), we get $Q_s^{c'} = RQ_s^c(1 \otimes u(s))R(1 \otimes u(s)^*) = RQ_s^c(id \otimes \theta_s)(R^*)$ for any $s \in \mathbf{R}$. This means that, for a.e. $x \in X_N$, we have

$$c(-s,x)^{-1} = p(x)^{-1}c(-s,x)p(F_{-s}^N x).$$

Therefore, c' is cohomologous to c □

Thanks to this proposition, just as in the case of the correspondence

$$c \in Z^1(\mathcal{R}, K) \longmapsto \alpha_c \in \{\text{coactions of } K \text{ on } W^*(\mathcal{R}, \sigma)\}$$

established in the previous section, the assignment $c \in Z^1(F^N, K) \longmapsto \beta_c^\phi$ passes to the map from $Z^1(F^N, K)$ modulo the cohomologous equivalence to the set of coactions of K on N modulo cocycle conjugacy.

Definition 3.4. Let $c \in Z^1(F^N, K)$. From Theorem 3.1 and Proposition 3.3, the dual covariant system $\{\widehat{K}\,{}_{\beta_c^\phi}\!\ltimes N,\ K,\ \widehat{(\beta_c^\phi)}\}$, where $\widehat{K}\,{}_{\beta_c^\phi}\!\ltimes N$ is the crossed product of N by β_c^ϕ and $\widehat{(\beta_c^\phi)}$ is the dual action of β_c^ϕ, depends only on the cohomology class $[c]$ of c, up to conjugacy. Following Izumi's terminology in [13], we call $\widehat{K}\,{}_{\beta_c^\phi}\!\ltimes N$ the skew-product of N by c and denote it by $N \otimes_c L^\infty(K)$.

Theorem 3.1 states that the fixed-point algebra of an extended modular coaction always contains the centralizer of the dominant weight in question. In the next theorem, we see that it is this property that characterizes extended modular coactions among all the coactions on N.

Theorem 3.5. Let N be a type III factor and β be a coaction of a locally compact group K on N. If the fixed-point algebra N^β contains the centralizer N_ϕ of some dominant weight ϕ on N, then there exists a $c \in Z^1(F^N, K)$ such that $\beta = \beta_c^\phi$. Namely, β is an extended modular coaction.

Sketch of proof. Let $\{u(s)\}_{s\in\mathbf{R}}$ be as in Theorem 3.1. Set $w(s) := (1\otimes u(s)^*)\beta(u(s)) \in W^*(K) \otimes N$. For any $x \in N_\phi$, we have

$$\begin{aligned}(1 \otimes x)w(s) &= (1 \otimes u(s)^*)(1 \otimes \theta_s(x))\beta(u(s)) \\ &= (1 \otimes u(s)^*)\beta(\theta_s(x)u(s)) \\ &= (1 \otimes u(s)^*)\beta(u(s)x) \\ &= w(s)(1 \otimes x).\end{aligned}$$

By the Relative Commutant Theorem, we get

$$w(s) \in W^*(K) \otimes N \cap (\mathbf{C} \otimes N_\phi)' = W^*(K) \otimes Z(N_\phi).$$

Hence $w(\cdot)$ belongs to $W^*(K) \otimes L^\infty(\mathbf{R}) \otimes Z(N_\phi)$, and thus can be regraded as a bounded Borel function from $\mathbf{R} \times X_N$ into $W^*(K)$.

In the meantime, we have

$$\begin{aligned}(\Delta_K \otimes id_N)(w(s)) &= (1 \otimes 1 \otimes u(s)^*)(\Delta_K \otimes id_N)(\beta(u(s)))\\ &= (1 \otimes 1 \otimes u(s)^*)(id_{W^*(K)} \otimes \beta)(\beta(u(s)))\\ &= (1 \otimes w(s))(1 \otimes \beta(u(s)^*))(id_{W^*(K)} \otimes \beta)(\beta(u(s)))\\ &= w(s)_{23}(id_{W^*(K)} \otimes \beta)(w(s))\\ &= w(s)_{23}w(s)_{13}.\end{aligned}$$

Thus $w(s,x)$ satisfies $\Delta_K(w(s,x)) = w(s,x) \otimes w(s,x)$ for any $s \in \mathbf{R}$ and $x \in X_N$. To sum up, there is a Borel map $c : \mathbf{R} \times X_N \to K$ such that $w(s,x) = \lambda_K(c(s,x)) \in W^*(K)$. Replacing c by a suitable Borel map almost everywhere equal to c if necessary, this c serves our purpose. Namely, we obtain $\beta = \beta_c^\phi$. □

Remark. As in Remark just after Theorem 2.1, we could strengthen Theorem 3.5 to the case of a locally compact quantum group action as follows. If a locally compact quantum group $\mathbb{G}$ admits a faithful action β on a type III factor N for which there exists a dominant weight ϕ on N satisfying $N_\phi \subseteq N^\beta$, then $\mathbb{G}$ must be cocommutative.

Next we examine the structure of the skew-product $N \otimes_c L^\infty(K)$. First, we state a criterion of when $N \otimes_c L^\infty(K)$ is a factor.

Theorem 3.6. Let N be a type III factor with a dominant weight ϕ and K be a locally compact group. If $c \in Z^1(F^N, K)$ satisfies $r^*(c) = K$, then the relative commutant of $\beta_c^\phi(N)$ in the skew-product $N \otimes_c L^\infty(K)$ is trivial. In particular, $N \otimes_c L^\infty(K)$ is a factor.

Remark. A 1-cocycle $c \in Z^1(F^N, K)$ is said to have dense range in K if it satisfies $r^*(c) = K$. If c has dense range, then, by [23], K is necessarily amenable. In the meantime, suppose that (X, μ) be a properly ergodic $\mathbf{R}$-space. Then, thanks to [1], [9] and [14], for any amenable locally compact group K, there exists a Borel 1-cocycle $c : \mathbf{R} \times X \to K$ having dense range.

The next theorem can be regarded as an extension of Izumi's result in [13] to the case of noncompact locally compact groups.

Theorem 3.7. Let N be a type III factor and K be an amenable locally compact group. Suppose that $c \in Z^1(F^N, K)$ has dense range. Then the skew-product $M := N \otimes_c L^\infty(K)$ is an infinite factor. Moreover,

(1) the smooth flow of weights $\{X_M, F^M\}$ on M is given as follows:

- the flow space: $X_M = K \times X_N$:
- the flow: $F_t^M(k,x) = (c(t,x)k, F_t^N x)$ $(\forall (k,x) \in X_M = K \times X_N)$. Namely, $\{F^M\}$ is exactly the so-called the skew-product action induced by c in ergodic theory.

(2) The Connes-Takesaki module $\mathrm{mod}(\widehat{(\beta_c^\phi)})$ of the dual action $\widehat{(\beta_c^\phi)}$ is given by

$$\mathrm{mod}(\widehat{(\beta_c^\phi)_k})(g,x) = (gk^{-1}, x) \qquad (k \in K,\ (g,x) \in X_M).$$

In particular, $\mathrm{Ker}(\mathrm{mod}(\widehat{(\beta_c^\phi)})) = \{e\}$.

We close this section with a remakr that, as in [13], the actions that arise as dual actions in the manner described above can be classified up to conjugacy when the skew-product algebra M is an AFD type III factor.

4 Canonical extension of β_c^ϕ

The present and the following sections consist of announcement of a few results studied as continuation of the research done in [2] and [22].

Let A be a von Neumann algebra with a faithful normal semifinite weight ω. For any action α of a locally compact group G on A, it is always possible to extend α canonically to the action $\tilde{\alpha}$ of G on the crossed product $\mathbf{R}_{\sigma^\omega} \ltimes A$ by the modular automorphism group $\{\sigma^\omega\}$, which is called the canonical extension of α (see [12]). This remains true even if an action is replaced by a coaction in the above statement, and this fact was actually verified in [21].

It is known that the canonical extension of an extended modular automorphism is always inner. Thus we might expect that the canonical extension of an extended modular coaction is also inner. The following theorem tells us that it is indeed the case.

Theorem 4.1. Let N be a type III factor with a dominant weight ϕ and K be a locally compact group. For any $c \in Z^1(F^N, K)$, the canonical extension Θ of β_c^ϕ to $\mathbf{R}_{\sigma^\phi} \ltimes N$ is inner in the sense that there exists a unitary $V \in W^*(K) \otimes (\mathbf{R}_{\sigma^\phi} \ltimes N)$ such that

- $(\Delta_K \otimes id)(V) = V_{23}V_{13}$;
- $\Theta(a) = V(1 \otimes a)V^*$ for all $a \in \mathbf{R}_{\sigma^\phi} \ltimes N$.

5 Relation between α_c and β_c^ϕ

In this final section, we show that there is some relation between the coactions constructed in Sections 2 and 3 at least when the operator algebra in question is a particular type of von Neumann algebra.

Before we state our main theorem of this section, let us explain the setting that we shall consider below.

Let M be a type III factor with a Cartan subalgebra D. Then we can choose a discrete measured equivalence relation $\mathcal{R}$ on a standard Borel probability measure space (X, μ) and a normalized $\mathbf{T}$-valued Borel 2-cocycle σ on $\mathcal{R}$ so that $(M \supseteq D) = (W^*(\mathcal{R}, \sigma) \supseteq L^\infty(X, \mu))$ ([8]).

Assume that we are given a Borel 1-cocycle $a \in Z^1(F^M, K)$, where K is a locally compact group.

Put $\bar{M} := B(L^2(\mathbf{R})) \otimes M$, $\bar{\mathcal{R}} := \mathbf{R}^2 \times \mathcal{R}$ and $\bar{X} := \mathbf{R} \times X$. We introduce an equivalence relation on $\bar{X}$ by saying that (s, x) is equivalent to (t, y) when $(x, y) \in \mathcal{R}$. Then the graph of the equivalence relation thus defined is $\bar{\mathcal{R}}$. If we define a $\mathbf{T}$-valued 2-cocycle $\bar{\sigma}$ on $\bar{\mathcal{R}}$ by $\bar{\sigma}((s,x),(t,y),(r,z)) := \sigma(x,y,z)$, we easily find that $\bar{M}$ can be identified with $W^*(\bar{\mathcal{R}}, \bar{\sigma})$, the von Neumann algebra associated with the principal measured groupoid $\bar{\mathcal{R}}$ (with continuous orbits). Note that, when we identify $\bar{M} = B(L^2(\mathbf{R})) \otimes M$ with $W^*(\bar{\mathcal{R}}, \bar{\sigma})$ represented in a standard form on $L^2(\bar{\mathcal{R}}) = L^2(\mathbf{R}) \otimes L^2(\mathbf{R}) \otimes L^2(\mathcal{R})$, this identification is realiized by the isomorphism $T \in B(L^2(\mathbf{R})) \otimes M \longmapsto T_{13} \in W^*(\bar{\mathcal{R}}, \bar{\sigma}) = B(L^2(\mathbf{R})) \otimes \mathbf{C} \otimes M$.

Let $\mathcal{D}$ be the diagonal subset of $\mathcal{R}$ and put $\omega := \omega_{\chi_{\mathcal{D}}}|_M$. Also denote by P the nonsingular positive self-adjoint operator on $L^2(\mathbf{R})$ satisfying $P^{it} = \rho_{\mathbf{R}}(t)$ for all $t \in \mathbf{R}$. Then the weight $\phi := \mathrm{Tr}(P\,\cdot) \otimes \omega$ is a dominant weight on $\bar{M}$. We have $\bar{M}_\phi = \mathbf{R}_{\sigma^\omega} \ltimes M$.

Note that, since M is of type III, the 1-cocycle a can be viewed as an element of $Z^1(F^{\bar{M}}, K)$. Hence, thanks to Section 3, we obtain an extended modular coaction β_a^ϕ of K on $\bar{M}$ satisfying $\bar{M}^{\beta_a^\phi} \supseteq \bar{M}_\phi = \mathbf{R}_{\sigma^\omega} \ltimes M$.

In the meantime, as is well-known, the smooth flow of weights $\{X_M, F^M\}$ on M is realized as follows. With δ as the Radon-Nikodym derivative associated with $\mathcal{R}$, we introduce an equivalence relation on $\mathbf{R} \times X$ by saying that $(s,t) \sim (t,y)$ iff $(x,y) \in \mathcal{R}$ and $t = s - \log\delta(x,y)$. Let $\{\theta'_t\}_{t\in\mathbf{R}}$ be the one-parameter automorphism group of $L^\infty(\mathbf{R}\times X)$ given by $\theta'_s(f)(t,x) := f(t-s,x)$. Then define C to be the von Neumann subalgebra of all functions in $L^\infty(\mathbf{R} \times X)$ invariant under the equivalence relation introduced right above, and $\{\theta_t\}_{t\in\mathbf{R}}$ to be the one-parameter automorphism group of C obtained by restricting $\{\theta'_t\}$ to C. Then $\{X_M, F^M\}$ is taken to be a measure-theoretical realization of $\{C, \{\theta_t\}\}$:

$$C = L^\infty(X_M), \qquad \theta_t(f) = f \circ F^M_{-t}.$$

Denote by $\pi_M : \mathbf{R} \times X \to X_M$ the $\mathbf{R}$-factor map which induces the embedding of $L^\infty(X_M)$ into $L^\infty(\mathbf{R} \times X)$. Thus we have $\pi_M(t,x) = \pi_M(t - \log\delta(x,y), y)$ for any $(x,y) \in \mathcal{R}$ and $t \in \mathbf{R}$. We also have $F^M_s\pi_M(t,x) = \pi_M(t+s,x)$.

Theorem 5.1. Let M be a factor of type III having a Cartan subalgebra D. Choose a discrete measured equivalence relation $\mathcal{R}$ on a standard Borel probability measure space (X, μ) and a normalized $\mathbf{T}$-valued Borel 2-cocycle σ on $\mathcal{R}$ so that $(D \subseteq M) = (L^\infty(X) \subseteq W^*(\mathcal{R}, \sigma))$. With $\mathcal{D}$ the diagonal subset of $\mathcal{R}$, put $\omega := \omega_{\chi_{\mathcal{D}}}|_M$ and $\phi := \mathrm{Tr}(P\,\cdot) \otimes \omega$, which is a dominant weight on $\bar{M} := B(L^2(\mathbf{R})) \otimes M$, where P is the nonsingular positive self-adjoint operator on $L^2(\mathbf{R})$ satisfying $P^{it} = \rho_{\mathbf{R}}(t)$ for all $t \in \mathbf{R}$.

Let $a \in Z^1(F^M, K)$, where K is a locally compact group. Then we have the following.

(1) The Borel mapping $c_a : \mathcal{R} \to K$ defined by

$$c_a(x,y) := a(\log\delta(y,x), \pi_M(0,y)) \qquad ((x,y) \in \mathcal{R})$$

belongs to $Z^1(\mathcal{R}, K)$. Here $\pi_M : \mathbf{R} \times X \longrightarrow X_M$ is the $\mathbf{R}$-factor map introduced before.

(2) A cocycle $a' \in Z^1(F^M, K)$ is cohomologous to a if and only if $c_{a'}$ is cohomologous to c_a.

(3) The coaction β_a^ϕ of K on $\bar{M}$ is cocycle conjugate to the one α_{c_a} on M.

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