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DIGEST OF THE CARTAN PAPER BY OZAWA AND POPA

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ABSTRACT. This is a digest of the paper [OP] by S. Popa and the author. We prove that the normalizer of any diffuse amenable subalgebra of a free group factor $L(\mathbb{F}_r)$ generates an amenable von Neumann subalgebra. We also sketch the proof of the fact that if a free ergodic measure preserving action of a free group $\mathbb{F}_r$, $2 \leq r \leq \infty$, on a probability space (X, μ) is profinite then the group measure space factor $L^\infty(X) \rtimes \mathbb{F}_r$ has unique Cartan subalgebra, up to unitary conjugacy.

1. INTRODUCTION

See [OP] for the historical background. We assume every finite von Neumann algebra comes together with a distinguished faithful normal tracial state and every action on a finite von Neumann algebra is trace-preserving. A von Neumann algebra is said to be *diffuse* if it does not have a non-zero minimal projection. In this note, we state theorems and lemmas in general forms, but prove them only in the case of $Q = \mathbb{C}1$.

Theorem. *Let $\mathbb{F}_r \curvearrowright Q$ be an action of a free group on a finite von Neumann algebra. Assume $M = Q \rtimes \mathbb{F}_r$ has the CMAP. If $P \subset M$ is a diffuse amenable subalgebra and N denotes the von Neumann algebra generated by its normalizer $\mathcal{N}_M(P)$, then either N is amenable relative to Q inside M, or a non-zero corner of P can be conjugated into Q inside M.*

We mention three interesting applications of the Theorem, each corresponding to a particular choice of $\mathbb{F}_r \curvearrowright Q$. Thus, taking $Q = \mathbb{C}1$, we get:

Corollary 1. *The normalizer of any diffuse amenable subalgebra P of a free group factor $L(\mathbb{F}_r)$ generates an amenable von Neumann algebra.*

This strengthens two well known in-decomposability properties of free group factors: Voiculescu's result in [Vo], showing that $L(\mathbb{F}_r)$ has no Cartan subalgebras, and the author's result in [Oz] that the commutant in $L(\mathbb{F}_r)$ of any diffuse subalgebra must be amenable.

If we take Q to be an arbitrary finite factor with $\Lambda_{\mathrm{cb}}(Q) = 1$ and let $\mathbb{F}_r$ act trivially on it, then $M = Q \bar{\otimes} L(\mathbb{F}_r)$ has the CMAP and Theorem implies:

Corollary 2. *If Q is a II_1 factor with the CMAP then $Q \bar{\otimes} L(\mathbb{F}_r)$ does not have Cartan subalgebras.*

This shows in particular that any factor of the form $L(\mathbb{F}_r) \bar{\otimes} R$ or $L(\mathbb{F}_{r_1}) \bar{\otimes} L(\mathbb{F}_{r_2}) \bar{\otimes} \cdots$ does not have a Cartan subalgebra.

Problem. Get rid of the assumption that Q has the CMAP.

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Finally, if we take $\mathbb{F}_r \curvearrowright X$ to be a profinite measure preserving action on a probability measure space (X, μ), i.e. an action with the property that $L^\infty(X)$ is a limit of an increasing sequence of $\mathbb{F}_r$-invariant finite dimensional subalgebras Q_n of $L^\infty(X)$, then $N = L^\infty(X) \rtimes \mathbb{F}_r$ is an increasing limit of the algebras $Q_n \rtimes \mathbb{F}_r$, each one of which is an amplification of $L(\mathbb{F}_r)$. Since the CMAP behaves well to amplifications and inductive limits, it follows that N has the CMAP, so by applying Theorem and (A.1 in [Po1]) we get:

Corollary 3. *If $\mathbb{F}_r \curvearrowright X$ is a free ergodic measure preserving profinite action, then $L^\infty(X)$ is the unique Cartan subalgebra of the II_1-factor $L^\infty(X) \rtimes \mathbb{F}_r$, up to unitary conjugacy.*

2. Preliminaries

2.1. Finite von Neumann algebras. We fix conventions for (semi-)finite von Neumann algebras, but before that we note that the symbol "Lim" will be used for a state on $\ell^\infty(\mathbb{N})$, or more generally on $\ell^\infty(I)$ with I directed, which extends the ordinary limit, and that the abbreviation "u.c.p." stands for "unital completely positive." We say a map is *normal* if it is ultraweakly continuous. Whenever a *finite* von Neumann algebra M is being considered, it comes equipped with a distinguished faithful normal tracial state, denoted by τ. Any group action on a finite von Neumann algebra is assumed to preserve the tracial state τ. If $M = L(\Gamma)$ is a group von Neumann algebra, then the tracial state τ is given by $\tau(x) = \langle x\delta_1, \delta_1\rangle$ for $x \in L(\Gamma)$. Any von Neumann subalgebra $P \subset M$ is assumed to contain the unit of M and inherits the tracial state τ from M. The unique τ-preserving conditional expectation from M onto P is denoted by E_P. We denote by $\mathcal{Z}(M)$ the center of M; by $\mathcal{U}(M)$ the group of unitary elements in M; and by

$$\mathcal{N}_M(P) = \{u \in \mathcal{U}(M) : (\mathrm{Ad}\, u)(P) = P\}$$

the normalizing group of P in M, where $(\mathrm{Ad}\, u)(x) = uxu^*$. A maximal abelian von Neumann subalgebra $A \subset M$ satisfying $\mathcal{N}_M(A)'' = M$ is called a *Cartan subalgebra.* We note that if $\Gamma \curvearrowright X$ is an ergodic essentially-free probability-measure-preserving action, then $A = L^\infty(X)$ is a Cartan subalgebra in the crossed product $L^\infty(X) \rtimes \Gamma$. (See [FM].)

We refer the reader to the section IX.2 of [Ta] for the details of the following facts on noncommutative L^p-spaces. Let $\mathcal{N}$ be a semi-finite von Neumann algebra with a faithful normal semi-finite trace Tr. For $1 \leq p < \infty$, we define the L^p-norm on $\mathcal{N}$ by $\|x\|_p = \mathrm{Tr}(|x|^p)^{1/p}$. By completing $\{x \in \mathcal{N} : \|x\|_p < \infty\}$ with respect to the L^p-norm, we obtain a Banach space $L^p(\mathcal{N})$. We only need $L^1(\mathcal{N})$, $L^2(\mathcal{N})$ and $L^\infty(\mathcal{N}) = \mathcal{N}$. The trace Tr extends to a contractive linear functional on $L^1(\mathcal{N})$. We occasionally write $\widehat{x}$ for $x \in \mathcal{N}$ when viewed as an element in $L^2(\mathcal{N})$. For any $1 \leq p, q, r \leq \infty$ with $1/p + 1/q = 1/r$, there is a natural product map

$$L^p(\mathcal{N}) \times L^q(\mathcal{N}) \ni (x, y) \mapsto xy \in L^r(\mathcal{N})$$

which satisfies $\|xy\|_r \leq \|x\|_p \|y\|_q$ for any x and y. The Banach space $L^1(\mathcal{N})$ is identified with the predual of $\mathcal{N}$ under the duality $L^1(\mathcal{N}) \times \mathcal{N} \ni (\zeta, x) \mapsto \mathrm{Tr}(\zeta x) \in \mathbb{C}$. The Banach space $L^2(\mathcal{N})$ is identified with the GNS-Hilbert space of $(\mathcal{N}, \mathrm{Tr})$. Elements in $L^p(\mathcal{N})$ can be regarded as closed operators on $L^2(\mathcal{N})$ which are affiliated with $\mathcal{N}$ and hence in addition to the above-mentioned product, there are well-defined notion of positivity, square root, etc. We will use many times the generalized Powers–Størmer inequality (Theorem XI.1.2 in [Ta]):

$$\|\eta - \zeta\|_2^2 \leq \|\eta^2 - \zeta^2\|_1 \leq \|\eta + \zeta\|_2 \|\eta - \zeta\|_2 \tag{2.1}$$

for every $\eta, \zeta \in L^2(\mathcal{N})_+$. The Hilbert space $L^2(\mathcal{N})$ is an $\mathcal{N}$-bimodule such that $\langle x\xi y, \eta\rangle = \mathrm{Tr}(x\xi y\eta^*)$ for $\xi, \eta \in L^2(\mathcal{N})$ and $x, y \in \mathcal{N}$. We recall that this gives the canonical identification between the commutant $\mathcal{N}'$ of $\mathcal{N}$ in $\mathbb{B}(L^2(\mathcal{N}))$ and the opposite von Neumann algebra $\mathcal{N}^{\mathrm{op}} = \{x^{\mathrm{op}} : x \in \mathcal{N}\}$ of $\mathcal{N}$. Moreover, the opposite von Neumann algebra $\mathcal{N}^{\mathrm{op}}$ is $*$-isomorphic to the complex conjugate von Neumann algebra $\bar{\mathcal{N}} = \{\bar{x} : x \in \mathcal{N}\}$ of $\mathcal{N}$ under the $*$-isomorphism $x^{\mathrm{op}} \mapsto \bar{x}^*$.

Whenever $\mathcal{N}_0 \subset \mathcal{N}$ is a von Neumann subalgebra such that the restriction of Tr to $\mathcal{N}_0$ is still semi-finite, we identify $L^p(\mathcal{N}_0)$ with the corresponding subspace of $L^p(\mathcal{N})$. Anticipating a later use, we consider the tensor product von Neumann algebra $(\mathcal{N} \bar{\otimes} M, \mathrm{Tr} \otimes \tau)$ of a semi-finite von Neumann algebra $(\mathcal{N}, \mathrm{Tr})$ and a finite von Neumann algebra (M, τ). Then, $\mathcal{N} \cong \mathcal{N} \bar{\otimes} \mathbb{C}1 \subset \mathcal{N} \bar{\otimes} M$ and the restriction of $\mathrm{Tr} \otimes \tau$ to $\mathcal{N}$ is Tr. Moreover, the conditional expectation $\mathrm{id} \otimes \tau \colon \mathcal{N} \bar{\otimes} M \to \mathcal{N}$ extends to a contraction from $L^1(\mathcal{N} \bar{\otimes} M) \to L^1(\mathcal{N})$.

Let $Q \subset M$ be finite von Neumann algebras. Then, the conditional expectation E_Q can be viewed as the orthogonal projection e_Q from $L^2(M)$ onto $L^2(Q) \subset L^2(M)$. It satisfies $e_Q x e_Q = E_Q(x) e_Q$ for every $x \in M$. The *basic construction* $\langle M, e_Q\rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M))$ generated by M and e_Q. We note that $\langle M, e_Q\rangle$ coincides with the commutant of the right Q-action in $\mathbb{B}(L^2(M))$. In particular, if $Q = \mathbb{C}1$, then $\langle M, \varepsilon_Q\rangle = \mathbb{B}(L^2(M))$. The linear span of $\{x e_Q y : x, y \in M\}$ is an ultraweakly dense $*$-subalgebra in $\langle M, e_Q\rangle$ and the basic construction $\langle M, e_Q\rangle$ comes together with the faithful normal semi-finite trace Tr such that $\mathrm{Tr}(x e_Q y) = \tau(xy)$. See Section 1.3 in [Po1] for more information on the basic construction.

2.2. **Relative amenability.** We adapt here Connes's characterization of amenable (injective) von Neumann algebras to the relative situation. Recall that for von Neumann algebras $N \subset \mathcal{N}$, a state φ on $\mathcal{N}$ is said to be N-*central* if $\varphi \circ \mathrm{Ad}(u) = \varphi$ for any $u \in \mathcal{U}(N)$, or equivalently if $\varphi(ax) = \varphi(xa)$ for all $a \in N$ and $x \in \mathcal{N}$.

Theorem 2.1. *Let Q and N be von Neumann subalgebras of a finite von Neumann algebra M. Then, the following are equivalent.*

(1) *There exists an N-central state φ on $\langle M, e_Q\rangle$ such that $\varphi|_M = \tau$.*
(2) *There exists a conditional expectation Φ from $\langle M, e_Q\rangle$ onto N such that $\Phi|_M = E_N$.*
(3) *There exists a net (ξ_n) of unit vectors in $L^2\langle M, e_Q\rangle$ such that $\lim\langle x\xi_n, \xi_n\rangle = \tau(x)$ for every $x \in M$ and $\lim \|[u, \xi_n]\|_2 = 0$ for every $u \in N$.*

Definition 2.2. Let $Q, N \subset M$ be finite von Neumann algebras. We say N is *amenable relative to Q inside M* if any of the conditions in Theorem 2.1 holds.

We note that if N is amenable relative to an amenable von Neumann subalgebra Q, then N is amenable; and that for $M = Q \rtimes \Gamma$, the von Neumann subalgebra $L(\Gamma) \subset M$ is amenable relative to Q inside M iff Γ is amenable.

Problem. Let $Q, N \subset M$. Prove that N is amenable relative to Q inside M if and only if the following condition holds:

(4) There exists a conditional expectation Ψ from $\langle M, e_Q\rangle$ onto $N' \cap \langle M, e_Q\rangle$ such that $\Psi \circ \mathrm{Ad}(u) = \Psi$ for every $u \in \mathcal{U}(N)$.

2.3. **Intertwining subalgebras inside II_1 factors.** We extract from [Po1, Po2] some results which are needed later.

Theorem 2.3. *Let M be a finite von Neumann algebra and $P, Q \subset M$ be von Neumann subalgebras. Then, the following are equivalent.*

(1) There exists a non-zero projection $e \in \langle M, e_Q\rangle$ with $\mathrm{Tr}(e) < \infty$ such that the ultraweakly closed convex hull of $\{w^*ew : w \in \mathcal{U}(P)\}$ does not contain 0.

(2) There exist non-zero projections $p \in P$ and $q \in Q$, a normal $*$-homomorphism $\theta\colon pPp \to qQq$ and a non-zero partial isometry $v \in M$ such that

$$\forall x \in pPp \quad xv = v\theta(x)$$

and $v^*v \in \theta(pPp)' \cap qMq$, $vv^* \in p(P' \cap M)p$.

Definition 2.4. Let $P, Q \subset M$ be finite von Neumann algebras. We say that P *embeds into Q inside M* if any of the conditions in Theorem 2.3 holds.

Let $\langle M, e_Q\rangle$ be the basic construction of finite von Neumann algebras $Q \subset M$. We define $\mathbb{K}\langle M, e_Q\rangle$ to be the norm-closed linear span of $\{xe_Qy : x, y \in M\}$.

Corollary 2.5. *Let $P, Q \subset M$ be finite von Neumann algebras. Assume there exists a P-central state φ on $\langle M, e_Q\rangle$ which is normal on M and such that $\varphi(\mathbb{K}\langle M, e_Q\rangle) \neq \{0\}$. Then, P embeds into Q inside M.*

Proof in the case of $Q = \mathbb{C}1$. We restrict φ to $\mathbb{K}\langle M, e_Q\rangle = \mathbb{K}(L^2(M))$ and view it as the trace class operator h, i.e., $\varphi(x) = \mathrm{Tr}(hx)$ for $x \in \mathbb{K}(L^2(M))$. It follows that h is a non-zero compact operator which commutes with P. This implies P contains a non-zero minimal projection, i.e., P embeds into $Q = \mathbb{C}1$ inside M. Indeed, if P is diffuse, then there is a sequence (u_n) of unitary elements in P which converges to zero ultraweakly and $0 = \text{SOT-}\lim u_n h u_n^* = h$ for every compact operator h which commutes with P. □

Finally, recall that A.1 in [Po1] shows the following:

Lemma 2.6. *Let A and B be Cartan subalgebras of a type II_1-factor M. If A embeds into B inside M, then there exists $u \in \mathcal{U}(M)$ such that $uAu^* = B$.*

2.4. **The complete metric approximation property.** Let Γ be a discrete group. For a function f on Γ, we write m_f for the multiplier on $\mathbb{C}\Gamma \subset L(\Gamma)$ defined by $m_f(g) = fg$ for $g \in \mathbb{C}\Gamma$. We simply write $\|f\|_{\mathrm{cb}}$ for $\|m_f\|_{\mathrm{cb}}$ and call it the Herz-Schur norm. We denote by

$$B_2(\Gamma) = \{f : \|f\|_{\mathrm{cb}} < \infty\}$$

the Banach space of Herz-Schur multipliers. Every $f \in B_2(\Gamma)$ (or precisely m_f) extends to a normal completely bounded map on $L(\Gamma)$ such that $m_f(\lambda(s)) = f(s)\lambda(s)$. We refer the reader to [BO] for an account of Herz-Schur multipliers.

Definition 2.7. A discrete group Γ is *weakly amenable* if there exist a constant $C \geq 1$ and a net (f_n) of finitely supported functions on Γ such that $\limsup \|f_n\|_{\mathrm{cb}} \leq C$ and $f_n \to 1$ pointwise. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(\Gamma)$ of Γ is defined as the infimum of the constant C for which a net (f_n) as above exists.

We say a finite von Neumann algebra M has the *(weak*) completely bounded approximation property* if there exist a constant $C \geq 1$ and a net (ϕ_n) of normal finite-rank maps on M such that $\limsup \|\phi_n\|_{\mathrm{cb}} \leq C$ and $\phi_n \to \mathrm{id}_M$ in the point-ultraweak topology. The Cowling-Haagerup constant $\Lambda_{\mathrm{cb}}(M)$ of M is defined as the infimum of the constant C for which a net (ϕ_n) as above exists. Also, we say that M has the *(weak*) complete metric approximation property (CMAP)* if $\Lambda_{\mathrm{cb}}(M) = 1$.

Haagerup proved that $\Lambda_{\mathrm{cb}}(M) = \Lambda_{\mathrm{cb}}(\Gamma)$ (the inequality $\leq$ is trivial: just take $\phi_n = m_{f_n}$). Thus, the following results imply the CMAP of $L(\mathbb{F}_r)$. For the following, we assume $r < \infty$ for simplicity.

Theorem 2.8. *Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $l(x)$ denote the canonical word length of $x \in \mathbb{F}_r$. Then, for every $z \in \mathbb{D}$, the function $\mathbb{F}_r \ni x \mapsto z^{l(x)} \in \mathbb{C}$ belongs to $B_2(\mathbb{F}_r)$ with*

$$\|z^l\|_{\mathrm{cb}} \leq \frac{|1-z|}{1-|z|}.$$

Moreover, $\mathbb{D} \ni z \mapsto z^l \in B_2(\mathbb{F}_r)$ is holomorphic and γ^l is positive definite for $\gamma \in \mathbb{R}$.

Proof. Every element $x \in \mathbb{F}_r = \langle g_1, \ldots, g_r\rangle$ is written uniquely as a reduced word $x = g_{k_1}^{\varepsilon_1} \cdots g_{k_n}^{\varepsilon_n}$, where $n \in \mathbb{N}_0$, $1 \leq k_i \leq n$ and $\varepsilon_n = \pm 1$ such that there is no consecutive $g_k^{-1} g_k$ nor $g_k g_k^{-1}$. The length $l(x)$ of the element x is n. We identify the free group $\mathbb{F}_r$ with its Cayley graph (w.r.t. the canonical generators), which is the $2r$-regular tree. The distance between $x, y \in \mathbb{F}_r$ is given by $d(x,y) = l(xy^{-1})$. (Warning: the choice $d(x,y) = l(x^{-1}y)$ is more common, but $d(x,y) = l(xy^{-1})$ is more compatible with the left regular representation.) A *geodesic path* in $\mathbb{F}_r$ is a finite or infinite sequence $x_0, x_1, \ldots$ of points in $\mathbb{F}_r$ such that $d(x_i, x_j) = |i-j|$ for all i and j.

We fix a point ω at infinity, i.e., ω is an infinite geodesic path (starting at the unit, say). For every $x \in \mathbb{F}_r$, there exists a unique geodesic path ω_x starting at x and eventually flows into ω, i.e., $\exists k \in \mathbb{Z}$ such that $\omega_x(i) = \omega(k+i)$ for sufficiently large i.

For $z \in \mathbb{D}$, we define $\zeta_z \in \ell^\infty(\mathbb{F}_r, \ell^2(\mathbb{F}_r))$ by

$$\zeta_z(x) = \sqrt{1-z^2} \sum_{i=0}^{\infty} z^i \delta_{\omega_x(i)},$$

where $\sqrt{1-z}$ is the principal branch of the square root. The series converges absolutely in z and the function $\mathbb{D} \ni z \mapsto \zeta_z \in \ell^\infty(\mathbb{F}_r, \ell^2(\mathbb{F}_r))$ is holomorphic. One has for every $x \in \mathbb{F}_r$ that

$$\|\zeta_z(x)\|^2 = |1-z^2| \sum_{i=0}^{\infty} |z|^{2i} = \frac{|1-z^2|}{1-|z|^2} \leq \frac{|1-z|}{1-|z|}$$

and that

$$\begin{aligned} \langle \zeta_z(y), \overline{\zeta_z(x)}\rangle &= (1-z^2) \sum_{i,j=1}^{\infty} z^{i+j} \delta_{\omega_x(i), \omega_y(j)} \\ &= (1-z^2) \sum_{n=0}^{\infty} z^n \sum_{i=0}^{n} \delta_{\omega_x(i), \omega_y(n-i)} \end{aligned}$$

now we observe that for every n one has $\omega_x(i) = \omega_y(n-i)$ for at most one i and $n - d(x,y) \in 2\mathbb{N}_0$; and hence

$$\begin{aligned} &= (1-z^2) \sum_{m=0}^{\infty} z^{d(x,y)+2m} \\ &= z^{d(x,y)}. \end{aligned}$$

We define $V_z, W_z \in \mathbb{B}(\ell^2(\mathbb{F}_r), \ell^2(F_r) \otimes \ell^2(\mathbb{F}_r))$ by

$$V_z \delta_x = \delta_x \otimes \overline{\zeta_z(x)} \text{ and } W_z \delta_y = \delta_y \otimes \zeta_z(y).$$

It follows that

$$\langle V_z^*(\lambda(s)\otimes 1)W_z\delta_y,\delta_x\rangle = \langle\delta_{sy},\delta_x\rangle\langle\zeta_z(y),\overline{\zeta_z(x)}\rangle = \langle\lambda(s)\delta_y,\delta_x\rangle z^{l(s)},$$

which implies $V_z^*(\lambda(s)\otimes 1)W_z = m_{z^l}(\lambda(s))$. Moreover, if $z\in\mathbb{R}$, then $V_z = W_z$ and m_{z^l} is u.c.p. Since

$$\|V_z\|^2 = \|W_z\|^2 = \|\zeta_z\|^2_{\ell^\infty(\mathbb{F}_r,\ell^2(\mathbb{F}_r))} \le \frac{|1-z|}{1-|z|},$$

we are done. □

Theorem 2.9 (De Cannière and Haagerup). $\Lambda_{\mathrm{cb}}(\mathbb{F}_r) = 1$.

Proof. Since $\|z^t\|_{\mathrm{cb}} = 1$ for $t\in(0,1)$ and $z^t\to 1$ pointwise as $t\to 1$, it suffices to show z^t can be approximated in $B_2(\mathbb{F}_r)$ by finitely supported functions. Let $B_2^0\subset B_2(\mathbb{F}_r)$ be the norm-closure of the finitely supported functions. Since $z^l\in\ell^1(\mathbb{F}_r)$ for $|z|<(2r-1)^{-1}$, one has $z^l\in B_2^0$ for $|z|<(2r-1)^{-1}$. The function $\mathbb{D}\ni z\mapsto z^l\in B_2(\mathbb{F}_r)/B_2^0$ is holomorphic on $\mathbb{D}$ and zero for $|z|<(2r-1)^{-1}$. Hence, by uniqueness of holomorphic extensions, it is zero everywhere. □

3. Weakly compact actions

For a finite von Neumann algebra P, let J be the conjugate unitary on $L^2(P)$ defined by $J\hat{x} = \widehat{x^*}$. Then, we have $P' = JPJ$ and P' is $*$-isomorphic to the complex-conjugate von Neumann algebra $\bar{P} = \{\bar{x} : x\in P\}$ via $JxJ\mapsto\bar{x}$.

Definition 3.1. Let σ be an action of a group Γ on a finite von Neumann algebra P. We say the action σ is *profinite* if there exists an increasing net (P_n) of Γ-invariant finite-dimensional von Neumann subalgebras of P such that $P = (\bigcup P_n)''$. We say the action σ is *weakly compact* if there exists a net (η_n) of unit vectors in $L^2(P\bar{\otimes}\bar{P})_+$ such that

- $\|\eta_n - (v\otimes\bar{v})\eta_n\|_2\to 0$ for every $v\in\mathcal{U}(P)$.
- $\|\eta_n - (\sigma_g\otimes\bar{\sigma}_g)(\eta_n)\|_2\to 0$ for every $g\in\Gamma$.
- $(\tau\otimes\mathrm{id})(\eta_n^2) = 1 = (\mathrm{id}\otimes\tau)(\eta_n^2)$ for every n.

Here, we identify σ as the corresponding unitary representation on $L^2(P)$.

Proposition 3.2. *Let σ be an action of a group Γ on a finite von Neumann algebra P and consider the following conditions.*

(1) The action σ is profinite.
(2) The action σ is compact and the von Neumann algebra P is amenable.
(3) there exists a net (μ_n) of normal states on $P\bar{\otimes}\bar{P}$ such that
- $\mu_n(v\otimes\bar{v})\to 1$ *for every* $v\in\mathcal{U}(P)$.
- $\|\mu_n - \mu_n\circ(\sigma_g\otimes\bar{\sigma}_g)\|\to 0$ *for every* $g\in\Gamma$.
- $\mu_n(x\otimes 1) = \tau(x) = \mu_n(1\otimes\bar{x}^*)$ *for every n and* $x\in P$.

(4) The action σ is weakly compact.
(5) There exists a state φ on $\mathbb{B}(L^2(P))$ such that $\varphi|_P = \tau$ and $\varphi\circ\mathrm{Ad}\,u = \varphi$ for all $u\in\mathcal{U}(P)\cup\sigma(\Gamma)$.

Then, one has (1) ⇒ (2) ⇒ (3) ⇔ (4) ⇔ (5).

We only prove (1) ⇒ (3) ⇔ (4) ⇒ (5).

Proof. (1) $\Rightarrow$ (3): Suppose that σ is profinite and take a net (P_n) as in the definition. Let n be fixed. We note that the τ-preserving conditional expectation E_n from P onto P_n is Γ-equivariant: $E_n \circ \mathrm{Ad}(u) = \mathrm{Ad}(u) \circ E_n$ for every $u \in \sigma(\Gamma)$. We define a state on the algebraic tensor product $P \otimes \bar{P}$ by

$$\mu_n(\sum_k a_k \otimes \bar{b}_k) = \sum_k \tau(E_n(a_k)b_k^*) = \langle E_n(a_k)\hat{1}b_k^*, \hat{1}\rangle_{L^2(P)}.$$

Since P_n is finite-dimensional, μ_n is contractive w.r.t. the minimal tensor norm and moreover μ_n extends to a normal state on $P \bar{\otimes} \bar{P}$. It is not difficult to see that (μ_n) satisfies the conditions in (3).

(3) $\Leftrightarrow$ (4): This follows from the Powers-Størmer inequality (2.1) and the inequality (3.1) below.

(4) $\Rightarrow$ (5): The state φ on $\mathbb{B}(L^2(P))$, defined by

$$\varphi(x) = \mathrm{Lim}\langle (x \otimes 1)\eta_n, \eta_n\rangle,$$

satisfies the condition (5). Indeed,

$$(\varphi \circ \mathrm{Ad}(u))(x) = \mathrm{Lim}\langle (x \otimes 1)(u \otimes \bar{u})\eta_n, (u \otimes \bar{u})\eta_n\rangle = \varphi(x)$$

for every $x \in \mathbb{B}(L^2(P))$ and $u \in \mathcal{U}(P) \cup \sigma(\Gamma)$. □

The following is the main theorem of this section.

Theorem 3.3. *Let M be a finite von Neumann algebra with the CMAP and $P \subset M$ be a von Neumann subalgebra. Then, the conjugation action of $\mathcal{N}_M(P)$ on P is weakly compact.*

We need the following consequence of Connes's theorem (Theorem 2.1).

Lemma 3.4. *Let M be a finite von Neumann algebra, $P \subset M$ be an amenable von Neumann subalgebra and $u \in \mathcal{N}_M(P)$. Then, the von Neumann algebra Q generated by P and u is amenable.*

Proof. Since P is injective, the τ-preserving conditional expectation E_P from M onto P extends to a u.c.p. map $\tilde{E}_P$ from $\mathbb{B}(L^2(M))$ onto P. We note that $\tilde{E}_P$ is a conditional expectation: $\tilde{E}_P(axb) = a\tilde{E}_P(x)b$ for every $a, b \in P$ and $x \in \mathbb{B}(L^2(M))$. We define a state φ on $\mathbb{B}(L^2(M))$ by

$$\varphi(x) = \mathrm{Lim}_n \frac{1}{n}\sum_{k=0}^{n-1} \tau(\tilde{E}_P(u^k x u^{-k})).$$

It is not hard to check that $\varphi|_M = \tau$, $\varphi \circ \mathrm{Ad}(u) = \varphi$ and $\varphi \circ \mathrm{Ad}(v) = \varphi$ for every $v \in \mathcal{U}(P)$. It follows that φ is a Q-central state with $\varphi|_Q = \tau$. By Connes's theorem, this implies that Q is amenable. □

Proof of Theorem 3.3. First we note the following general fact: Let ω be a state on a C*-algebra N and $u \in \mathcal{U}(N)$. We define $\omega_u(x) = \omega(xu^*)$ for $x \in N$. Then, one has

$$\max\{\|\omega - \omega_u\|, \|\omega - \omega \circ \mathrm{Ad}(u)\|\} \le 2\sqrt{2|1 - \omega(u)|}. \tag{3.1}$$

Indeed, one has $\|\xi_\omega - u^*\xi_\omega\|^2 = 2(1 - \Re\omega(u)) \le 2|1 - \omega(u)|$, where ξ_ω is the GNS-vector for ω.

Let (ϕ_n) be a net of normal finite rank maps on M such that $\limsup \|\phi_n\|_{\mathrm{cb}} \le 1$ and $\|x - \phi_n(x)\|_2 \to 0$ for all $x \in M$. We observe that the net $(\tau \circ \phi_n)$ converges to τ weakly in M_*. Hence by the Hahn-Banach separation theorem, one may assume, by passing to convex combinations, that $\|\tau-\tau\circ\phi_n\|\to 0$. Let μ be the $*$-representation of the algebraic tensor product $M\otimes\bar{M}$ on $L^2(M)$ defined by

$$\mu(\sum_k a_k\otimes\bar{b}_k)\xi=\sum_k a_k\xi b_k^*.$$

We define a linear functional μ_n on $M\otimes\bar{M}$ by

$$\mu_n(\sum_k a_k\otimes\bar{b}_k)=\langle\mu(\sum_k\phi_n(a_k)\otimes\bar{b}_k)\hat{1},\hat{1}\rangle_{L^2(M)}=\tau(\sum_k\phi_n(a_k)b_k^*).$$

Since ϕ_n is normal and of finite rank, μ_n extends to a normal linear functional on $M\bar{\otimes}\bar{M}$, which is still denoted by μ_n. For an amenable von Neumann subalgebra $Q\subset M$, we denote by μ_n^Q the restriction of μ_n to $Q\bar{\otimes}\bar{Q}$. Since Q is amenable, the $*$-representation μ is continuous with respect to the spatial tensor norm on $Q\otimes\bar{Q}$ and hence $\|\mu_n^Q\|\le\|\phi_n\|_{\mathrm{cb}}$. We denote $\omega_n^Q=\|\mu_n^Q\|^{-1}|\mu_n^Q|$. Since $\limsup\|\mu_n^Q\|\le 1$ and $\lim\mu_n^Q(1\otimes 1)=1$, the inequality (3.1), applied to ω_n^Q, implies that

$$\limsup_n\|\mu_n^Q-\omega_n^Q\|=0. \tag{3.2}$$

Now, consider the case $Q=P$. By (3.2), one has

$$\lim_n\omega_n^P(v\otimes\bar{v})=\lim_n\mu_n^P(v\otimes\bar{v})=\lim_n\tau(\phi_n(v)v^*)=1 \tag{3.3}$$

for any $v\in\mathcal{U}(P)$. Now, let $u\in\mathcal{N}(P)$ and consider the case $Q=\langle P,u\rangle$, which is amenable by Lemma 3.4. Since $\mu_n^{\langle P,u\rangle}(u\otimes\bar{u})=\tau(\phi_n(u)u^*)\to 1$, one has

$$\limsup_n\|\mu_n^{\langle P,u\rangle}-\mu_n^{\langle P,u\rangle}\circ\mathrm{Ad}(u\otimes\bar{u})\|=0 \tag{3.4}$$

by (3.1) and (3.2). But since $(\mu_n^{\langle P,u\rangle}\circ\mathrm{Ad}(u\otimes\bar{u}))|_{P\bar{\otimes}\bar{P}}=\mu_n^P\circ\mathrm{Ad}(u\otimes\bar{u})$, one has

$$\limsup_n\|\omega_n^P-\omega_n^P\circ\mathrm{Ad}(u\otimes\bar{u})\|=0 \tag{3.5}$$

by (3.2) and (3.4). Therefore, (ω_n^P) satisfies Proposition 3.2.(3). □

4. Main Results

Theorem 4.1. *Let $M=Q\rtimes\mathbb{F}_r$ be the crossed product of a finite von Neumann algebra Q and the free group $\mathbb{F}_r$ of rank $2\le r\le\infty$ acting on Q (need not be ergodic nor free). Let $P\subset M$ and assume that the conjugation action of $\mathcal{N}_N(P)$ on P is weakly compact. (This is automatic if M has the CMAP.) Then, either P embeds into Q inside M, or $\mathcal{N}_M(P)''$ is amenable relative to Q inside M.*

For the proof of Theorem 4.1, recall from [Po3] the construction of 1-parameter automorphisms α_t ("malleable deformation") of $L(\mathbb{F}_r*\tilde{\mathbb{F}}_r)$, where $\tilde{\mathbb{F}}_r$ is a copy of $\mathbb{F}_r$. Let $a_1,a_2,\ldots$ (resp. $b_1,b_2,\ldots$) be the standard generators of $\mathbb{F}_r$ (resp. $\tilde{\mathbb{F}}_r$) viewed as unitary elements in $L(\mathbb{F}_r*\tilde{\mathbb{F}}_r)$. Let $b_k^t=\exp(t\log b_k)$, where where log is the principal branch of the complex logarithm ($\sqrt{-1}\log z\in(-\pi,\pi]$ for $z\in\mathbb{C}$ with $|z|=1$). The $*$-automorphism α_t is defined by $\alpha_t(a_k)=a_kb_k^t$ and $\alpha_t(b_k)=b_k$.

In this paper, we adapt this construction to $\mathbb{F}_r\curvearrowright Q$ and $M=Q\rtimes\mathbb{F}_r$. We extend the action $\mathbb{F}_r\curvearrowright Q$ to that of $\mathbb{F}_r*\tilde{\mathbb{F}}_r$, by letting $\tilde{\mathbb{F}}_r$ act trivially on Q. We consider

$$\tilde{M}=Q\rtimes(\mathbb{F}_r*\tilde{\mathbb{F}}_r)=M*_Q(Q\bar{\otimes}L(\tilde{\mathbb{F}}_r))$$

and redefine the $*$-homomorphism $\alpha_t\colon M \to \tilde{M}$ by $\alpha_t(x) = x$ for $x \in Q$ and $\alpha_t(a_k) = a_k b_k^t$ for each k. (We can define α_t on $\tilde{M}$, but we do not need it.)

Let

$$\gamma(t) = \tau(b_k^t) = \frac{1}{2}\int_{-1}^{1} \exp(t\pi\sqrt{-1}s)\, ds = \frac{\sin(t\pi)}{t\pi}$$

and $\phi_{\gamma(t)}\colon L(\mathbb{F}_r) \to L(\mathbb{F}_r)$ be the Haagerup multiplier (Theorem 2.8) associated with the positive type function $g \mapsto \gamma(t)^{l(g)}$ on $\mathbb{F}_r$. We may extend $\phi_{\gamma(t)}$ to M by defining $\phi_{\gamma(t)}(x\lambda(g)) = x\phi_{\gamma(t)}(\lambda(g))$ for $x \in Q$ and $\lambda(g) \in L(\mathbb{F}_r)$. We relate α_t and $\phi_{\gamma(t)}$ as follows.

Lemma 4.2. *One has $E_M \circ \alpha_t = \phi_{\gamma(t)}$.*

Proof by Example. Let $Q = \mathbb{C}1$ and $x = a_1 a_1 a_2^{-1}$. Then, $\alpha_t(x) = a_1 b_1^t a_1 b_1^t b_2^{-t} a_2^{-1}$. Since the von Neumann algebras $W^*(a_1), \ldots, W^*(b_1), \ldots$ are mutually free, one has

$$(E_M \circ \alpha_t)(x) = a_1\tau(b_1^t)a_1\tau(b_1^t b_2^{-t})a_2^{-1} = \tau(b_1^t)\tau(b_1^t)\tau(b_2^{-t})a_1 a_1 a_2^{-1} = \gamma(t)^3 x.$$

□

In particular, the τ-preserving u.c.p. map $E_N \circ \alpha_t$ on M is compact as an operator on $L^2(M)$. (Assume $r < \infty$ for simplicity.)

Let $Q \subset M \subset \tilde{M}$ be as above, and consider the basic construction $\langle M, e_Q\rangle$ of $(Q \subset M)$. Then, $L^2\langle M, e_Q\rangle$ is naturally an M-bimodule.

Lemma 4.3. *Let $Q \subset M \subset \tilde{M}$ be as above. Then, $L^2(\tilde{M}) \ominus L^2(M)$ is isomorphic as an M-bimodule to a multiple of $L^2\langle M, e_Q\rangle$.*

Proof in the case of $Q = \mathbb{C}1$. Let X be the subset of $\mathbb{F}_r * \tilde{\mathbb{F}}_r$ consisting those elements whose initial and last letters in the reduced forms come from $\tilde{\mathbb{F}}_r$. It follows that every element of $\mathbb{F}_r * \tilde{\mathbb{F}}_r \setminus \mathbb{F}_r$ is uniquely written as sxt, where $s, t \in \mathbb{F}_r$ and $x \in X$. Now, one has

$$L^2(\tilde{M}) \ominus L^2(M) = \ell^2(\mathbb{F}_r * \tilde{\mathbb{F}}_r \setminus \mathbb{F}_r) \cong \ell^2(\mathbb{F}_r) \otimes \ell^2(X) \otimes \ell^2(\mathbb{F}_r)$$

as $L(\mathbb{F}_r)$-bimodule. □

In particular, when $Q = \mathbb{C}1$ and $M = L(\mathbb{F}_r)$, the representation of $L(\mathbb{F}_r) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$ on $L^2(\tilde{M}) \ominus L^2(M)$, defined by

$$(a \otimes b^{\mathrm{op}})(\xi) = a\xi b$$

for $a, b \in L(\mathbb{F}_r)$ and $\xi \in L^2(\tilde{M}) \ominus L^2(M)$, naturally extends to a representation of $\mathbb{B}(\ell^2(\mathbb{F}_r)) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$.

Proof of Theorem 4.1 in the case $Q = \mathbb{C}1$. Let $M = L(\mathbb{F}_r)$ and $P \subset M$ be a diffuse amenable von Neumann subalgebra. We will prove that $\mathcal{N}_M(P)''$ is amenable. Let a finite subset $F \subset \mathcal{N}_M(P)$ and $\varepsilon > 0$ be given.

We choose and fix $t > 0$ such that $\alpha = \alpha_t$ satisfies $\|u - \alpha(u)\|_2 < \varepsilon/4$ for every $u \in F$. Let (η_n) be the net of unit vectors in $L^2(P \bar{\otimes} \bar{P})_+$ satisfying the conditions in Definition 3.1. Let v_α be α viewed as an isometry from $L^2(M)$ into $L^2(\tilde{M})$ and consider $\eta_n^\alpha = (v_\alpha \otimes 1)\eta_n$. We note that

$$\langle (x \otimes 1)\eta_n^\alpha, \eta_n^\alpha\rangle = \tau(\alpha^{-1}(E_{\alpha(M)}(x))) = \tau(x) \tag{4.1}$$

for every n and $x \in \tilde{M}$. It follows that

$$\limsup_n \|[u \otimes \bar{u}, \eta_n^\alpha]\|_2 < \varepsilon/2 + \limsup_n \|[\alpha(u) \otimes \bar{u}, \eta_n^\alpha]\|_2 = \varepsilon/2 \tag{4.2}$$

for every $u \in F$. Let e_M be the orthogonal projection from $L^2(\tilde{M})$ onto $L^2(M)$ and

$$\zeta_n = ((1 - e_M) \otimes 1)\eta_n^\alpha \in \left(L^2(\tilde{M}) \ominus L^2(M)\right) \bar{\otimes} L^2(\bar{P}).$$

We note $T = e_M v_\alpha \in \mathbb{K}(L^2(M))$, by Lemma 4.2. Since η_n is approximately P-central, Corollary 2.5 implies

$$\lim_n \|\eta_n^\alpha - \zeta_n\|_2 = \lim_n \|(T \otimes 1)\eta_n\|_2 = 0. \tag{4.3}$$

By Lemma 4.3, the representation σ of $L(\mathbb{F}_r) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$ on $L^2(\tilde{M}) \ominus L^2(M)$ naturally extends to $\mathbb{B}(\ell^2(\mathbb{F}_r)) \otimes L(\mathbb{F}_r)^{\mathrm{op}}$. Now, we define a state $\varphi_{F,\varepsilon}$ on $\mathbb{B}(\ell^2(\mathbb{F}_r))$ by

$$\varphi_{F,\varepsilon}(x) = \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes 1) \otimes 1)\zeta_n, \zeta_n \rangle$$

We note that if $x \in L(\mathbb{F}_r)$, then (4.1) and (4.3) imply

$$\varphi_{F,\varepsilon}(x) = \operatorname*{Lim}_{n\to\infty} \langle (x \otimes 1)\zeta_n, \zeta_n \rangle = \tau(x).$$

Moreover, if $u \in F$, then (4.2) and (4.3) imply

$$\begin{aligned}
\varphi_{F,\varepsilon}(u^* x u) &= \operatorname*{Lim}_{n\to\infty} \langle (\sigma(u^* x u \otimes 1) \otimes \bar{u}^* \bar{u})\zeta_n, \zeta_n \rangle \\
&= \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes 1) \otimes 1)(u \otimes \bar{u})\zeta_n, (u \otimes \bar{u})\zeta_n \rangle \\
&\approx_\varepsilon \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes 1) \otimes 1)\big(\zeta_n(u \otimes \bar{u})\big), \big(\zeta_n(u \otimes \bar{u})\big) \rangle \\
&= \operatorname*{Lim}_{n\to\infty} \langle (\sigma(x \otimes (u^{\mathrm{op}})^* u^{\mathrm{op}}) \otimes \bar{u}^* \bar{u})\zeta_n, \zeta_n \rangle \\
&= \varphi_{F,\varepsilon}(x)
\end{aligned}$$

for all contractions $x \in \mathbb{B}((\ell^2(\mathbb{F}_r))$. It follows that the state φ on $\mathbb{B}((\ell^2(\mathbb{F}_r))$, defined by

$$\varphi(x) = \operatorname*{Lim}_{F,\varepsilon} \varphi_{F,\varepsilon}(x),$$

satisfies $\varphi|_M = \tau$ and $\varphi \circ \mathrm{Ad}(u) = \varphi$ for all $u \in \mathcal{N}_M(P)$. It follows that $\varphi(ax) = \varphi(xa)$ for all a in the linear span of $\mathcal{N}_M(P)$ and $x \in \mathbb{B}(\ell^2(\mathbb{F}_r))$. By Cauchy-Schwarz inequality, this implies that $\varphi(ax) = \varphi(xa)$ for all $a \in \mathcal{N}_M(P)''$ and $x \in \mathbb{B}(\ell^2(\mathbb{F}_r))$, i.e., φ is an $\mathcal{N}_M(P)''$-central state such that $\varphi|_{\mathcal{N}_M(P)''} = \tau$. This implies that $\mathcal{N}_M(P)''$ is amenable (Theorem 2.1). □

References

[BO] N. Brown and N. Ozawa, *C^*-algebras and Finite-Dimensional Approximations.* Graduate Studies in Mathematics, 88. American Mathematical Society, Providence, RI, 2008.

[FM] J. Feldman and C. C. Moore, Ergodic equivalence relations, cohomology, and von Neumann algebras. I and II. *Trans. Amer. Math. Soc.* **234** (1977), 289–359.

[Ha] U. Haagerup, An example of a nonnuclear C*-algebra, which has the metric approximation property, *Invent. Math.* **50** (1978/79), 279–293.

[Oz] N. Ozawa, Solid von Neumann algebras. *Acta Math.* **192** (2004), 111–117.

[OP] N. Ozawa and S. Popa; On a class of II_1 factors with at most one Cartan subalgebra. *Preprint.*

[Po1] S. Popa, On a class of type II_1 factors with Betti numbers invariants. *Ann. Math.* **163** (2006), 809–899.

[Po2] S. Popa, Strong rigidity of II_1 factors arising from malleable actions of w-Rigid groups I. *Invent. Math.* **165** (2006), 369–408.

[Po3] S. Popa, On Ozawa's property for free group factors. *Int. Math. Res. Not.* To appear. arXiv:math.OA/0607561.

[Ta] M. Takesaki, *Theory of operator algebras. II.* Encyclopaedia of Mathematical Sciences, 125. Operator Algebras and Non-commutative Geometry, 6. Springer-Verlag, Berlin, 2003.

[Vo] D. Voiculescu, The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras. *Geom. Funct. Anal.* **6** (1996), 172–199.

5. (おまけ) Gaboriau's theorem after Lück, Sauer and Thom

5.1. Background in homological algebra. Throughout this section, R is a unital ring and V is a left R-module.

Definition 5.1. A *complex* $\mathbf{V}$ consists of sequences of modules and morphisms

$$\mathbf{V}: \qquad \cdots \longrightarrow V_{n+1} \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \longrightarrow \cdots$$

such that $\partial_n \circ \partial_{n+1} = 0$ for all n. The n-th homology module of $\mathbf{V}$ is defined to be $H_n(\mathbf{V}) = \ker \partial_n / \operatorname{ran} \partial_{n+1}$. The complex $\mathbf{V}$ is *exact* if $H_n(\mathbf{V}) = 0$ for all n.

A *morphism* $\boldsymbol{\varphi}\colon \mathbf{V} \to \mathbf{W}$ consists of a sequence of morphisms $\varphi_n\colon V_n \to W_n$ such that $\varphi_n \circ \partial_{n+1} = \partial'_{n+1} \circ \varphi_{n+1}$ for all n. Since $\varphi_n(\operatorname{ran} \partial_{n+1}) \subset \operatorname{ran} \partial'_{n+1}$ and $\varphi_n(\ker \partial_{n+1}) \subset \ker \partial'_{n+1}$, the morphism $\boldsymbol{\varphi}$ induces morphisms $\varphi_{*,n}\colon H_n(\mathbf{V}) \to H_n(\mathbf{W})$.

A morphism $\boldsymbol{\varphi}\colon \mathbf{V} \to \mathbf{W}$ is *null-homotopic* if there is a sequence of morphisms $h_n\colon V_n \to W_{n+1}$ such that $\varphi_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$:

$$\begin{tikzcd}
\cdots \arrow[r] & V_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "\varphi_{n+1}"'] \arrow[dl] & V_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] \arrow[dl, "h_n"'] & V_{n-1} \arrow[r] \arrow[d, "\varphi_{n-1}"'] \arrow[dl, "h_{n-1}"'] & \cdots \\
\cdots \arrow[r] & W_{n+1} \arrow[r, "\partial'_{n+1}"] & W_n \arrow[r, "\partial'_n"] & W_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Morphisms $\boldsymbol{\varphi}$, $\boldsymbol{\psi}\colon \mathbf{V} \to \mathbf{W}$ are *homotopic* if $\boldsymbol{\varphi} - \boldsymbol{\psi}$ is null-homotopic.

Lemma 5.2. *If $\boldsymbol{\varphi}$ and $\boldsymbol{\psi}$ are homotopic, then $\varphi_{*,n} = \psi_{*,n}$ for all n.*

Proof. If $\boldsymbol{\varphi}$ is null-homotopic, then $\varphi_n(\ker \partial_n) = (\partial'_{n+1} \circ h_n)(\ker \partial_n) \subset \operatorname{ran} \partial'_{n+1}$ and hence $\varphi_{*,n} = 0$. The general case follows from this. □

Theorem 5.3. *Let complexes $\mathbf{V}$, $\mathbf{W}$ and a morphism $\varphi\colon V \to W$ be given*

$$\begin{tikzcd}
\mathbf{V}: & \cdots \arrow[r] & V_n \arrow[r, "\partial_n"] & V_{n-1} \arrow[r] & \cdots \arrow[r] & V_0 \arrow[r, two heads] & V \arrow[d, "\varphi"] \\
\mathbf{W}: & \cdots \arrow[r] & W_n \arrow[r, "\partial'_n"] & W_{n-1} \arrow[r] & \cdots \arrow[r] & W_0 \arrow[r, two heads] & W
\end{tikzcd}$$

such that every V_n ($n \geq 0$) is projective and $\mathbf{W}$ is exact. Then, there exists a morphism $\boldsymbol{\varphi}\colon \mathbf{V} \to \mathbf{W}$ which extends φ. Moreover, the extension $\boldsymbol{\varphi}$ is unique up to homotopy.

Proof. (Existence.) We proceed by induction. Let $\varphi_{-1} = \varphi$ and $\varphi_{-2} = 0$, and suppose we have constructed $\varphi_{-2}, \cdots, \varphi_{n-1}$ satisfying $\varphi_{m-2} \circ \partial_{m-1} = \partial'_{m-1} \circ \varphi_{m-1}$ for $m \leq n$:

$$\begin{tikzcd}
V_n \arrow[r, "\partial_n"] \arrow[d, dashed] & V_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\varphi_{n-1}"'] & V_{n-2} \arrow[d, "\varphi_{n-2}"'] \\
W_n \arrow[r, "\partial'_n"] & W_{n-1} \arrow[r, "\partial'_{n-1}"] & W_{n-2}
\end{tikzcd}$$

Since $\partial'_{n-1} \circ \varphi_{n-1} \circ \partial_n = \varphi_{n-2} \circ \partial_{n-1} \circ \partial_n = 0$, one has $\operatorname{ran} \varphi_{n-1} \circ \partial_n \subset \operatorname{ran} \partial'_n$ by exactness. Since V_n is projective, there is a morphism $\varphi_n\colon V_n \to W_n$ which lifts $\varphi_{n-1} \circ \partial_n$ through ∂'_n, i.e., $\partial'_n \circ \varphi_n = \varphi_{n-1} \circ \partial_n$.

(Uniqueness.) It suffices to show that any extension φ of $\varphi = 0$ is null-homotopic. Let $h_{-1} = 0$ and $h_{-2} = 0$, and suppose we have constructed $h_{-2}, \dots, h_{n-1}$ satisfying $\varphi_{m-1} = \partial'_m \circ h_{m-1} + h_{m-2} \circ \partial_{m-1}$ for $m \leq n$:

$$\begin{tikzcd}
 & V_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"] \arrow[dl, dashed] & V_{n-1} \arrow[r, "\partial_{n-1}"] \arrow[d, "\varphi_{n-1}"] \arrow[dl, "h_{n-1}"'] & V_{n-2} \arrow[dl, "h_{n-2}"'] \\
W_{n+1} \arrow[r, "\partial'_{n+1}"] & W_n \arrow[r, "\partial'_n"] & W_{n-1} &
\end{tikzcd}$$

Since $\partial'_n \circ \varphi_n = \varphi_{n-1} \circ \partial_n = (\partial'_n \circ h_{n-1} + h_{n-2} \circ \partial_{n-1}) \circ \partial_n = \partial'_n \circ h_{n-1} \circ \partial_n$, one has $\operatorname{ran}(\varphi_n - h_{n-1} \circ \partial_n) \subset \operatorname{ran} \partial'_{n+1}$ by exactness. Since V_n is projective, there is a morphism $h_n \colon V_n \to W_{n+1}$ such that $\partial'_{n+1} \circ h_n = \varphi_n - h_{n-1} \circ \partial_n$. $\square$

Definition 5.4. For a module V, a *projective resolution* of V is an exact complex

$$\mathbb{V}: \qquad \cdots \longrightarrow V_n \longrightarrow \cdots \longrightarrow V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} V \longrightarrow 0$$

with all V_n $(n \geq 0)$ projective.

Definition 5.5. For a right R-module M and a left R-module V, define

$$\operatorname{Tor}_n^R(M, V) = H_n(M \otimes_R \mathbb{V}_{\geq 0}),$$

where $\mathbb{V}$ is any projective resolution of V and $M \otimes_R \mathbb{V}_{\geq 0}$ is the complex

$$M \otimes_R \mathbb{V}_{\geq 0}: \qquad \cdots \longrightarrow M \otimes_R V_n \longrightarrow \cdots \longrightarrow M \otimes_R V_1 \xrightarrow{\partial_1} M \otimes_R V_0 \longrightarrow 0.$$

Note that $M \otimes_R \mathbb{V}_{\geq 0}$ is given by omitting the term $M \otimes_R V$ from $M \otimes_R \mathbb{V}$.

Remark 5.6. Every module V has a projective (or even free) resolution, and the projective resolution is unique up to homotopy. It follows that the complex $M \otimes_R \mathbb{V}_{\geq 0}$ used to define $\operatorname{Tor}_\bullet^R(M, V)$ is also unique up to homotopy and hence $\operatorname{Tor}_\bullet^R(M, V)$ does not depends on the choice of a projective resolution of V.

We recall that the relative tensor product $M \otimes_R V$ is defined to be the $\mathbb{Z}$-module generated by $\{a \otimes \xi : a \in M,\ \xi \in V\}$ and factored out by the relations $a \otimes \xi + b \otimes \xi - (a+b) \otimes \xi$, $a \otimes \xi + a \otimes \eta - a \otimes (\xi + \eta)$, and $ar \otimes \xi - a \otimes r\xi$. If M is an S-R-module, then $M \otimes_R V$ is naturally a left S-module. We note that the relative tensor product operation $\otimes_R$ is associative and distributive w.r.t. a direct sum.

Examples. $M \otimes_R R = M$ and $R \otimes_R V = V$.

The module $\operatorname{Tor}_n^R(M, V)$ can be non-zero because $M \otimes_R \cdot$ needs not be a short exact functor. Namely, $V_2 \rightarrowtail V_1$ does not imply $M \otimes_R V_2 \rightarrowtail M \otimes_R V_1$. (The symbol $\rightarrowtail$ is used for injection.) However the functor $M \otimes_R \cdot$ is always right exact.

Lemma 5.7 (Right exactness). *Let M be arbitrary. If $V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \to 0$ is exact, then $M \otimes_R V_2 \xrightarrow{\mathrm{id} \otimes \partial_2} M \otimes_R V_1 \xrightarrow{\mathrm{id} \otimes \partial_1} M \otimes_R V_0 \to 0$ is exact.*

Proof. Exactness at $M \otimes_R V_0$ is clear. Since $(\mathrm{id} \otimes \partial_1) \circ (\mathrm{id} \otimes \partial_2) = \mathrm{id} \otimes (\partial_1 \circ \partial_2) = 0$, the morphism $\mathrm{id} \otimes \partial_1$ induces a morphism $\tilde{\partial}_1 \colon M \otimes_R V_1 / \operatorname{ran}(\mathrm{id} \otimes \partial_2) \twoheadrightarrow M \otimes_R V_0$. It is left to show that $\tilde{\partial}_1$ is injective. For this, it suffices to construct the left inverse σ of $\tilde{\partial}_1$: For $\sum a_i \otimes \xi_i \in M \otimes_R V_0$, define $\sigma(\sum a_i \otimes \xi_i) = \sum a_i \otimes \tilde{\xi}_i + \operatorname{ran}(\mathrm{id} \otimes \partial_2)$, where $\tilde{\xi}_i \in V_1$ is any lift of ξ_i. Then, σ is a well-defined morphism with $\sigma \circ \tilde{\partial}_1 = \mathrm{id}$. $\square$

Definition 5.8. A right S-module N is *flat* if $N \otimes_S \cdot$ is an exact functor.

Note that free modules and projective modules are flat.

Lemma 5.9. *For a right S-module N, the following are equivalent.*

(1) N is flat.
(2) $\ker(\mathrm{id}\otimes\varphi) = N\otimes_S \ker\varphi$ for any morphism $\varphi\colon W\to V$.
(3) $H_\bullet(N\otimes_S \mathbb{V}) = N\otimes_S H_\bullet(\mathbb{V})$ for any complex $\mathbb{V}$ of S-modules.
(4) $N\otimes_S V \rightarrowtail N\otimes_S F$ for every f.g. modules $V\subset F$ with F free.

In particular, if N is flat, then for any S-R-module M and any left R-module V,

$$N\otimes_S \operatorname{Tor}^R_\bullet(M,V) = \operatorname{Tor}^R_\bullet(N\otimes_S M, V).$$

Proof. It is routine to check the equivalence of the conditions (1)–(3). (Use right exactness.) We only prove the implication (4)⇒(1). We first observe that the f.g. assumption on V and F can be dropped by continuity of a tensor product w.r.t. inductive limits. Let $\iota\colon W_1 \hookrightarrow W_2$ be given. We will show $N\otimes_S W_1 \rightarrowtail N\otimes_S W_2$. Take a free S-module F and a surjection $\pi\colon F\twoheadrightarrow W_2$, and set $V=\ker\pi$. Then, we have a commuting diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \ker(\mathrm{id}\otimes\iota) \arrow[d] & \\
N\otimes_S V \arrow[r] \arrow[d, equal] & N\otimes_S \pi^{-1}(W_1) \arrow[r] \arrow[d] & N\otimes_S W_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & N\otimes_S V \arrow[r] \arrow[d] & N\otimes_S F \arrow[r] & N\otimes_S W_2 \\
 & 0 & &
\end{tikzcd}$$

which is exact everywhere. By Snake Lemma, one has $\ker(\mathrm{id}\otimes\iota)=0$. □

For the later purpose, we need the following. A (full) subcategory $\mathcal{D}$ of modules is a *Serre subcategory* if for every short exact sequence $0\to V_2\to V_1\to V_0\to 0$, one has $V_1\in\mathcal{D} \Leftrightarrow V_0, V_2\in\mathcal{D}$. A morphism $\varphi\colon V\to W$ is an *isomorphism modulo $\mathcal{D}$* if both $\ker\varphi$ and $\operatorname{coker}\varphi = V/\operatorname{ran}\varphi$ are in $\mathcal{D}$.

Lemma 5.10. *Let $\mathcal{D}$ be a Serre subcategory. Let $\mathbb{V}$ and $\mathbb{W}$ be complexes of modules and $\varphi\colon\mathbb{V}\to\mathbb{W}$ be a morphism consisting of isomorphisms modulo $\mathcal{D}$. Then all $\varphi_{*,\bullet}\colon H_\bullet(\mathbb{V})\to H_\bullet(\mathbb{W})$ are also isomorphisms modulo $\mathcal{D}$.*

Proof. Consider the following commuting exact diagram:

$$\begin{tikzcd}
0 \arrow[r] & \ker\partial_n \arrow[r, hook] \arrow[d, "\varphi_n"'] & V_n \arrow[r, "\partial_n"] \arrow[d, "\varphi_n"'] & \operatorname{ran}\partial_n \arrow[r] \arrow[d, "\varphi_{n-1}"'] & 0 \\
0 \arrow[r] & \ker\partial'_n \arrow[r, hook] & W_n \arrow[r, "\partial'_n"] & \operatorname{ran}\partial'_n \arrow[r] & 0
\end{tikzcd}$$

Since φ_n is an isomorphism modulo $\mathcal{D}$ and $\ker\varphi_{n-1}\cap\operatorname{ran}\partial_n$ is in $\mathcal{D}$, Snake Lemma implies that other two column morphisms are also isomorphisms modulo $\mathcal{D}$. Now, applying Snake

Lemma again to the following commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{ran}\partial_{n+1} \arrow[r, hook] \arrow[d, "\varphi_n"] & \ker\partial_n \arrow[r] \arrow[d, "\varphi_n"] & H_n(\mathbb{V}) \arrow[r] \arrow[d, "\varphi_{*,n}"] & 0 \\ 0 \arrow[r] & \operatorname{ran}\partial_{n+1} \arrow[r, hook] & \ker\partial_n \arrow[r] & H_n(\mathbb{W}) \arrow[r] & 0 \end{tikzcd}$$

one sees that $\varphi_{*,n}$ is an isomorphism modulo $\mathcal{D}$. □

5.2. **Dimension function (after Lück).** Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra and recall that $\mathrm{Proj}(\mathcal{M})$ is a lattice such that $\tau(p)+\tau(q)=\tau(p\vee q)+\tau(p\wedge q)$ for every $p,q\in\mathrm{Proj}(\mathcal{M})$. Throughout this section, a module means a left $\mathcal{M}$-module. Note that

$$\mathrm{Mor}(\mathcal{M}^{\oplus m},\mathcal{M}^{\oplus n})=\mathbb{M}_{m,n}(\mathcal{M}) \text{ by the right multiplication.}$$

Definition 5.11. A module V is *finitely generated and projective* (abbreviated as f.g.p.) if $V\cong\mathcal{M}^{\oplus m}P$ for some $m\in\mathbb{N}$ and some idempotent $P\in\mathbb{M}_m(\mathcal{M})$.

Remark 5.12. In the original definition, a module V is *projective* if every surjection onto it splits. We note that a concrete realization $\mathcal{M}^{\oplus m}P$ of V is *not* among the structures of V. We can take P to be self-adjoint, because if we set $P_0=l(P)$, then $P=SP_0S^{-1}$ for $S=I+P_0-P$. For the following, we generally assume that P is self-adjoint.

A ring R is said to be "semi-hereditary" if every f.g. R-submodule of a free R-module is projective. Every von Neumann algebra has this property.

Lemma 5.13. *(1) Every weakly closed submodule V of $\mathcal{M}^{\oplus m}$ is of the form $\mathcal{M}^{\oplus m}P$.*
(2) For every $\varphi\in\mathrm{Mor}(\mathcal{M}^{\oplus m},\mathcal{M}^{\oplus n})$, both $\ker\varphi$ and $\operatorname{ran}\varphi$ are f.g.p.
(3) Every f.g. submodule V of $\mathcal{M}^{\oplus m}$ is projective.

Proof. Ad(1): One observes that $V=\mathcal{M}^{\oplus m}P$ for the orthogonal projection P in $\mathbb{M}_m(\mathcal{M})$ from $L^2\mathcal{M}^{\oplus m}$ onto the L^2-norm closure of V.
Ad(2): $\ker\varphi=\mathcal{M}^{\oplus m}P$ by (1) and $\operatorname{ran}\varphi\cong\mathcal{M}^{\oplus m}P^{\perp}$ by Isomorphism Theorem.
Ad(3): If V is f.g., then $V=\operatorname{ran}\varphi$ for some $\varphi\in\mathrm{Mor}(\mathcal{M}^{\oplus n},\mathcal{M}^{\oplus m})$. □

Definition 5.14. For a f.g.p. module $V\cong\mathcal{M}^{\oplus m}P$, define $\dim_{\mathcal{M}}V=(\mathrm{Tr}\otimes\tau)(P)$.

Remark 5.15. The $\mathcal{M}$-dimension $\dim_{\mathcal{M}}V$ is well-defined: If $\mathcal{M}^{\oplus m}P\cong\mathcal{M}^{\oplus n}Q$, then $(\mathrm{Tr}\otimes\tau)(P)=(\mathrm{Tr}\otimes\tau)(Q)$. In particular, if $W\cong V$ (resp. $W\subset V$) are f.g.p. modules, then $\dim_{\mathcal{M}}W=\dim_{\mathcal{M}}V$ (resp. $\dim_{\mathcal{M}}W\le\dim_{\mathcal{M}}V$).

Definition 5.16. For every module V, we define the $\mathcal{M}$-dimension of V by

$$\dim_{\mathcal{M}}V=\sup\{\dim_{\mathcal{M}}W : W\subset V \text{ f.g.p. submodule}\}\in[0,\infty].$$

Note that the definitions are consistent for f.g.p. modules. The dimension function is *continuous* in the following sense: if $V=\bigcup V_i$ is a directed union of modules, then one has $\dim_{\mathcal{M}}V=\lim\dim_{\mathcal{M}}V_i$.

For $V\subset\mathcal{M}^{\oplus m}$, we denote by $\overline{V}$ the weak closure of V. Although there is a way defining $\overline{V}$ purely algebraically for arbitrary module V, we do not elaborate it.

Proposition 5.17. *Let $V\subset\mathcal{M}^{\oplus m}$ be a submodule with $\overline{V}=\mathcal{M}^{\oplus m}P$. Then, there exists a net of projections $P_i\in\mathbb{M}_m(\mathcal{M})$ such that $\mathcal{M}^{\oplus m}P_i\subset V$ and $P_i\to P$. In particular, one has $\dim_{\mathcal{M}}V=\dim_{\mathcal{M}}\overline{V}$.*

Proof. Let $V \subset \mathcal{M}^{\oplus m}$ be given. Let $i = (W, \varepsilon)$ be a pair of f.g. submodule $W \subset V$ and $\varepsilon > 0$. We choose $n \in \mathbb{N}$ and $T \in \mathbb{M}_{n,m}(\mathcal{M})$ such that $W = \mathcal{M}^{\oplus n}T$, and $\delta > 0$ such that $P_i = \chi_{[\delta,1]}(T^*T) \in \mathbb{M}_m(\mathcal{M})$ satisfies $\tau(r(T) - P_i) < \varepsilon$. Since $P_i = ST$ for $S = \chi_{[\delta,1]}(T^*T)(T^*T)^{-1}T^* \in \mathbb{M}_{m,n}(\mathcal{M})$, we have $\mathcal{M}^{\oplus m}P_i \subset \mathcal{M}^{\oplus n}T \subset V$. It is not hard to see $P_i \nearrow P$. This implies that $\dim_{\mathcal{M}} V \geq \sup \dim_{\mathcal{M}} \mathcal{M}^{\oplus m}P_i = \dim_{\mathcal{M}} \overline{V}$. The converse inequality is trivial. □

Theorem 5.18 (Lück). *For every short exact sequence* $0 \to V_2 \xrightarrow{\iota} V_1 \xrightarrow{\pi} V_0 \to 0$, *one has* $\dim_{\mathcal{M}} V_1 = \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} V_2$.

Proof. Let $W \subset V_0$ be any f.g.p. submodule. Then, one has $\pi^{-1}(W) \cong W \oplus \iota(V_2)$ by the projectivity of W. Hence,

$$\dim_{\mathcal{M}} V_1 \geq \dim_{\mathcal{M}} \pi^{-1}(W) \geq \dim_{\mathcal{M}} W + \dim_{\mathcal{M}} \iota(V_2).$$

Taking the supremum over all $W \subset V_0$, one gets $\dim_{\mathcal{M}} V_1 \geq \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} V_2$. In particular, we have proved that $\dim_{\mathcal{M}}$ decreases under a surjection. To prove the converse inequality, let $W \subset V_1$ be any f.g.p. submodule. We realize W as $\mathcal{M}^{\oplus m}P$. Then, one has $\overline{\iota(V_2) \cap W} = \mathcal{M}^{\oplus m}Q$ for some projection $Q \in \mathbb{M}_m(\mathcal{M})$ with $Q \leq P$. This implies that $W/\overline{\iota(V_2) \cap W} \cong \mathcal{M}^{\oplus m}(P - Q)$. It follows by Proposition 5.17 that

$$\begin{aligned}
\dim_{\mathcal{M}} W &= \dim_{\mathcal{M}} W/\overline{\iota(V_2) \cap W} + \dim_{\mathcal{M}} \overline{\iota(V_2) \cap W} \\
&\leq \dim_{\mathcal{M}} W/(\iota(V_2) \cap W) + \dim_{\mathcal{M}} \iota(V_2) \cap W \\
&\leq \dim_{\mathcal{M}} V_0 + \dim_{\mathcal{M}} \iota(V_2),
\end{aligned}$$

where we have applied the first part to $W/(\iota(V_2) \cap W) \twoheadrightarrow W/\overline{\iota(V_2) \cap W}$. □

We call V a *torsion module* if $\dim_{\mathcal{M}} V = 0$. Torsion modules form a Serre subcategory and every module V has the unique largest torsion submodule $V_T \subset V$.

Corollary 5.19. *For every f.g. module* V, *one has* $V \cong V_P \oplus V_T$, *where* V_P *is f.g.p. with* $\dim_{\mathcal{M}} V_P = \dim_{\mathcal{M}} V$.

Proof. We prove that the f.g. module $V_P = V/V_T$ is projective (and hence there is a splitting $V_P \hookrightarrow V$). Take a surjection $\varphi\colon \mathcal{M}^{\oplus m} \twoheadrightarrow V_P$. Since $\overline{\ker\varphi}/\ker\varphi$ is a torsion submodule of $\mathcal{M}^{\oplus m}/\ker\varphi \cong V_P$, it is zero. It follows that $\ker\varphi$ is closed and $V_P \cong \mathcal{M}^{\oplus m}/\ker\varphi$ is projective. □

Although we do not use it explicitly, this corollary, in combination with continuity, is useful to reduce the proof of dimensional equations to those for f.g.p. modules.

Definition 5.20. A morphism $\varphi\colon V \to W$ is a $\dim_{\mathcal{M}}$*-isomorphism* if it is an isomorphism modulo torsion modules, i.e., $\dim_{\mathcal{M}} \ker\varphi = 0 = \dim_{\mathcal{M}} \operatorname{coker}\varphi$.

Lemma 5.21. *The morphism* $\mathcal{M} \hookrightarrow L^2\mathcal{M}$ *is a* $\dim_{\mathcal{M}}$*-isomorphism.*

Proof. Let $\xi \in L^2\mathcal{M}$ be given. We view it as a closed square-integrable operator affiliated with $\mathcal{M}$. Then, for $p_n = \chi_{[0,n]}(\xi\xi^*) \in \mathcal{M}$, one has $p_n \to 1$ and $p_n\xi \in \mathcal{M}$. We note that $p_n\xi \in \mathcal{M}$ means that $p_n\xi = 0$ in $L^2\mathcal{M}/\mathcal{M}$. □

Remark 5.22. From this lemma, one observes that $\dim_{\mathcal{M}}$ agrees with the von Neumann dimension function for normal Hilbert $\mathcal{M}$-modules.

5.3. **Definition of the ℓ_2-Betti numbers (after Lück).**

Definition 5.23. For a discrete group Γ, we define the n-th ℓ_2-Betti number of Γ by

$$\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(\mathcal{L}\Gamma, \mathbb{C}),$$

where $\mathbb{C}$ is the trivial $\mathbb{C}\Gamma$-module: $f \cdot z = \sum_{s\in\Gamma} f(s)z$.

Exercise. Prove that $\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} \operatorname{Tor}_n^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C})$. (Hint: You have to show that the functor $\ell_2\Gamma \otimes_{\mathcal{L}\Gamma} \cdot$ is exact and $\dim_{\mathcal{L}\Gamma}$-preserving.)

Example. For $d = 1, 2, \ldots$, one has

$$\beta_n^{(2)}(\mathbb{F}_d) = \begin{cases} d-1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}.$$

Proof. Let $g_1, \ldots, g_d$ be the canonical generators of $\mathbb{F}_d$. We consider the complex

$$\mathbb{V}: \qquad 0 \longrightarrow (\mathbb{C}\mathbb{F}_d)^{\oplus d} \xrightarrow{\partial_1} \mathbb{C}\mathbb{F}_d \xrightarrow{\partial_0} \mathbb{C} \longrightarrow 0,$$

where $\partial_0(\xi) = \sum_{s\in\mathbb{F}_d} \xi(s)$ and $\partial_1((\xi_i)_{i=1}^d) = \sum_{i=1}^d \xi_i \cdot g_i - \xi_i$. (We define $(\xi \cdot s)(t) = \xi(ts)$.) We will show that the complex $\mathbb{V}$ is exact. We check $\ker\partial_1 = 0$. Let $\chi_j \in \ell_\infty\mathbb{F}_d$ be the characteristic function of the subset of reduced words starting at g_j. It is not hard to see that $\chi_j \cdot g_i^{-1} = \chi_j + \delta_{i,j}\delta_e$ for every i, j. If $(\xi_i)_{i=1}^d \in \ker\partial_1$, then for every $s \in \Gamma$ and j, one has

$$0 = \langle \sum_{i=1}^d \xi_i \cdot g_i - \xi_i,\, s \cdot \chi_j \rangle = \sum_{i=1}^d \langle \xi_i,\, s \cdot (\chi_j \cdot g_i^{-1} - \chi_j) \rangle = \xi_j(s)$$

and $(\xi_i)_{i=1}^d = 0$. We next check $\operatorname{ran}\partial_1 = \ker\partial_0$. It is easy to see $\partial_0 \circ \partial_1 = 0$. Let $\chi_i^\vee \in \ell_\infty\mathbb{F}_d$ be the characteristic function of the subset of reduced words ending at g_i^{-1}. We observe that $\chi_i - s \cdot \chi_i^\vee$ is finitely supported for every $s \in \mathbb{F}_d$. (Indeed, it suffices to check this for $g_1, \ldots, g_d$.) Moreover, since $\chi_i^\vee \cdot g_i - \chi_i^\vee$ is the characteristic function of the reduced words ending at other than $g_i^{\pm 1}$, one has $\sum_{i=1}^d \chi_i^\vee \cdot g_i - \chi_i^\vee = (d-1)\mathbf{1} + \delta_e$. Now, suppose $\xi \in \ker\partial_0$. Then, since

$$\xi = -(\sum_{s\neq e} \xi(s)\delta_e) + \sum_{s\neq e} \xi(s)\delta_s = \sum_{s\neq e} \xi(s)(\delta_e - \delta_s),$$

$\xi_i = \xi * \chi_i^\vee \in \mathbb{C}\Gamma$ by the above observation, and since $\xi * \mathbf{1} = 0$, one has

$$\partial_1((\xi_i)_{i=1}^d) = \sum_{i=1}^d \xi_i \cdot g_i - \xi_i = \sum_{i=1}^d \xi * (\chi_i^\vee \cdot g_i - \chi_i^\vee) = \xi.$$

We have proved that $\mathbb{V}$ is a projective resolution of $\mathbb{C}$. Since

$$\mathcal{L}\mathbb{F}_d \otimes_{\mathbb{C}\mathbb{F}_d} \mathbb{V}_{\geq 0}: \qquad 0 \longrightarrow (\mathcal{L}\mathbb{F}_d)^{\oplus d} \xrightarrow{\partial_1} \mathcal{L}\mathbb{F}_d \longrightarrow 0,$$

one has

$$\operatorname{Tor}_n^{\mathbb{C}\mathbb{F}_d}(\mathcal{L}\mathbb{F}_d, \mathbb{C}) = \begin{cases} \mathcal{L}\mathbb{F}_d / \operatorname{ran}\partial_1 & \text{if } n = 0 \\ \ker\partial_1 & \text{if } n = 1 \\ 0 & \text{if } n \geq 2 \end{cases}.$$

Since $\lambda(s) - \lambda(t) \in \operatorname{ran}\partial_1$ and $\lambda(t) \to 0$ weakly as $t \to \infty$, one has $\lambda(s) \in \overline{\operatorname{ran}\partial_1}$ for every $s \in \mathbb{F}_d$ and hence $\overline{\operatorname{ran}\partial_1} = \mathcal{L}\mathbb{F}_d$. It follows that $\beta_0^{(2)}(\mathbb{F}_d) = 0$ and $\beta_1^{(2)}(\mathbb{F}_d) = \dim_{\mathcal{L}\mathbb{F}_d} \ker\partial_1 = d - \dim_{\mathcal{L}\mathbb{F}_d} \operatorname{ran}\partial_1 = d - 1$. $\square$

Below, we sketch an argument showing that the above definition of ℓ_2-Betti numbers is consistent with another(?). We denote by $\mathcal{F}(\Gamma, X)$ the set of functions from a set Γ into X. Now Γ be a discrete group and consider ℓ_Γ as a right Γ-module. The is a natural complex

$$0 \longrightarrow \ell_2\Gamma \xrightarrow{\partial_0} \mathcal{F}(\Gamma, \ell_2\Gamma) \xrightarrow{\partial_1} \mathcal{F}(\Gamma^2, \ell_2\Gamma) \longrightarrow \cdots,$$

where $(\partial_0 f)(s) = f - f \cdot s$ and $(\partial_1 b)(s,t) = b(t) - b(st) + b(s) \cdot t$, etc. We then define the ℓ_2-cohomology $H_n(\Gamma, \ell_2\Gamma)$ by $H_n(\Gamma, \ell_2\Gamma) = \ker \partial_n / \operatorname{ran} \partial_{n+1}$. Since ∂_n commutes with the $\mathcal{L}\Gamma$-action on $\ell_2\Gamma$, the ℓ_2-cohomology $H_n(\Gamma, \ell_2\Gamma)$ is naturally an $\mathcal{L}\Gamma$-module. We define $\beta_n^{(2)}(\Gamma) = \dim_{\mathcal{L}\Gamma} H_n(\Gamma, \ell_2\Gamma)$. Let us calculate $\beta_n^{(2)}(\Gamma)$ for $n = 0, 1$. Since $H_0 \subset \ell_2\Gamma$ is the subspace of constant functions, one has $\beta_0^{(2)}(\Gamma) = |\Gamma|^{-1}$. We note that $D(\Gamma) = \ker \partial_1$ is the space of derivations and $D_0(\Gamma) = \operatorname{ran} \partial_0$ is the space of inner derivations. To see what $\beta_1^{(2)}(\Gamma)$ is, we assume that Γ is generated by a finite subset $\{s_1, \ldots, s_d\}$. Then, there is an $\mathcal{L}\Gamma$-module map

$$D(\Gamma) \ni b \longmapsto (b(s_i))_{i=1}^d \in \bigoplus_{i=1}^d \ell_2\Gamma,$$

which is an isomorphism onto a closed subspace. We note that $D_0(\Gamma)$ is closed in $\bigoplus_{i=1}^d \ell_2\Gamma$ iff Γ is finite or non-amenable, and that $\dim_{\mathcal{L}\Gamma} \overline{D_0(\Gamma)} = \dim_{\mathcal{L}\Gamma}(\ker \partial_0)^\perp = 1 - |\Gamma|^{-1}$. Hence, one has $\dim_{\mathcal{L}\Gamma} D(\Gamma) = \beta_1^{(2)}(\Gamma) + \dim_{\mathcal{L}\Gamma} D_0(\Gamma) = \beta_1^{(2)}(\Gamma) - \beta_0^{(2)}(\Gamma) + 1$. We view ∂_0 as a map from $\ell_2\Gamma$ into $\bigoplus_{i=1}^d \ell_2\Gamma$ and consider

$$\partial_0^* \colon \bigoplus_{i=1}^d \ell_2\Gamma \ni (\xi_i) \longmapsto \sum_i \xi_i - \xi_i \cdot s_i^{-1} \in \ell_2\Gamma.$$

Lemma 5.24. *One has* $(\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp = D(\Gamma)$.

Proof. We note that the scalar product $\langle \cdot, \cdot \rangle$ is defined consistently on $\mathbb{C}\Gamma \times \mathcal{F}(\Gamma, \mathbb{C})$ and on $\ell_2\Gamma \times \ell_2\Gamma$. Moreover, $\mathcal{F}(\Gamma, \mathbb{C})$ is the algebraic dual of $\mathbb{C}\Gamma$ w.r.t. this scalar product. Suppose that $b \in D(\Gamma)$. It is not hard to show that $b = f - f \cdot s$ for some $f \in \mathcal{F}(\Gamma, \mathbb{C})$. It follows that for every $\xi \in \ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma$, one has

$$\langle \xi, b \rangle = \sum \langle \xi_i, b(s_i) \rangle = \sum_i \langle \xi_i - \xi_i \cdot s_i^{-1}, f \rangle = 0.$$

Conversely, if $b \in \ell_2\Gamma$ is such that $b \perp (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)$, then the linear functional $\langle \cdot, b \rangle$ on $\bigoplus \mathbb{C}\Gamma$ factors through ∂_0^* and there is $f \in \mathcal{F}(\Gamma, \mathbb{C})$ such that $\langle \xi, b \rangle = \langle \partial_0^*(\xi), f \rangle$ for every $\xi \in \bigoplus \mathbb{C}\Gamma$. It follows that $b(s) = f - f \cdot s$ and $b \in D(\Gamma)$. $\square$

Since $(\ker \partial_0^*)^\perp = \overline{\operatorname{ran} \partial_0} = \overline{D_0(\Gamma)}$, one has

$$\begin{aligned} D(\Gamma)/\overline{D_0(\Gamma)} &\cong (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp \ominus (\ker \partial_0^*)^\perp \\ &= \ker \partial_0^* \cap (\ker \partial_0^* \cap \bigoplus \mathbb{C}\Gamma)^\perp \quad \cong \operatorname{Tor}_1^{\mathbb{C}\Gamma}(\ell_2\Gamma, \mathbb{C}). \end{aligned}$$

The last isomorphism follows from the following observation:

$$\mathbb{V}: \qquad \cdots \longrightarrow \bigoplus_{i=1}^d \mathbb{C}\Gamma \xrightarrow{\partial_0^*} \mathbb{C}\Gamma \longrightarrow \mathbb{C}$$

is a free resolution of the trivial left $\mathbb{C}\Gamma$-module $\mathbb{C}$ and

$$\ell_2\Gamma \otimes_{\mathbb{C}\Gamma} \mathbb{V}_{\geq 0}: \qquad \cdots \longrightarrow \bigoplus_{i=1}^{d} \ell_2\Gamma \xrightarrow{\partial_0^*} \ell_2\Gamma \longrightarrow 0$$

with $\operatorname{ran}\partial_1^* = \ell_2\Gamma \otimes_{\mathbb{C}\Gamma} \ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma}) \subset \overline{\mathbb{C}\Gamma \otimes_{\mathbb{C}\Gamma} \ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma})} = \overline{\ker(\partial_0^*|_{\bigoplus \mathbb{C}\Gamma})}$.

5.4. Rank metric (after Thom).

Definition 5.25. Let V be a left $\mathcal{M}$-module. For $\xi \in V$, we define its *rank norm* by

$$[\xi] = \inf\{\tau(p) : p \in \operatorname{Proj}(A),\ p\xi = \xi\} \in [0,1].$$

We record several basic properties of the rank norm.

Lemma 5.26. *For a left $\mathcal{M}$-module V, the following are true.*

(1) Triangle inequality: $[\xi + \eta] \leq [\xi] + [\eta]$ for every $\xi, \eta \in V$.
(2) $[x\xi] \leq \min\{[x], [\xi]\}$ for every $x \in \mathcal{M}$ and $\xi \in V$.
(3) $V_T = \{\xi \in V : [\xi] = 0\}$.
(4) A submodule $W \subset V$ is dense in rank norm if and only if $\dim_{\mathcal{M}} V/W = 0$.
(5) Every $\varphi \in \operatorname{Mor}(V, W)$ is a rank contraction: $[\varphi(\xi)] \leq [\xi]$.
(6) For every $\varphi \in \operatorname{Mor}(V, W)$, $\eta \in \operatorname{ran}\varphi$ and $\varepsilon > 0$, there exists $\xi \in \varphi^{-1}(\eta)$ such that $[\eta] \leq [\xi] + \varepsilon$.

Proof. The triangle inequality follows from the fact that $\tau(p \vee q) \leq \tau(p) + \tau(q)$. The second assertion follows from the fact that $p\xi = \xi$ implies $xp\xi = \xi$ and $[x\xi] \leq \tau(l(xp)) \leq \tau(p)$. For the third assertion, we observe that $[\xi] = 0$ iff $\mathcal{M}\xi$ is a torsion submodule. Indeed, the "if" part is rather easy and the "only if" part follows by considering the morphism $\varphi\colon M \ni x \mapsto x\xi \in V$. Since $\ker\varphi$ is a left ideal with $\dim_{\mathcal{M}} \ker\varphi = 1$, i.e., $\overline{L} = \mathcal{M}$, Proposition 5.17 implies that there is a net $p_i \in L$ of projections such that $p_i \to 1$. This means $[\xi] = 0$. The rest are trivial. □

We recall that the *completion* of a metric space (X, d) is the metric space of all equivalence classes of Cauchy sequences in X. Here, two Cauchy sequences $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ are equivalent if $d((x_n), (y_n)) := \lim_n d(x_n, y_n) = 0$.

Definition 5.27. The *rank completion* of a left $\mathcal{M}$-module V is the completion $C(V)$ of V w.r.t. the *rank metric* d, where $d(\xi, \eta) = [\xi - \eta]$ for $\xi, \eta \in V$. We observe that

$$C(V) = \{\text{ Cauchy sequences in V }\}/\{\text{ Null sequences }\}$$

and that $C(V)$ is naturally a left $\mathcal{M}$-module (thanks to Lemma 5.26).

The rank metric is actually a pseudo-metric. More precisely, it is a metric on V/V_T. The constant "embedding" $c\colon V \to C(V)$ is a $\dim_{\mathcal{M}}$-isomorphism and it induces a canonical inclusion $V/V_T \hookrightarrow C(V)$. Moreover, $C(V)$ is the unique torsion-free complete $\mathcal{M}$-module containing V/V_T as a dense submodule. Indeed, one has:

Lemma 5.28. *Let V, W be $\mathcal{M}$-modules with W torsion-free and complete. Then, every $\varphi \in \operatorname{Mor}(V, W)$ extends to $\tilde{\varphi} \in \operatorname{Mor}(C(V), W)$, i.e., $\tilde{\varphi} \circ c = \varphi$.*

Proposition 5.29. *The rank completion c is an exact functor.*

Proof. Let a short exact sequence $0 \longrightarrow V_2 \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \longrightarrow 0$ be given.

Exactness at $C(V_0)$. Let $\xi \in C(V_0)$ and choose a representing Cauchy sequence $(\xi_n)_n$ in V_0 such that $d(\xi_n, \xi_{n+1}) < 2^{-(n+1)}$. We will construct $\eta_1, \eta_2, \dots$ such that $\partial_1(\eta_n) = \xi_n$ and $d(\eta_n, \eta_{n+1}) < 2^{-n}$. Suppose $\eta_1, \dots, \eta_n$ have been chosen. Lift $\xi_{n+1} - \xi_n \in V_0$ to $\zeta_{n+1} \in V_1$ with $[\zeta_{n+1}] \leq [\xi_{n+1} - \xi_n] + 2^{-(n+1)}$. Set $\eta_{n+1} = \eta_n + \zeta_{n+1}$ and we are done. Now the sequence $(\eta_n)_n$ is Cauchy in V_1 and hence converges to an element η in $C(V_1)$ such that $\partial_1(\eta) = \xi$.

Exactness at $C(V_1)$. It is clear that $C(\partial_1) \circ C(\partial_2) = 0$ by continuity. Let $\xi \in \ker C(\partial_1)$ be given and choose $(\xi_n)_n$ in V_1 such that $\xi_n \to \xi$. Since $\partial_1(\xi_n) \to C(\partial_1)(\xi) = 0$, the sequence $(\partial_1(\xi_n))_n$ is null. Hence, one can lift $(\partial_1(\xi_n))_n$ to a null sequence $(\eta_n)_n$ in V_1. It follows that $(\xi_n - \eta_n)_n$ is a Cauchy sequence in $\ker \partial_1 = \operatorname{ran} \partial_2$. Therefore,

$$\xi = \lim_{n\to\infty} \xi_n = \lim_{n\to\infty} (\xi_n - \eta_n) \in \overline{\operatorname{ran} \partial_2} = \operatorname{ran} C(\partial_2),$$

where we used the result of the previous paragraph for the last equality.

Exactness at $C(V_2)$. Since ∂_2 is an isometry, $C(\partial_2)$ is an isometry as well. Since $C(V_2)$ does not have a non-zero torsion element, $C(\partial_2)$ is injective. □

5.5. Gaboriau's theorem (after Sauer and Thom).

Proposition 5.30. *Let $\mathcal{M} \subset \mathcal{N}$ be finite von Neumann algebras with $\tau_{\mathcal{M}} = \tau_{\mathcal{N}}|_{\mathcal{M}}$. Then, $\mathcal{N}$ is a flat $\mathcal{M}$-module and $\dim_{\mathcal{M}} V = \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$ for any $\mathcal{M}$-module V.*

Proof. We use Lemma 5.9 to prove flatness. Let $V \subset \mathcal{M}^{\oplus m}$ be a f.g. submodule. It follows that there is $T \in \mathbb{M}_{n,m}(\mathcal{M})$ such that $V = \mathcal{M}^{\oplus n}T$. Let P be the left support of T and observe that $V = \mathcal{M}^{\oplus n}T \ni \xi T \mapsto \xi P \in \mathcal{M}^{\oplus n}P$ is an isomorphism. Since $\mathcal{M}^{\oplus n}P$ is a direct summand of $\mathcal{M}^{\oplus n}$, one has the following kosher identifications

$$\mathcal{N} \otimes_{\mathcal{M}} V \cong \mathcal{N} \otimes_{\mathcal{M}} (\mathcal{M}^{\oplus n}P) \cong \mathcal{N}^{\oplus n}P \cong \mathcal{N}^{\oplus n}T \subset \mathcal{N}^{\oplus m} \cong \mathcal{N} \otimes_{\mathcal{M}} \mathcal{M}^{\oplus m}.$$

It follows from Lemma 5.9 that $\mathcal{N}$ is flat.

Since the dimension function is continuous w.r.t. inductive limits, it suffices to check the identity $\dim_{\mathcal{M}} V = \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$ for a f.g. V. Since $\mathcal{M}^{\oplus m}P \hookrightarrow V$ implies $\mathcal{N}^{\oplus m}P \hookrightarrow \mathcal{N} \otimes_{\mathcal{M}} V$, one has $\dim_{\mathcal{M}} V \leq \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V$. To prove the converse inequality, take a surjection $\pi \colon \mathcal{M}^{\oplus n} \twoheadrightarrow V$. Then, $\mathrm{id} \otimes \pi \colon \mathcal{N}^{\oplus n} \twoheadrightarrow \mathcal{N} \otimes_{\mathcal{M}} V$ is also a surjection such that $\ker(\mathrm{id} \otimes \pi) = \mathcal{N} \otimes_{\mathcal{M}} \ker \pi$ by flatness. It follows that

$$\dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} V = n - \dim_{\mathcal{N}} \ker(\mathrm{id} \otimes \pi) \leq n - \dim_{\mathcal{M}} \ker \pi = \dim_{\mathcal{M}} V$$

by the previous inequality. □

Let $\Gamma \curvearrowright (X, \mu)$ be an essentially-free probability-measure-preserving action. Let $\mathcal{A} = L^\infty(X, \mu)$, $\mathcal{M} = \mathcal{L}\Gamma$ and $\mathcal{N} = \mathcal{A} \rtimes \Gamma$. Let $R_0 \subset \mathcal{N}$ (resp. $R \subset \mathcal{N}$) be the $\mathbb{C}$-algebra generated by $\mathcal{A}$ and Γ (resp. by $\mathcal{A}$ and the full group $[\Gamma]$). Then,

$$\mathcal{A} \subset R_0 \subset R \subset \mathcal{N}$$

and $\mathcal{A}$ is a left R-module: $a\varphi \cdot f = a\varphi_*(f)$ for $a, f \in \mathcal{A}$ and $\varphi \in [\Gamma]$. Now, Gaboriau's theorem that $\beta_\bullet^{(2)}(\Gamma)$ is an invariant of $[\Gamma]$ follows from the following equalities:

$$\begin{aligned}
\beta_\bullet^{(2)}(\Gamma) &= \dim_{\mathcal{M}} \mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{M}, \mathbb{C}) & \\
&= \dim_{\mathcal{N}} \mathcal{N} \otimes_{\mathcal{M}} \mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{M}, \mathbb{C}) & \text{by Proposition 5.30} \\
&= \dim_{\mathcal{N}} \mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{N} \otimes_{\mathcal{M}} \mathcal{M}, \mathbb{C}) & \text{since } \mathcal{N} \text{ is flat over } \mathcal{M} \\
&= \dim_{\mathcal{N}} \mathrm{Tor}_\bullet^{R_0}(\mathcal{N}, \mathcal{A}) & (\spadesuit) \\
&= \dim_{\mathcal{N}} \mathrm{Tor}_\bullet^{R}(\mathcal{N}, \mathcal{A}) & (\heartsuit)
\end{aligned}$$

The proof of ($\spadesuit$) is rather routine: Since $\mathcal{N}$ is also a right R_0-module and R_0 is a free left $\mathbb{C}\Gamma$-module (Consider the conditional expectation onto $\mathcal{A}$), one has

$$\mathrm{Tor}_\bullet^{\mathbb{C}\Gamma}(\mathcal{N}, V) = \mathrm{Tor}_\bullet^{R_0}(\mathcal{N}, R_0 \otimes_{\mathbb{C}\Gamma} V)$$

for any $\mathbb{C}\Gamma$-module V. Indeed, if $\mathbb{V}$ is a projective resolution of V, then $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{V}$ is a projective resolution of $R_0 \otimes_{\mathbb{C}\Gamma} V$ with $\mathcal{N} \otimes_{R_0} (R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{V}) \cong \mathcal{N} \otimes_{\mathbb{C}\Gamma} \mathbb{V}$. We then observe that $R_0 \otimes_{\mathbb{C}\Gamma} \mathbb{C} \cong \mathcal{A}$ as an R_0-module. The proof of ($\heartsuit$) is more involved, but reduces to the fact that $R_0 \subset R$ is dense in an appropriate sense.

We write $[\xi]_{\mathcal{A}}$ (resp. $[\xi]_{\mathcal{N}}$) for the rank norm w.r.t. $\mathcal{A}$ (resp. $\mathcal{N}$) and note that $[\xi]_{\mathcal{N}} \le [\xi]_{\mathcal{A}}$. In particular, one has $[x]_{\mathcal{A}} = \inf\{\tau(p) : p \in \mathrm{Proj}(\mathcal{A}),\ px = x\}$ for $x \in \mathcal{N}$. For $x \in \mathcal{N}$, we define

$$\|[x]\|_{\mathcal{A}} = \sup\{[xp]_{\mathcal{A}}/[p]_{\mathcal{A}} : p \in \mathrm{Proj}(\mathcal{A})\} \in [0, \infty].$$

We record several basic properties of this norm.

Lemma 5.31. *(1)* $\|[\alpha x]\|_{\mathcal{A}} = \|[x]\|_{\mathcal{A}}$ *for every* $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $x \in \mathcal{N}$.
(2) $\|[v]\|_{\mathcal{A}} = 1$ *for every non-zero pseudo-normalizer* v *of* $\mathcal{A}$ *in* $\mathcal{N}$.
(3) $\|[x+y]\|_{\mathcal{A}} \le \|[x]\|_{\mathcal{A}} + \|[y]\|_{\mathcal{A}}$ *and* $\|[xy]\|_{\mathcal{A}} \le \|[x]\|_{\mathcal{A}} \|[y]\|_{\mathcal{A}}$ *for every* $x, y \in \mathcal{N}$.
(4) $\|[x]\|_{\mathcal{A}} < \infty$ *for every* $x \in R$.
(5) For every $x \in R$, *there is a sequence* $(x_n)_n$ *in* R_0 *such that* $[x_n - x]_{\mathcal{A}} \to 0$ *and* $\sup \|[x_n]\|_{\mathcal{A}} < \infty$.
(6) If V *is an* R_0*-module, then* $[x\xi]_{\mathcal{A}} \le \|[x]\|_{\mathcal{A}} [\xi]_{\mathcal{A}}$ *for every* $x \in R_0$ *and* $\xi \in V$. *The same thing holds for* R.

Lemma 5.32. *Let* V *be a left* R_0*-module. Then, the rank completion* $C(V)$ *w.r.t.* $\mathcal{A}$ *is naturally a left* R*-module. Moreover,* C *is a natural functor from the category of* R_0*-modules into the category of complete* R*-modules.*

Proof. By the previous lemma, one knows that $C(V)$ is naturally an R_0-module. Let $x \in R$ and $\xi \in C(V)$ be given. Choose a sequence $(x_n)_n$ in R_0 such that $[x - x_n]_{\mathcal{A}} \to 0$. Then, $(x_n\xi)_n$ is a Cauchy sequence in $C(V)$ and has a limit $x\xi$ in $C(V)$. We note that the limit is independent of the choice of $(x_n)_n$. Moreover, if $\|[x_n]\|_{\mathcal{A}}$ is bounded and $[y_m - y]_{\mathcal{A}} \to 0$, then $[x_n y_m - xy]_{\mathcal{A}} \to 0$. This shows $(xy)\xi = x(y\xi)$. □

Lemma 5.33. *Let* V *be a left* R_0*-module. Then the constant embedding*

$$\mathrm{id} \otimes c \colon \mathcal{N} \otimes_{R_0} V \to \mathcal{N} \otimes_{R_0} C(V)$$

is a $\dim_{\mathcal{N}}$*-isomorphism. The same thing holds for* R.

Proof. Suppose that $\sum_{i=1}^n x_i \otimes \xi_i \in \ker(\mathrm{id} \otimes c)$ and $\varepsilon > 0$ be given. Then, one has

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_j b_j r_j \otimes \eta_j - b_j \otimes r_j \eta_j \quad \text{in } \mathcal{N} \otimes_{\mathbf{C}} C(V).$$

Choose $p_j \in \mathrm{Proj}(\mathcal{A})$ such that $p_j^\perp \eta_j \in V$ and $n \sum (1 + \|[r_j]\|_{\mathcal{A}}) \tau(p_j) < \varepsilon$. It follows that there is $p \in \mathrm{Proj}(\mathcal{A})$ such that $pp_j = p_j$, $pr_j p_j = r_j p_j$ and $\tau(p) < \varepsilon/n$. Since $\sum_j b_j r_j \otimes p_j^\perp \eta_j - b_j \otimes r_j p_j^\perp \eta_j$ is zero in $\mathcal{N} \otimes_{R_0} V$, subtracting it from the both sides of the above equation, we may assume that

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_j b_j r_j \otimes p_j \eta_j - b_j \otimes r_j p_j \eta_j \quad \text{in } \mathcal{N} \otimes_{\mathbf{C}} C(V).$$

It follows that $\sum x_i \otimes \xi_i = \sum x_i \otimes p\xi_i$ in $\mathcal{N} \otimes_{\mathbf{C}} C(V)$, and *a fortiori* in $\mathcal{N} \otimes_{\mathbf{C}} V$ since $\mathcal{N} \otimes_{\mathbf{C}} V \subset \mathcal{N} \otimes_{\mathbf{C}} C(V)$ (recall any module over a field is free). Hence, one has

$$\sum_{i=1}^n x_i \otimes \xi_i = \sum_{i=1}^n x_i \otimes p\xi_i = \sum_{i=1}^n x_i p \otimes \xi_i \quad \text{in } \mathcal{N} \otimes_{R_0} V.$$

This implies that $[\sum_{i=1}^n x_i \otimes \xi_i]_{\mathcal{N}} \le \sum [x_i p]_{\mathcal{N}} < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, one sees that $\ker(\mathrm{id} \otimes c)$ is a torsion submodule. That $\mathrm{ran}(\mathrm{id} \otimes c)$ is dense in $\mathcal{N} \otimes_{R_0} C(V)$ follows from the fact that $[x \otimes \xi]_{\mathcal{N}} \le [\xi]_{\mathcal{A}}$ for every $x \in \mathcal{N}$ and $\xi \in C(V)$. □

We omit the proof of the next lemma, which is similar to that of the previous one.

Lemma 5.34. *Let V be a left R-module, then the surjection*

$$N \otimes_{R_0} V \twoheadrightarrow N \otimes_R V$$

is a $\dim_{\mathcal{N}}$*-isomorphism.*

We are now in position to complete the proof of Gaboriau's theorem.

Proof of (♡). Let $\mathbf{V}$ (resp. $\mathbf{W}$) be a projective resolution of $\mathcal{A}$ as an R_0-module (resp. as an R-module). Then, by Theorem 5.3 (and Proposition 5.29), the identity morphism $\mathrm{id}_{\mathcal{A}}\colon \mathcal{A} \to \mathcal{A}$ (resp. the constant embedding $c\colon \mathcal{A} \to C(\mathcal{A})$) extends to a morphism $\varphi\colon \mathbf{V} \to \mathbf{W}$ (resp. a morphism $\psi\colon \mathbf{W} \to C(\mathbf{V})$):

$$\begin{array}{lccccccc}
\mathbf{V}: & \cdots \longrightarrow & V_n & \longrightarrow \cdots \longrightarrow & V_0 & \twoheadrightarrow & \mathcal{A} \\
 & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_0} & & \|{\scriptstyle \mathrm{id}_{\mathcal{A}}} \\
\mathbf{W}: & \cdots \longrightarrow & W_n & \longrightarrow \cdots \longrightarrow & W_0 & \twoheadrightarrow & \mathcal{A} \\
 & & \downarrow{\scriptstyle \psi_n} & & \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle c} \\
C(\mathbf{V}): & \cdots \longrightarrow & C(V_n) & \longrightarrow \cdots \longrightarrow & C(V_0) & \twoheadrightarrow & C(\mathcal{A}) \\
 & & \downarrow{\scriptstyle \tilde{\varphi}_n} & & \downarrow{\scriptstyle \tilde{\varphi}_0} & & \|{\scriptstyle \mathrm{id}_{C(\mathcal{A})}} \\
C(\mathbf{W}): & \cdots \longrightarrow & C(W_n) & \longrightarrow \cdots \longrightarrow & C(W_0) & \twoheadrightarrow & C(\mathcal{A})
\end{array}$$

By the uniqueness part of Theorem 5.3, the compositions $\psi \circ \varphi$ and $\tilde{\varphi} \circ \psi$ are homotopic to the morphisms of the constant embeddings. Taking tensor products, one has

$$\begin{tikzcd}[row sep=large]
\mathcal{N} \otimes_{R_0} \mathbb{V}_{\geq 0}: & \cdots \arrow[r] & \mathcal{N} \otimes_{R_0} V_n \arrow[r] \arrow[d, "\mathrm{id}\otimes\varphi_n"] & \cdots \\
\mathcal{N} \otimes_{R} \mathbb{W}_{\geq 0}: & \cdots \arrow[r] & \mathcal{N} \otimes_{R} W_n \arrow[r] \arrow[d, "\mathrm{id}\otimes\psi_n"] & \cdots \\
\mathcal{N} \otimes_{R} C(\mathbb{V})_{\geq 0}: & \cdots \arrow[r] & \mathcal{N} \otimes_{R} C(V_n) \arrow[r] \arrow[d, "\mathrm{id}\otimes\tilde{\varphi}_n"] & \cdots \\
\mathcal{N} \otimes_{R} C(\mathbb{W})_{\geq 0}: & \cdots \arrow[r] & \mathcal{N} \otimes_{R} C(W_n) \arrow[r] & \cdots
\end{tikzcd}$$

The morphism from $\mathcal{N} \otimes_{R_0} \mathbb{V}_{\geq 0}$ to $\mathcal{N} \otimes_R C(\mathbb{V})_{\geq 0}$ and the morphism from $\mathcal{N} \otimes_R \mathbb{W}_{\geq 0}$ to $\mathcal{N} \otimes_R C(\mathbb{W})_{\geq 0}$ are homotopic to the morphisms of constant embeddings. Since constant embeddings are $\dim_{\mathcal{N}}$-isomorphisms by Lemmas 5.33 and 5.34, the induced morphisms on the homology modules are all $\dim_{\mathcal{N}}$-isomorphisms by Lemma 5.10. It follows that $\varphi_{*,\bullet}\colon \mathrm{Tor}^{R_0}_{\bullet}(\mathcal{N}, \mathcal{A}) \to \mathrm{Tor}^{R}_{\bullet}(\mathcal{N}, \mathcal{A})$ are all $\dim_{\mathcal{N}}$-isomorphisms. $\square$

Let $p \in \mathcal{N}$ be a projection and V be an $\mathcal{N}$-module. It is not hard to check that $p\mathcal{N} \otimes_{\mathcal{N}} V \cong pV$ and $\dim_{p\mathcal{N}p} p\mathcal{N} \otimes_{\mathcal{N}} V = \tau(p)^{-1} \dim_{\mathcal{N}} V$, where one uses the normalized trace $\tau(p)^{-1}\tau(\,\cdot\,)$ for $p\mathcal{N}p$. If $p \in \mathrm{Proj}(\mathcal{A})$ is a projection such that $\sum_i v_i p v_i^* = 1$ for some pseudo-normalizers $v_1, \ldots, v_n$, then $\mathcal{N} \otimes_R V \cong \mathcal{N}p \otimes_{pRp} pV$ for every R-module V. whose central support in $\mathcal{N}$ is 1. It follows that

$$\begin{aligned}
\dim_{\mathcal{N}} \mathrm{Tor}^{R}_{\bullet}(\mathcal{N}, \mathcal{A}) &= \tau(p) \dim_{p\mathcal{N}p} p\mathcal{N} \otimes_{\mathcal{N}} \mathrm{Tor}^{R}_{\bullet}(\mathcal{N}, \mathcal{A}) \\
&= \tau(p) \dim_{p\mathcal{N}p} \mathrm{Tor}^{pRp}_{\bullet}(p\mathcal{N}p, p\mathcal{A}).
\end{aligned}$$

With little more analysis, one can show the above equation for every $p \in \mathrm{Proj}(\mathcal{A})$ with full central support.

References

[1] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory.* A Series of Modern Surveys in Mathematics 44. Springer-Verlag, Berlin, 2002.

[2] R. Sauer, L^2-Betti numbers of discrete measured groupoids. *Internat. J. Algebra Comput.* **15** (2005), 1169–1188.

[3] A. Thom, L^2-invariants and rank metric. Preprint. arXiv:math/0607263.

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