数理解析研究所講究録
第 1607 巻 2008 年 54-64

Eigenvalues of Dirac operators at the thresholds

YOSHIMI SAITŌ

Department of mathematics,

University of Alabama at Birmingham, USA

This article is based on the talk given by the author at the meeting "Spectral and Scattering Theory and Related Topics" held at Research Institute for mathematical Sciences Kyoto University (2008.1.15 $\sim$ 1.17).

The talk consisted of the following seven sections:

§1. Dirac operators.

§2. Limiting absorption principle for the free Dirac operator H_0.

§3. Singular integral operator A.

§4. Asymptotic boundedness of zero modes of $H = H_0 + Q$.

§5. Asymptotic limit of zero modes of $H = H_0 + Q$.

§6. Eigenfunctions at the thresholds of Dirac operator with mass $m > 0$.

§7. Dirac-Sobolev inequality and zero modes.

- §1 $\sim$ §6 are based on the joint work with Tomio Umeda (The University of Hyogo, Japan):

  [SU1] Y. Saitō and T. Umeda, The zero modes and zero resonances of massless Dirac operators, to appear in Hokkaido Mathematical Journal.

  [SU2] Y. Saitō and T. Umeda, The asymptotic limits of zero modes of massless Dirac operators, Letters in Mathematical Physics, 83 (2008), 97-106.

  [SU3] Y. Saitō and T. Umeda, Eigenfunctions at the threshold energies of Dirac operators with positive mass, Preprint. 2008.

- §7 are based on the joint work A. A. Balinsky and W. D. Evans (Cardiff University, Wales, U.K.):

  [BES] A. Balinsky, W. D. Evans and Y. Saitō, Dirac-Sobolev inequalities and estimates for the zero modes of massless Dirac operators, to appear in J. Mathematical Physics.

- You can find information of this and related topics in the references of the above papers.

1 Dirac operators

1.1. Massless Dirac operators H.

- Massless Dirac operators. The massless Dirac operator H is (formally) defined by

$$H = \alpha \cdot D + Q(x), \quad D = \frac{1}{i}\nabla_x, \quad x \in \mathbb{R}^3,$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is the triple of 4×4 Dirac matrices

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix} \qquad (j = 1, 2, 3)$$

with the 2×2 zero matrix $\mathbf{0}$ and the triple of 2×2 Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and $Q(x)$ is a 4×4 Hermitian matrix-valued function decaying at infinity.

The free Dirac operator H_0 is given by

$$H_0 = \alpha \cdot D\,.$$

Thus we have (formally) $H = H_0 + Q(x)$.

- Weyl-Dirac operators. Define the operator H_A by

$$H_A = \alpha \cdot \big(D - A(x)\big),$$

where $A(x) = (A_1(x), A_2(x), A_3(x))$ is an magnetic potential. The operator H_A has the form

$$\alpha \cdot \big(D - A(x)\big) = \begin{pmatrix} \mathbf{0} & \sigma \cdot (D - A(x)) \\ \sigma \cdot (D - A(x)) & \mathbf{0} \end{pmatrix}.$$

The component $H_w = \sigma \cdot (D - A(x))$ is called the Weyl-Dirac operator.

1.2. Dirac operators H_m with mass m > 0. The Dirac operators with mass $m > 0$ are (formally) defined by

$$\begin{cases} H_{m,A} = \alpha \cdot (D - A(x)) + m\beta, \\ H_{m,A,Q} = \alpha \cdot (D - A(x)) + m\beta + Q(x), \end{cases}$$

where β is a 4×4 matrix given by

$$\beta = \begin{pmatrix} I_2 & \mathbf{0} \\ \mathbf{0} & -I_2 \end{pmatrix}$$

with 2×2 identity matrix I_2.

1.3. Some background.

1) The zero resonances and zero modes play dominant roles in the asymptotic behavior of the resolvent $(H-z)^{-1}$ as $z \to 0$, cf. Jensen-Kato [17] for the Schrödinger operator.

2) As is shown in the works by Fröhlich-Lieb-Loss [16] and Loss-Yau [19], the existence of a pair of a vector potential $A(x) \in [L^6]^3$ and the zero modes of the corresponding Weyl-Dirac operator is equivalent to the stability of the Coulomb system with magnetic field described by the Pauli operator.

3) It has been known that the study of the zero modes of the Dirac operator has important implication to quantum electrodynamics as has been mentioned in the recent works by Adam-Muratori-Nash [1], [2] and [3].

1.4. Self-adjoint realization of the Dirac operators.

- **Assumption.** Here and in the sequel (up to the end of §5) it is assumed that each element $q_{jk}(x)$ $(j,\ k = 1, \cdots, 4)$ of $Q(x)$ is a measurable function satisfying
$$|q_{jk}(x)| \le C\langle x\rangle^{-\rho} \quad (\rho > 1),$$
where C is a positive constant. In the case of the operator H_A we assume that each element $\widetilde{q}_{jk}(x)$ of $-\alpha \cdot A(x) + Q(x)$ satisfies the same condition as in $q_{jk}(x)$.

- Function spaces $\mathcal{L}^2$ and $\mathcal{H}^1$. We set $\mathcal{L}^2 = [L^2(\mathbb{R}^3)]^4$ with inner product
$$(f, g)_{\mathcal{L}^2} := \sum_{j=1}^4 (f_j,\ g_j)_{L^2(\mathbb{R}^3)}$$
$$(f = {}^t(f_1, f_2, f_3, f_4),\ g = {}^t(g_1, g_2, g_3, g_4) \in \mathcal{L}^2).$$
Similarly we set $\mathcal{H}^1(\mathbb{R}^3) = [H^1(\mathbb{R}^3)]^4$ with inner product
$$(f,\ g)_{\mathcal{H}^1} = \sum_{j=1}^4 (f_j,\ g_j)_{H^1(\mathbb{R}^3)}$$
$$(f = {}^t(f_1, f_2, f_3, f_4),\ g = {}^t(g_1, g_2, g_3, g_4) \in \mathcal{H}^1(\mathbb{R}^3)).$$

- **Proposition.** *The operators H_0, H, $H_{m,A}$ and $H_{m,A,Q}$ defined on $\mathcal{H}^1$ are self-adjoint operators in $\mathcal{L}^2$.*

2 Limiting absorption principle for the free Dirac operator $\mathbf{H}_0$

2.1. vector-valued weighted L^2 spaces and weighted Sobolev spaces.

- Weighted spaces $\mathcal{L}^{2,s}$. For $s \in \mathbb{R}$ a vector-valued weighted L^2 space $\mathcal{L}^{2,s}$ is given by

$$\begin{cases} \mathcal{L}^{2,s} = [L^{2,s}(\mathbb{R}^3)]^4, \\ L^{2,s}(\mathbb{R}^3) := \{\, u \mid \langle x\rangle^s u \in L^2(\mathbb{R}^3)\,\}, \end{cases}$$

where $\langle x\rangle = \sqrt{1+|x|^2}$. The inner products $(u,\, v)_{L^{2,s}(\mathbb{R}^3)}$ of $L^{2,s}(\mathbb{R}^3)$ and $(f,\, g)_{\mathcal{L}^{2,s}}$ of $\mathcal{L}^{2,s}$ are defined by

$$\begin{cases} (u,\, v)_{L^{2,s}(\mathbb{R}^3)} := \int_{\mathbb{R}^3} \langle x\rangle^{2s} u(x)\, \overline{v(x)}\, dx, \\ (f,\, g)_{\mathcal{L}^{2,s}} := \sum_{j=1}^4 (f_j,\, g_j)_{L^{2,s}(\mathbb{R}^3)} \\ \qquad (f = {}^t(f_1, f_2, f_3, f_4),\ g = {}^t(g_1, g_2, g_3, g_4) \in \mathcal{L}^{2,s}, \end{cases}$$

respectively.

- Weighted Sobolev paces $\mathcal{H}^{\mu,s}$. For $\mu,\, s \in \mathbb{R}$ a vector-valued weighted Sobolev space $\mathcal{H}^{\mu,s}$ is given by

$$\begin{cases} \mathcal{H}^{\mu,s} = [H^{\mu,s}(\mathbb{R}^3)]^4, \\ H^{\mu,s}(\mathbb{R}^3) := \{\, u \in S'(\mathbb{R}^3) \mid \langle x\rangle^s \langle D\rangle^\mu u \in L^2(\mathbb{R}^3)\,\}, \end{cases}$$

where $\langle D\rangle = \sqrt{1-\Delta}$. The inner products $(u,\, v)_{H^{\mu,s}(\mathbb{R}^3)}$ of $H^{\mu,s}(\mathbb{R}^3)$ and $(f,\, g)_{\mathcal{H}^{\mu,s}}$ of $\mathcal{H}^{\mu,s}$ are defined by

$$\begin{cases} (u,\, v)_{H^{\mu,s}(\mathbb{R}^3)} := (\langle x\rangle^s \langle D\rangle^\mu u(x),\ \langle x\rangle^s \langle D\rangle^\mu v(x))_{L^2(\mathbb{R}^3)}, \\ (f,\, g)_{\mathcal{H}^{\mu,s}} := \sum_{j=1}^4 (f_j,\, g_j)_{\mathcal{H}^{\mu,s}(\mathbb{R}^3)} \\ \qquad (f = {}^t(f_1, f_2, f_3, f_4),\ g = {}^t(g_1, g_2, g_3, g_4) \in \mathcal{H}^{\mu,s}, \end{cases}$$

respectively. We have $\mathcal{H}^{0,s} = \mathcal{L}^{2,s}$.

2.2. Limiting absorption principle for the free Laplacian.

- The resolvent $(-\Delta - z)^{-1}$ of the free Laplacian $-\Delta$ can be expressed as

$$(-\Delta - z)^{-1} u(x) = \Gamma_0(z) u(x) = \int_{\mathbb{R}^3} \frac{e^{i\sqrt{z}|x-y|}}{4\pi|x-y|} u(y)\, dy, \qquad u \in L^2(\mathbb{R}^3)$$

for $z \in \mathbb{C} \setminus [0, +\infty)$, where $\mathrm{Im}\sqrt{z} > 0$.

- **Proposition** (limiting absorption principle for $-\Delta$). *Let*

$$\Pi_{(0,+\infty)} = \big(\mathbb{C} \setminus (0,+\infty)\big) \cup \{ z = \lambda + i0 \mid \lambda > 0 \} \cup \{ z = \lambda - i0 \mid \lambda > 0 \},$$

and let $s, s' > 1/2$ with $s + s' > 2$. Define $\widetilde{\Gamma}_0(z)$ for $z \in \Pi_{(0,+\infty)}$ by

$$\widetilde{\Gamma}_0(z) = \begin{cases} \Gamma_0(z) & \text{if } z \in \mathbb{C} \setminus [0,+\infty), \\ \Gamma_0^+(\lambda) & \text{if } z = \lambda + i0,\ \lambda \geq 0, \\ \Gamma_0^-(\lambda) & \text{if } z = \lambda - i0,\ \lambda \geq 0, \end{cases}$$

where

$$\Gamma_0^{\pm}(\lambda) = \lim_{\epsilon \downarrow 0} \Gamma_0(\lambda \pm i\epsilon)$$

Then $\widetilde{\Gamma}_0(z)$ is well-defined and continous on $\Pi_{(0,+\infty)}$ in $\mathbb{B}(H^{-1,s}\,;\,H^{1,-s'})$.

- **Lemma.** *Let $s, s' > 1/2$ and $s + s' > 2$. Then*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} \langle x \rangle^{-2s'} \frac{1}{|x-y|^2} \langle y \rangle^{-2s}\, dxdy < +\infty.$$

2.3. Limiting absorption principle for the free Dirac operator H_0

- Operator $\Omega_0^{\pm}(z)$. Let

$$\mathbb{C}_+ := \{ z \in \mathbb{C} \mid \operatorname{Im} z > 0 \}, \qquad \mathbb{C}_- := \{ z \in \mathbb{C} \mid \operatorname{Im} z < 0 \}.$$

Let $s, s' > 1/2$ with $s + s' > 2$. Then define a $\mathbb{B}(\mathcal{H}^{-1,s}\,;\,\mathcal{H}^{1,-s'})$-valued continuous functions $\Omega_0^+(z)$ on $\overline{\mathbb{C}}_+$ and $\Omega_0^-(z)$ on $\overline{\mathbb{C}}_-$ by

$$\Omega_0^{\pm}(z) = \widetilde{\Gamma}_0(z^2) \qquad (z \in \overline{\mathbb{C}}_{\pm}),$$

respectively, where $\widetilde{\Gamma}_0(z^2)$ should be interpreted as a copy acting on vector-valued function $f = {}^t(f_1, f_2, f_3, f_4)$ as

$$\widetilde{\Gamma}_0(z^2) f = {}^t(\widetilde{\Gamma}_0(z^2) f_1, \widetilde{\Gamma}_0(z^2) f_2, \widetilde{\Gamma}_0(z^2) f_3, \widetilde{\Gamma}_0(z^2) f_4).$$

In other words

$$\Omega_0^{\pm}(z) = \begin{cases} \Gamma_0(z^2) & \text{if } z \in \mathbb{C}_{\pm}, \\ \Gamma_0^{\pm}(\lambda^2) & \text{if } z = \lambda \geq 0, \\ \Gamma_0^{\mp}(\lambda^2) & \text{if } z = \lambda \leq 0. \end{cases}$$

Note that $\Omega_0^+(0) = \Omega_0^-(0) = \widetilde{\Gamma}_0(0)$.

- **Proposition.** *Let s, μ be in $\mathbb{R}$. Then,* $H_0 \in \mathbb{B}(\mathcal{H}^{\mu,s}\,;\,\mathcal{H}^{\mu-1,s})$.

- **Proposition** (limiting absorption principle for H_0). *Let $R_0(z)$, $z \in \mathbb{C}_\pm$, be the resolvent of the free Dirac operator. Let s, $s' > 1/2$, and $s + s' > 2$. Then $R_0(z) \in \mathbb{B}(\mathcal{H}^{-1,s}\,;\,\mathcal{H}^{0,-s'})$ is continuous in $z \in \mathbb{C}_\pm$. Moreover, they can possess continuous extensions $R_0^\pm(z)$ to $\overline{\mathbb{C}}_\pm$, respectively, as $\mathcal{B}(-1,s\,;\,0,-s')$-valued functions, and*

$$R_0^\pm(z) = (H_0 + z)\Omega_0^\pm(z), \quad z \in \overline{\mathbb{C}}_\pm.$$

In particular,

$$R_0^+(0) = R_0^-(0) = H_0\widetilde{\Gamma}(0) \quad \text{in } \mathbb{B}(\mathcal{H}^{-1,s}\,;\,\mathcal{H}^{0,-s'}).$$

3 Singular integral operator A

3.1. Singular integral operator A

- The operator A. Define the Singular integral operator A by

$$Af(x) = \int_{\mathbb{R}^3} i\,\frac{\alpha\cdot(x-y)}{4\pi|x-y|^3} f(y)\,dy$$

for $f(x) = {}^t(f_1(x), f_2(x), f_3(x), f_4(x))$.

- **Proposition.** *For $f \in \mathcal{L}^2$, $Af(x)$ is well-defined for a.e. $x \in \mathbb{R}^3$. The operator A satisfies $A \in \mathbb{B}(\mathcal{L}^2, \mathcal{L}^6)$ and $A \in \mathbb{B}(\mathcal{L}^{2,s}, \mathcal{L}^2)$ for $s \le 1$. Further, we have*

$$\|Af\|_{\mathcal{L}^\infty} \le C_{pq}\big(\|f\|_{\mathcal{L}^p} + \|f\|_{\mathcal{L}^q}\big) \qquad (f \in \mathcal{L}^p \cap \mathcal{L}^q),$$

where $1 < p < 3 < q < \infty$.

- Remark. By noting that the resolvent $R_0(z)$ of the free Dirac operator has an integral expressed

$$R_0(z)f(x) = \int_{\mathbb{R}^3} \Big(i\,\frac{\alpha\cdot(x-y)}{|x-y|^2} \pm z\,\frac{\alpha\cdot(x-y)}{|x-y|} + zI_4\Big)\frac{e^{\pm iz|x-y|}}{4\pi|x-y|} f(y)\,dy$$

for $z \in \mathbb{C}_\pm$ and $f \in \mathcal{S} = [S(\mathbb{R}^3)]^4$, the operator A can be (formally) seen as $A = R_0(0)$.

3.2. Identity $AH_0f = f$.

- **Lemma.** *Let s, $s' > 1/2$, and $s + s' > 2$. Then A can be continuously extended to an operator in $\mathbb{B}(\mathcal{H}^{-1,s}\,;\,\mathcal{H}^{0,-s'})$, and we have, for $f \in \mathcal{H}^{-1,s}$,*

$$R_0^+(0)f = R_0^-(0)f = Af \qquad \text{in } \mathcal{H}^{0,-s'}.$$

- **Proposition.** *Let $s > 1/2$. Then,*

$$H_0 A g = g$$

for all $g \in \mathcal{L}^{2,s}$.

- **Lemma (Jensen-Kato).** *Let $s > 1/2$. Then*

  (i) $(-\Delta)\widetilde{\Gamma}_0(0)g = g$ *for all* $g \in \mathcal{H}^{-1,s}$.

  (ii) $\widetilde{\Gamma}_0(0)(-\Delta)f = f$ *if* $f \in \mathcal{L}^{2,-3/2}$ *and* $(-\Delta)f \in \mathcal{H}^{-1,s}$.

- **Lemma.** *Let $s > 1/2$. Then $\widetilde{\Gamma}_0(0)H_0\,g = Ag$ for all $g \in \mathcal{L}^{2,s}$.*

- **Theorem.** *If $f \in \mathcal{L}^{2,-3/2}$ and $H_0 f \in \mathcal{L}^{2,s}$ for some $s > 1/2$, then $AH_0f = f$.*

- Remark. Note that we have $H_0 f(x) = -Q(x)f(x)$ when f is a resonance or zero mode of a massless Dirac operator H. Thus the above theorem will used to give an integral expression

$$f(x) = -\int_{\mathbb{R}^3} i\,\frac{\alpha\cdot(x-y)}{4\pi|x-y|^3}\,Q(y)f(y)\,dy$$

for f (see §4 and §5).

4 Asymptotic boundedness of zero modes of $\mathbf{H = H_0 + Q}$

- **Theorem.** *Suppose that $Q(x) = O(|x|^{-\rho})$ $(\rho > 1)$ is satisfied. Let f be a zero mode of the operator the massless Dirac operator H. Then*

  (i) *the inequality*

$$|f(x)| \le C\langle x\rangle^{-2}$$

  holds for all $x \in \mathbb{R}^3$, where the constant $C(= C_f)$ depends only on the zero mode f;

  (ii) *the zero mode f is a continuous function on $\mathbb{R}^3$.*

- **Lemma.** *We have*

$$\int_{\mathbb{R}^3} \frac{1}{|x-y|^2 \langle y \rangle^\gamma}\, dy \le C_\gamma \begin{cases} \langle x \rangle^{-\gamma+1} & \text{if } 1<\gamma<3, \\ \langle x \rangle^{-2} \log(1+\langle x \rangle) & \text{if } \gamma = 3, \\ \langle x \rangle^{-2} & \text{if } \gamma > 3. \end{cases}$$

- Sketch of the proof of the theorem: We have

f is a zero mode

$$\Longrightarrow f \in \mathcal{L}^2 \cap \mathcal{L}^6 \quad \text{(Proposition in 3.1)}$$

$$\Longrightarrow \|f\|_\infty < \infty \quad \text{(Proposition in 3.1)}$$

$$\Longrightarrow f = O(\langle x \rangle^{-\rho+1}) \quad \text{(the above lemma).}$$

Then we can repeat this argument.

- **Theorem.** *Suppose that $Q(x) = O(|x|^{-\rho})$ with with $\rho > 3/2$. If f belongs to $\mathcal{L}^{2,-s}$ for some s with $0 < s \le \min\{3/2,\, \rho - 1\}$ and satisfies $Hf = 0$ in the distributional sense, then $f \in \mathcal{H}^1$.*

5 Asymptotic limit of zero modes of $\mathbf{H = H_0 + Q}$.

- **Theorem.** *Suppose that $|Q(x)| \le C\langle x \rangle^{-\rho}$ with $\rho > 1$. Let f be a zero mode of the massless Dirac operator H. Then for any $\omega \in \mathbb{S}^2$*

$$\lim_{r\to+\infty} r^2 f(r\omega) = -\frac{i}{4\pi}(\alpha\cdot\omega)\int_{\mathbb{R}^3} Q(y)f(y)\,dy,$$

where the convergence being uniform with respect to $\omega \in \mathbb{S}^2$.

- Sketch of the proof. It follows from the integral equation $f = -AQf$ that

$$f(x) = -\frac{i}{4\pi}\int_{\mathbb{R}^3} \frac{\alpha\cdot(x-y)}{|x-y|^3} Q(y)f(y)\,dy,$$

which implies that

$$r^2 f(r\omega) = -\frac{i}{4\pi}\int_{\mathbb{R}^3} \frac{\alpha\cdot(\omega - r^{-1}y)}{|\omega - r^{-1}y|^3} Q(y)f(y)\,dy.$$

Thus we have only to show that

$$r^2 f(r\omega) + \frac{i}{4\pi}\int_{\mathbb{R}^3} (\alpha\cdot\omega)\,Q(y)f(y)\,dy$$

$$= \frac{i}{4\pi}\int_{\mathbb{R}^3} \alpha\cdot\left\{\omega - \frac{\omega - r^{-1}y}{|\omega - r^{-1}y|^3}\right\} Q(y)f(y)\,dy \to 0$$

as $r \to \infty$.

- <u>Corollary.</u> *For any* $\omega \in \mathbb{S}^2$

$$\lim_{r\to+\infty} r^2|f(r\omega)| = \frac{1}{4\pi}\left|\int_{\mathbb{R}^3} Q(y)f(y)\,dy\right|.$$

6 Eigenfunctions at the thresholds of Dirac operator with mass m > 0

- <u>Dirac operators H_w and $H_{m,A}$</u> For $m > 0$ let

$$\begin{cases} H_{m,A} = \alpha\cdot(D - A(x)) + m\beta \quad (\mathcal{D}(H_{m,A}) = \mathcal{H}^1[H^1(\mathbb{R}^3)]^4), \\ H_w = \sigma\cdot(D - A(x)) \quad (\mathcal{D}(H_w) = [H^1(\mathbb{R}^3)]^2). \end{cases}$$

- <u>Theorem.</u> *Assume that* $A(x) = {}^t(A_1(x), A_2(x), A_3(x))$ *is a real measurable vector-valued function such that*

$$|A(x)| \le C\langle x\rangle^{-\rho} \qquad (x \in \mathbb{R}^3)$$

with constants $C > 0$ *and* $\rho > 1$. *Then,* $H_{m,A}$ *and* H_w *are selfadjoint and*

$$\begin{cases} \mathrm{Ker}\,(H_{m,A} - m) = \mathrm{Ker}\,(H_w) \oplus \{0\}\,, \\ \mathrm{Ker}\,(H_{m,A} + m) = \{0\} \oplus \mathrm{Ker}\,(H_w)\,. \end{cases}$$

In other words, let $f = {}^t(\psi_+, \psi_-) \in \mathcal{D}(H_{m,A})$ *such that* $\psi_\pm \in [H^1(\mathbb{R}^3)]^2$. *Then,* f *is an eigenfunction of* $H_{m,A}$ *associated with the eigenvalue* m $[\ -m\ ]$ *if and only if* $\psi_- = 0$ $[\ \psi_+ = 0\ ]$ *and* ψ_+ $[\ \psi_-\ ]$ *is a zero mode of* H_w.

- <u>Some extensions</u> The above theorem can be extended in the following cases:
  (1) Case that $A(x) \in [L^3(\mathbb{R}^3)]^3$ (cf. Balinsky-Evans [2001, 02, 03]).
  (2) Case that $A(x) = o(|x|^{-1})$ (cf. Elton [2002]).

7 Dirac-Sobolev inequality and zero modes

7.1. Transformation of the Dirac operator H by the involution.

- Involution.

$$Inv : \mathbb{R}^3 \setminus B_1 \ni x \to y = \frac{x}{|x|^2} \in B_1,$$

where B_1 is the unit ball with center the origin. We have

$$\frac{\partial(x_1, x_2, x_3)}{\partial(y_1, y_2, y_3)} = -|y|^6 .$$

- The map M through Involution Inv. Defined the map M by

$$M : \psi(x) \to (M\psi)(y) = \widetilde{\psi}(y) = \psi\Big(\frac{y}{|y|^2}\Big) \qquad (y \in B_1),$$

where ψ is a function on $\mathbb{R}^3 \setminus B_1$. Note that $\widetilde{\psi} = \psi \circ Inv^{-1}$.

- Map $\mathbf{\Psi(y) = -X(y)^{-1}\widetilde{\psi}}$. Let $X(y)$ be a unitary matrix in $\mathbb{C}^4$ given by

$$X(y) = \begin{pmatrix} X_0(y) & 0 \\ 0 & X_0 \end{pmatrix},$$

where

$$X_0(y) = \begin{pmatrix} -i\omega_3 & \omega_2 + i\omega_1 \\ \omega_2 - i\omega_1 & i\omega_3 \end{pmatrix} \qquad (\omega = y/|y|),$$

and consider the transformation

$$\Psi(y) = -X(y)^{-1}\widetilde{\psi}.$$

- Proposition. *We have*

$$M\{(\alpha \cdot D)\psi\}(y) = |y|^2 x(y)\{\alpha \cdot D_y)\Psi(y) + Y(y)\Psi(y),$$

where

$$Y(y) = \sum_{k=1}^{3} \alpha_k X(y)^{-1} \left(\frac{1}{i}\frac{\partial}{\partial y_k} X(y)\right).$$

Consequently, for a week solution ψ of $H\psi = 0$ on $\mathbb{R}^3 \setminus B_1$, Ψ defined above satisfies (weakly)

$$(\alpha \cdot D_y)\Psi(y) + Z(y)\Psi(y) = 0 \qquad (\text{in } B_1),$$

where

$$Z(y) = Y(y) - |y|^{-2} X(y)^{-1} \widetilde{Q}(y) X(y).$$

- Remark. Note that $Y(y) = O(|y|^{-1})$ at $y = 0$, and $Z(y) = O(|y|^{-1})$ at $y = 0$ if $Q(x) = O(|x|^{-1})$ at $x = \infty$.

7.2. Dirac-Sobolev inequalities.

- Space $\mathcal{H}^{1,p}_{0,d}(\Omega)$, $\Omega \subset \mathbb{R}^3$.

$\mathcal{H}^{1,p}_{0,d}(\Omega)$:= completion of $[C_0^\infty(\Omega)]^4$ with respect to the norm

$$\|f\|_{d,1,p;\Omega} := \left\{ \int_\Omega (|(\alpha \cdot D)f|^p + |f|^p)dx \right\}^{1/p}.$$

- **Theorem.** *Let Ω be bounded, $1 \le p < q < \infty$ and $r := 3(\frac{q}{p} - 1) \in [1,p]$. If $f \in \mathcal{H}^{1,p}_{0,d}(\Omega)$, then we have that for any $k \in (0,q)$ and $\theta = p/q$*

$$\|f\|_{k,\Omega} \le C\|(\alpha \cdot D)f\|^{\theta}_{p,\Omega}\|f\|^{1-\theta}_{r,B_1}.$$

- **Remark 1.** (i) The proof is inspired by a work by M. Ledoux, "On improved Sobolev embedding theorems" (Mathematical Research Letters, **10** (2003)).

- **Corollary.** *Let $1 \le p < \infty$. Then, for $k \in [1,p]$, we have*

$$\|f\|_{k,\Omega} \le C\|(\alpha \cdot D)f\|_{p,\Omega} \qquad (f \in \mathcal{H}^{1,p}_{0,d}(\Omega)).$$

- **Remark 2.** For $p = 2$ the above inequality is the same as the usual Poincaré inequality.

7.3. Estimate for zero modes.

- **Theorem 1.** *Let $Q(x) = O(|x|^{-1})$ in B_1^c. Let $\psi \in L^2(B_1^c)$ such that $(\alpha \cdot D)\psi \in L^2(B_1^c)$ and ψ is a solution of $((\alpha \cdot D) + Q(x))\psi = 0$. Then, by setting $\phi(x) = |x|^2\psi(x)$, we have*

$$\int_{B_1^c} |\phi(x)|^k|x|^{-6}dx < \infty$$

for any $k \in [1, 10/3)$.

- **Theorem 2.** *Let $\phi^{(t)}(y) = |x|^{2+t}\psi(x)$. Then*

$$\int_{B_1^c} |\phi^{(t)}(x)|^k|x|^{-6}dx < \infty$$

for any $k \in [1, 4/3)$ and $t < 11/10$.

- **Remark.** The result of this theorem does not look as good as the one in §4 though the method is quite different and the assumption on $Q(x)$ allows a Coulomb type $Q(x)$.