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ON STAR MOMENT SEQUENCE OF OPERATORS

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Abstract

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. We call "an operator T acting on $\mathcal{H}$ has a star moment sequence supported on a set K" when there exist nonzero vectors u and v in $\mathcal{H}$ and a positive Borel measure μ such that $\langle T^{*j}T^k u, v\rangle = \int_K \bar{z}^j z^k d\mu$ for all $j, k \geq 0$. We obtain a characterization to find a representing star moment measure and discuss some related properties.

1. Introduction and Preliminaries

The results of this article will be appeared in other journal. And so we omit some detail proof here. Let $\mathcal{X}$ be a (real or complex) Banach space, and denoted by $\mathcal{L}(\mathcal{X})$ be the algebra of all bounded linear operators on $\mathcal{X}$. Let $\mathcal{X}^*$ be a dual space of $\mathcal{X}$. Following [AG], we say that a T in $\mathcal{L}(\mathcal{X})$ has a *moment sequence* if there exist nonzero vectors $x \in \mathcal{X}$ and $y \in \mathcal{X}^*$ and a positive Borel measure supported on the spectrum $\sigma(T)$ of T (and, of course, $\sigma(T) \subset \mathbb{R}$ if $\mathcal{X}$ is a real Banach space) such that

$$y(T^n x) = \int_{\sigma(T)} \lambda^n d\mu_{x,y}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

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where $\mathbb{N}_0$ denotes, in usual, the set of nonnegative integers.

Atzmon and Godefroy then showed in [AG] that if $\mathcal{X}$ is real and satisfies some additional conditions, that every operator in $\mathcal{L}(\mathcal{X})$ that has a moment sequence (as in (1.1)), has, in fact, a nontrivial invariant subspace. This immediately raises the following question.

Question 1.1. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. Which classes of operators have moment sequences?

Of course, at present, it is not known, in this context, whether having a moment sequence implies the existence of invariant subspace. This led the authors of [FJKP] and [CJKP] to undertake a study of this equation in the case that $\mathcal{X} = \mathcal{H}$, and they showed that various classes of operators in $\mathcal{L}(\mathcal{H})$ do have moment sequences.

In this note we consider a related question and define a new definition of moment sequence.

Definition 1.2. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space. An operator $T \in \mathcal{L}(\mathcal{H})$ has a *$*$-moment sequence* supported on $K \subset \mathbb{C}$ if there exist nonzero vectors u and v in $\mathcal{H}$ and a positive Borel measure μ such that

$$\langle T^{*j}T^k u, v\rangle = \int_K \bar{z}^j z^k d\mu, \qquad j,k \in \mathbb{N}_0. \tag{1.2}$$

(Throughout this note K will be $\sigma(T)$ in almost cases.)

In this note, we will establish some necessary and sufficient conditions for which an operator T in $\mathcal{L}(\mathcal{H})$ has a *-moment sequence, and in condition which equivalent to *-moment sequence.

2. Some Results

Let N be a normal operator in $\mathcal{L}(\mathcal{H})$. Then obviously N has a *-moment sequence. Moreover, this fact can be improved to the case of subnormal operator as following.

Proposition 2.1. *Every subnormal operator S in $\mathcal{L}(\mathcal{H})$ have a *-moment sequence on $\sigma(S)$.*

We now characterize operators having a *-moment sequence, which is the main theorem of this note.

Theorem 2.2. *Let $T \in \mathcal{L}(\mathcal{H})$. Then T has a *-moment sequence supported on $\sigma(T)$ if and only if there exist nonzero vectors u and v in $\mathcal{H}$ such that $\langle u, v\rangle \geq 0$ and*

$$|\langle p(T^*, T)u, v\rangle| \leq \langle u, v\rangle \, \|p\|_\infty, \qquad p \in \mathbb{C}[\bar{z}, z],$$

where $\|p\|_\infty$ ($:=\sup_{z,\bar{z}\in\sigma(T)}|p(z,\bar{z})|$ for $p \in C[\bar{z}, z]$) is the sup norm supported on $\sigma(T)$ and

$$|\langle p(T^*, T)u, v\rangle| = \sum a_{jk} T^{*j} T^k \quad \textit{for } p(\bar{z}, z) = \sum a_{jk} \bar{z}^j z^k, \;\; j,k \in \mathbb{N}_0.$$

(*Note that* $p(T^*, T) \equiv \sum a_{jk} T^{*j} T^k$ *which is not equal to* $\sum a_{jk} T^k T^{*j}$ *in general.*)

Sketch of Proof. Suppose $T \in \mathcal{L}(\mathcal{H})$ has a *-moment sequence supported on $\sigma(T)$. Then there exist nonzero vectors u and v and a positive Borel measure μ such that

$$\langle T^{*j} T^k u, v \rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu, \qquad j, k \in \mathbb{N}_0.$$

For $j = k = 0$, we have

$$\langle u, v \rangle = \int_{\sigma(T)} 1 d\mu = \mu(\sigma(T)).$$

Then

$$\begin{aligned} |\langle p(T^*, T) u, v \rangle| &= \left| \int_{\sigma(T)} p(\bar{z}, z) d\mu \right| \leq \int_{\sigma(T)} |p(\bar{z}, z)| \, d\mu \\ &\leq \|p\|_\infty \left| \int_{\sigma(T)} 1 \, d\mu \right| = \|p\|_\infty \mu(\sigma(T)) = \|p\|_\infty \langle u, v \rangle . \end{aligned}$$

Conversely, without loss of generality we may assume that there exist nonzero vectors u, v such that $\|u\| = \|v\| = 1$, $\langle u, v \rangle \geq 0$ and

$$|\langle p(T^*, T) u, v \rangle| \leq \langle u, v \rangle \|p\|_\infty , \qquad p \in C[\bar{z}, z].$$

Define $\tau : \mathbb{C}[\bar{z}, z] \to \mathbb{C}$ by

$$\tau(p(\bar{z}, z)) = \langle p(T^*, T) u, v \rangle .$$

Then τ is obviously linear and $\tau(1) = \langle u, v \rangle \geq 0$. So τ is positive. By the Hahn Banach theorem, there exists a continuous linear mapping $\tau_{ext} : C(\sigma(T)) \to \mathbb{C}$ such that $\tau_{ext}|_{\mathbb{C}[\bar{z}, z]} = \tau$ and $\|\tau_{ext}\| = \|\tau\| = \tau(1)$, which implies that τ_{ext} is positive. By Riesz representation theorem, there exists a positive Borel measure μ supported on $\sigma(T)$ such that

$$\tau_{ext}(p(\bar{z}, z)) = \int_{\sigma(T)} p(\bar{z}, z) d\mu.$$

Take $p(\bar{z}, z) = \bar{z}^i z^j$, $i, j \in \mathbb{N}_0$ and since

$$\langle p(T^*, T) u, v \rangle = \tau(p(\bar{z}, z)) = \tau_{ext}(p(\bar{z}, z)),$$

we have that

$$\langle T^{*j} T^k u, v \rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu, \quad j, k \in \mathbb{N}_0.$$

Hence T has a *-moment sequence supported on $\sigma(T)$. ■

Corollary 2.3. *If an operator T has a *-moment sequence on $\sigma(T) \subset \mathbb{C}$, then also T^* has a *-moment sequence supported on $\overline{\sigma(T)} = \sigma(T^*)$.*

The following corollary follows immediately from Theorem 2.2 and Corollary 2.3.

Corollary 2.4. *If an operator T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$ and $p(\bar{z}, z)$ is a polynomial, then $p(T)$ has a *-moment sequence supported on $p(\sigma(T)) \cup \overline{p(\sigma(T))}$.*

Remark 2.5. If an operator T has two nonzero invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ with $\mathcal{M} \perp \mathcal{N}$, then T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$. (*Indeed,* take $u \in \mathcal{M}$ and $v \in \mathcal{N}$. Then $T^k u \in \mathcal{M}$ and $T^j v \in \mathcal{N}$. Then the zero measure μ satisfies

$$\langle T^{*j} T^k u, v \rangle = \langle T^k u, T^j v \rangle = 0 = \int_{\sigma(T)} \bar{z}^j z^k d\mu, \quad j, k \in \mathbb{N}_0$$

for any set $\sigma(T)$. So we have this remark.)

The following Proposition 2.6 improves Proposition 1.6 in [FJKP].

Proposition 2.6. *Let $T \in \mathcal{L}(\mathcal{H})$. Suppose that $\{u_n\}$ and $\{v_n\}$ are sequences in $\mathcal{H}$ converging in norm to nonzero vectors u_0 and v_0, respectively, and for every $n \in \mathbb{N}_0$ there exists a Borel measure μ_n supported on the compact set $\sigma(T) \subset \mathbb{C}$ such that*

$$\langle T^{*j} T^k u_n, v_n \rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu_n, \; n \in \mathbb{N}_0, \; j, k \in \mathbb{N}_0.$$

Then there exists a Borel measure μ_0 supported on compact set $\sigma(T) \subset \mathbb{C}$ such that

$$\langle T^{*j} T^k u_0, v_0 \rangle = \int_{\sigma(T)} \bar{z}^j z^k d\mu_0, \; n \in \mathbb{N}_0, \; j, k \in \mathbb{N}_0.$$

3. Remarks and Problems

We close this note an open problem and related some remarks. Recall from [FJKP] that if an operator of the form $T = N + K$, where N is a normal operator and K is a compact operator, then T has a moment sequence in version of [FJKP]. But we do not know the following.

Problem 3.1. Let $T = N + K$, where N is a normal operator and K is a compact operator. Does T have a *-moment sequence supported on $\sigma(T)$?

Let $\mathbf{K} := \mathbf{K}(\mathcal{H})$ be the set of compact operators on $\mathcal{H}$. Let $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathbf{K}$ be the Calkin map. Problem 3.1 is so interesting because of the following remark.

Remark 3.2. Assume that Problem 3.1 is true. If an operator T is essentially normal (i.e., $\pi(T)$ is normal), then T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$ or T has a nontrivial invariant subspace. (*Indeed*, by BDF-theorem ([BDF]), if T is biquasitriangular, T has a *-moment sequence supported on $\sigma(T) \subset \mathbb{C}$. If T is not biquasitriangular, by AFV-theorem ([AFV]), T has a nontrivial invariant subspace.)

Remark 3.3. An operator $T \in \mathcal{L}(\mathcal{H})$ is called *almost hyponomal* if $T^*T - TT^*$ can be written as $P + K$, where $P \geq 0$ and $K \in \mathcal{C}_1(\mathcal{H})$, the ideal of trace-class operators in $\mathcal{L}(\mathcal{H})$ (cf. [V]). It follows from [V] that if $T \in \mathcal{L}(\mathcal{H})$ is almost hyponormal, $X \in \mathcal{C}_2(\mathcal{H})$, where $\mathcal{C}_2(\mathcal{H})$ is the Hilbert-Schmidt class, and $T^*T - TT^* \notin \mathcal{C}_1(\mathcal{H})$, then $T + X$ has a nontrivial invariant subspace. Thus, if every operator in $\mathcal{L}(\mathcal{H})$ of the form $T + X$, where T is almost hyponormal and $X \in \mathcal{C}_2(\mathcal{H})$ admits a *-moment sequence or has a nontrivial invariant subspace. (*Indeed*, if $T^*T - TT^* \notin \mathcal{C}_1(\mathcal{H})$, $T + X$ has a nontrivial invariant subspace. If $T^*T - TT^* \in \mathcal{C}_1(\mathcal{H})$, $T + X$is essentially normal. By Remark 3.2, $T + X$ has a *-moment sequence or $T + X$ has a nontrivial invariant subspace.)

The following is an open problem which is more general 3.1.

Problem 3.4. Does every essential normal operator have *-moment sequence ?

Finally, we close this note with the following open problem.

Problem 3.5. Let T be an invertible operator in $\mathcal{L}(\mathcal{H})$ with a *-moment sequence. Does T^{-1} have *-moment sequence ?

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