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On Entangled Markov Chains

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Abstract

Entangled Markov chains, which can be recognized as quantum Markov chains in the sense of [1], were introduced by Accardi and Fidaleo in order to extend the notion of classical random walk to quantum systems. In [3], using our entanglement criterion for pure states [16] which is based on the notion of the quantum mutual information, we proved that the vector states defining the EMC's on infinite tensor products of matrix algebra "generically" are entangled.

On the other hand we showed that this entanglement condition for pure state is sufficient condition for entanglement in the case of mixtures [4]. This fact was then applied to prove that EMC with unitarily implementable transition operator induce a mixture entangled state on any local algebra.

Interestingly it was also shown that all these local states provide a new class of examples (in any dimension) of entangled states which nevertheless satisfy the PPT condition [4].

In this article we report the above results.

1 Introduction and preliminaries

In the recent development of quantum information many people have discussed the problem of finding a satisfactory quantum generalization of the classical random walks. Motivated by such situation Accardi and Fidaleo introduced the notion of entangled Markov chains which includes that of quantum random walk [2]. They listed requirement that should be fulfilled by any candidate definition of a quantum random walk.

(1) It should be a quantum Markov chain in the sense of [1] (locality),

(2) it should be purely generated in the sense of [10] (pure entanglement),

(3) its restriction on at least one maximal abelian subalgebra, should be a classical random walk (quantum extension property),

(4) it should be uniquely determined, up to arbitrary phases, by its classical restriction (amplitude condition).

In order to give an intuitive idea of the connection of their construction with entanglement, let us note that the key characteristic of entanglement is the superposition principle and the corresponding interpretation of the amplitudes as "complex square roots of probabilities". This suggest an approach in which, given a homogeneous classical Markov chain with finite state space S, determined by a stochastic matrix P and an initial distribution described by a row vector p, one can construct such a quantum Markov chain. The construction is as follows.

We consider a classical Markov chain (S_n) with state space $S = \{1, 2, \cdots, d\}$, initial distribution $p = (p_j)$ and transition probability matrix $P = (p_{ij})$ (i.e. $p_{ij} \geq 0$, $\sum\limits_j p_{ij} = 1$). Let $\{|e_i\rangle\}_{i \leq d}$ be an orthogonal basis (ONB for short) of $\mathbb{C}^{|S|}$. For fixed a vector $|e_0\rangle$ in this basis, denote

$$\mathcal{H}_{\mathbb{N}} := \bigotimes_{\mathbb{N}}^{(|e_0\rangle)} \mathbb{C}^{|S|} \tag{1}$$

the infinite tensor product of $\mathbb{N}$-copies of the Hilbert space $\mathbb{C}^{|S|}$ with respect to the constant sequence (e_0). An orthogonal basis of $\mathcal{H}_{\mathbb{N}}$ is given by the vectors

$$|e_{j_0}, \cdots, e_{j_n}\rangle := \left(\otimes_{\alpha \in [0,\, n]} |e_{j_\alpha}\rangle\right) \otimes \left(\otimes_{\alpha \in [0,\, n]^c} |e_0\rangle\right).$$

Note: For any Hilbert space $\mathcal{H}$ we denote $\mathcal{H}^*$ its dual and $\xi \in \mathcal{H} \mapsto \xi^* \in \mathcal{H}^*$ the canonical embedding. Thus, if $\xi \in \mathcal{H}$ is a unit vector, $\xi\xi^*$ denotes the projection onto the subspace generated by ξ.

Let M_d denote the $d \times d$ complex matrix algebra and let $\mathcal{A} := M_d \otimes M_d \otimes \cdots = \bigotimes\limits_{\mathbb{N}} M_d$ be the C*-infinite tensor product of $\mathbb{N}$-copies of M_d.

An element $A_\Lambda \in \mathcal{A}$ (observable) will be said to be localized in a finite region $\Lambda \subseteq \mathbb{N}$ if there exists an operator $\overline{A}_\Lambda \in \otimes_\Lambda M_d$ such that

$$A_\Lambda = \overline{A}_\Lambda \otimes 1_{\Lambda^c}.$$

In the following we will identify $A_\Lambda = \overline{A}_\Lambda$ and we denote $\mathcal{A}_\Lambda$ the local algebra at Λ.

Let $\sqrt{p_i}$ (resp. $\sqrt{p_{ij}}$) $(\in \mathbb{C})$ be any complex square root of p_i (resp. p_{ij}) (i.e. $\left|\sqrt{p_i}\right|^2 = p_i$ (resp. $\left|\sqrt{p_{ij}}\right|^2 = p_{ij}$)). Define the vector

$$|\Psi_n\rangle = \sum_{j_0, \cdots, j_n} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} \sqrt{p_{j_\alpha j_{\alpha+1}}}\, |e_{j_0}, \cdots, e_{j_n}\rangle. \tag{2}$$

Although the limit $\lim\limits_{n\to\infty} |\Psi_n\rangle$ will not exist the basic property of $|\Psi_n\rangle$ is the following:

Lemma 1 *There exists a unique quantum Markov chain ψ on $\mathcal{A}$ such that, for every $k \in \mathbb{N}$ and for ever $A \in \mathcal{A}_{[0,\, k]}$, one has*

$$\langle \Psi_{k+1},\ A\Psi_{k+1}\rangle = \lim_{n\to\infty} \langle \Psi_n,\ A\Psi_n\rangle =: \psi(A). \tag{3}$$

Moreover ψ is stationary if and only if the associated classical Markov chain $\{p = (p_i),\ P = (p_{ij})\}$ is stationary, i.e. for any j

$$\sum_i p_i p_{ij} = p_j. \tag{4}$$

Accardi and Fidaleo [2] called "entangled Markov chains" the family of quantum Markov chains that can be obtained by the above construction. However they did not prove that such quantum Markov chains are entangled. In [3], using the degree of entanglement (DEN for short) obtained in [16], we proved that EMC ψ in (3) "generically" satisfies the entanglement condition in terms of our criterion (see Definitions below).

On the other hand, using the PPT (Positive Partial Transpose) criterion [11, 18], Miyadera showed [14] that the finite volume restrictions of a class of EMC on infinite tensor products of 2×2 matrix algebras is indeed entangled. On the one hand we showed that the degree of entanglement gives the sufficient condition for entanglement in the case of mixtures (for pure states this condition is necessary and sufficient) [4]. This fact allows us to prove that the restriction of EMC's, generated by a unitarily implementable provides a new class of examples (in any dimension) of entangled states which nevertheless satisfy the PPT condition [4]. In that argument we use an another criterion which is the recently established equivalence between the Blelavkin-Ohya and PPT condition [12].

2 Notions of multiple entanglement and degree of entanglement

Definition 2 *Let $\mathcal{A}_j$ $(j \in \{1, 2, \cdots, n\})$ with $n < \infty$ be C^*-algebras and let $\mathcal{A} = \bigotimes_{j=1}^{n} \mathcal{A}_j$ be a tensor product of C^*-algebras. A state $\omega \in \mathcal{S}\left(\bigotimes_{j=1}^{n} \mathcal{A}_j\right)$ is called separable if*

$$\omega \in \overline{\mathbf{Conv}}\left\{\bigotimes_{j=1}^{n} \omega_j; \omega_j \in \mathcal{S}(\mathcal{A}_j), j \in \{1, 2, \cdots, n\}\right\}$$

where $\overline{\mathbf{Conv}}$ denotes norm closure of the convex hull.

A non-separable state is called entangled.

Notice that the notion of separability may depend on the choice the tensor product of C*-algebras. Unless otherwise specified, one realizes the C*-algebras on Hilbert spaces and one considers the induced tensor product. In any case a separable pure state must be a product of pure states.

Definition 3 *[3] In the notations of Definition 2 a state $\omega \in \mathcal{S}(\mathcal{A})$ is called 2-separable if*

$$\omega \in \overline{\mathbf{Conv}}\left\{\omega_{k]} \otimes \omega_{(k}\ ;\ \omega_{k]} \in \mathcal{S}\left(\mathcal{A}_{k]}\right),\ \omega_{(k} \in \mathcal{S}\left(\mathcal{A}_{(k}\right)\right\}, \forall k \in \{1, 2, \cdots, n-1\}$$

where $\mathcal{A} = \mathcal{A}_{k]} \otimes \mathcal{A}_{(k} := \mathcal{A}_{[1,k]} \otimes \mathcal{A}_{(k,n]}$.

A non-2-separable state is called 2-entangled.

Remark 4 *Notice that, for* $n = 2$*, 2-entanglement is equivalent to usual entanglement. For* $n > 2$*, 2-entanglement is a strictly stronger property than usual entanglement.*

Definition 5 *Let* $\mathcal{H}_1$*,* $\mathcal{H}_2$ *be separable Hilbert spaces and let* θ *be a density matrix in* $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$*,* ρ *and* σ *be marginal densities of* θ *in* $\mathcal{B}(\mathcal{H}_1)$*,* $\mathcal{B}(\mathcal{H}_2)$ *respectively.*

The quantum quasi mutual entropy of ρ *and* σ *w.r.t* θ *[15] is defined by*

$$I_\theta(\rho, \sigma) := tr\theta(\log\theta - \log\rho \otimes \sigma). \tag{5}$$

The degree of entanglement of θ*, denoted by* $D_{EN}(\theta)$ *[16], is defined by*

$$D_{EN}(\theta) := \frac{1}{2}\{S(\rho) + S(\sigma)\} - I_\theta(\rho, \sigma) \tag{6}$$

where $S(\cdot)$ *is the von-Neumann entropy.*

In the following we identify normal states on $\mathcal{B}(\mathcal{H})$ ($\mathcal{H}$: some separable Hilbert space) with their density matrices and, if ρ is such a state, we will use indifferently notations

$$\rho(A) = tr\rho A, \ A \in \mathcal{B}(\mathcal{H}).$$

Recalling that, for density operators ρ and σ in $\mathcal{B}(\mathcal{H})$, the relative entropy (or the information divergency) of the state ρ with respect to a reference state σ is defined by

$$R(\rho|\sigma) := tr\rho(\log\rho - \log\sigma). \tag{7}$$

We see that the quasi mutual entropy is defined as the relative entropy of the compound state θ on $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with respect to the product state of its marginal states ρ and σ. This quantity, generalizing the classical mutual information corresponding to the case of Abelian algebras, describes an information gain in a quantum system $(\mathcal{B}(\mathcal{H}_1), \rho)$ or $(\mathcal{B}(\mathcal{H}_2), \sigma)$ via a compound state θ with a quantum correlation between ρ and σ. It is natural treated as a measure of the strength of the entanglement having zero value only for completely disentangled state $\theta = \rho \otimes \sigma$. Using the quasi mutual entropy we can define the entanglement criterion as a kind of symmetrized quantum conditional entropy by (6). In the classical case the conditional entropy always takes non-negative value, however our new criterion D_{EN} can be negative according to the strength of quantum correlation between ρ and σ. Actually the degree of entanglement D_{EN} has good properties to judge the separability of compound state as follows (see Appendices):

Theorem 6 *[3, 16] For a pure state* θ*,*

(1) θ *is separable iff* $D_{EN}(\theta) = 0$,

(2) θ *is not separable, i.e. entangled iff* $D_{EN}(\theta) < 0$.

Theorem 7 *[4] For a mixture state θ, if θ is separable, then $D_{EN}(\theta) \geq 0$.*
Equivalently: a sufficient condition for θ to be entangled is that $D_{EN}(\theta) < 0$.

The degree of entanglement criterion, being based on a numerical inequality, is in many case easier to verify than the positivity condition required by the PPT criterion.

3 EMC "generically" satisfies the entanglement condition

Throughout this paper we assume the stationality of the EMC ψ corresponding to the condition of (4).

The vector $|\Psi_n\rangle$ defined by (2) induces the state $|\Psi_n\rangle\langle\Psi_n|$ which can be recognized as a pure state on a local algebra $\mathcal{A}_{[0,n]}$. In order to measure the degree of entanglement of EMC ψ in (3) we define the D_{EN} of $|\Psi_n\rangle\langle\Psi_n|$ as follows:

$$\underline{D_{\mathrm{EN}}}(|\Psi_n\rangle\langle\Psi_n|) := \inf_{\mu\in[0,n-1]}\left\{\frac{1}{2}\left(S(\rho_{\mu]}) + S\left(\sigma_{(\mu}\right)\right) - I_{|\Psi_n\rangle\langle\Psi_n|}\left(\rho_{\mu]},\sigma_{(\mu}\right)\right\} \tag{8}$$

where $\rho_{\mu]}$ and $\sigma_{(\mu}$ are marginal states of the pure state $|\Psi_n\rangle\langle\Psi_n|$ with respect to the Hilbert space $\mathcal{H}_{\mu]} = \otimes_{j\in[0,\mu]}\mathcal{H}_j$ and $\mathcal{H}_{(\mu} = \otimes_{j\in(\mu,n]}\mathcal{H}_j$ respectively. Then the following definition introduces to a natural way to measure analytically the strength of entanglement of EMC ψ.

Definition 8 *Let ψ be the EMC in (3). The D_{EN} of ψ is defined by*

$$D_{EN}(\psi) := \lim_{n\to\infty}\underline{D_{EN}}(|\Psi_n\rangle\langle\Psi_n|) \tag{9}$$

Using the above definition we can "generically" estimate the entanglement of EMC ψ as follows:

Theorem 9 *[3] To the stochastic matrix P we associate the density matrix σ_P given as*

$$\sigma_P := \sum_i p_i |f_i\rangle\langle f_i|, \tag{10}$$

where $|f_i\rangle = \sum_k \sqrt{p_{ik}}\,|e_k\rangle$ and $p = (p_i)$ is the initial distribution of ψ. Then

(1) the state $|\Psi_n\rangle\langle\Psi_n|$ is a pure 2-separable state for any $n < \infty$ iff

$$S(\sigma_P) = 0.$$

(2) The state $|\Psi_n\rangle\langle\Psi_n|$ is a pure 2-entangled state for any $n < \infty$ iff

$$S(\sigma_P) > 0.$$

(3) There always exists the D_{EN} of ψ such that

$$-H(p) \leq D_{EN}(\psi) = -S(\sigma_P) \leq 0,$$

where $H(p)$ is the Shannon entropy of the probability measure p.

In the above theorem if the stochastic matrix $P = (p_{ij})$ is unitarily implementable, i.e. there exists a unitary matrix $U = (u_{ij})$ such that $\sqrt{p_{ij}} = u_{ij}$ for any i and j, then the set $\{|f_i\rangle\}$ giving the decomposition of σ_P by (10) becomes an ONB, i.e.

$$\langle f_j, f_i \rangle = \sum_k u_{jk}^* u_{ik} = (UU^*)_{ij} = \delta_{i,j}$$

where u_{jk}^* is the complex conjugate of u_{jk}. Thus the following corollary holds.

Corollary 10 *If EMC ψ is a stationary with a unitarily implementable matrix P, then the DEN of ψ exists and is equal to:*

$$D_{EN}(\psi) = -H(p) \tag{11}$$

where p is the initial distribution of ψ.

4 Entanglement of EMC generated by a unitarily implementable stochastic matrix on local algebra

We discuss the entanglement of the finite volume restrictions of a class of EMC on infinite tensor products of $d \times d$ matrix algebras. By restricting an EMC to some local algebra one obtains a mixed state to which our entanglement criterion D_{EN} is applicable because of Theorem 7. This allows to prove the restriction of EMC's, generated by a unitarily implementable $d \times d$ stochastic matrix, to algebras localized on arbitrary intervals are entangled.

Finally, using the recently established equivalence [12] between the Belavkin-Ohya entanglement condition and PPT entanglement condition we prove that the above mentioned restrictions of EMC's satisfy the PPT condition.

4.1 D_{EN} of localized density

Let denote the unitarily implementable EMC state restricted to a finite region $[0,n]$ by $\theta_{[0,n]}$, then for every local observable $A \in \mathcal{A}_{[0,n]}$ one has $\theta_{[0,n]}(A) = \langle \Psi_{n+1}, (A \otimes I)\Psi_{n+1} \rangle$. Its corresponding density operator $\theta_{[0,n]}$ is given by taking the partial trace as follows:

$$\begin{aligned}
\theta_{[0,n]} &= tr_{\mathcal{H}_{n+1}} |\Psi_{n+1}\rangle \langle \Psi_{n+1}| \\
&= \sum_{\substack{j_0, \cdots, j_n, l \\ i_0, \cdots, i_n}} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-1} u_{i_\alpha i_{\alpha+1}}^* u_{j_\alpha j_{\alpha+1}} \\
&\quad u_{i_n l}^* u_{j_n l} |e_{j_0}, \cdots, e_{j_n}\rangle \langle e_{i_0}, \cdots, e_{i_n}|
\end{aligned}$$

From the unitarity of $U = (u_{ij})$ one has $\sum_l u^*_{i_n l} u_{j_n l} = \delta_{i_n, j_n}$. So that

$$\begin{aligned}
\theta_{[0,n]} &= \sum_{\substack{j_0, j_1, \cdots, j_{n-1}, \\ i_0, i_1, \cdots, i_{n-1}, k}} \sqrt{p_{i_0}}^* \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-2} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}} \\
&\quad u^*_{i_{n-1}k} u_{j_{n-1}k} \left| e_{j_0}, e_{j_1}, \cdots, e_{j_{n-1}}, e_k \right\rangle \left\langle e_{i_0}, e_{i_1}, \cdots, e_{i_{n-1}}, e_k \right| \\
&= \sum_k p_k \left| e_{[0,n]}(k) \right\rangle \left\langle e_{[0,n]}(k) \right| \qquad (12)
\end{aligned}$$

where $\left| e_{[0,n]}(k) \right\rangle := \frac{1}{\sqrt{p_k}} \sum_{j_0, j_1, \cdots, j_{n-1}} \sqrt{p_{j_0}} \prod_{\alpha=0}^{n-2} u_{j_\alpha j_{\alpha+1}} u_{j_{n-1}k} \left| e_{j_0}, \cdots, e_{j_{n-1}}, e_k \right\rangle$. It is easy to check that $\left\{ \left| e_{[0,n]}(k) \right\rangle \right\}$ becomes an ONB. The normality is as follows:

$$\begin{aligned}
\left\| \left| e_{[0,n]}(k) \right\rangle \right\|^2 &= \frac{1}{p_k} \sum_{j_0, j_1, \cdots, j_{n-1}} p_{j_0} \prod_{\alpha=0}^{n-2} p_{j_\alpha j_{\alpha+1}} p_{j_{n-1}k} \\
&= \frac{1}{p_k} \sum_{j_{n-1}} p_{j_{n-1}} p_{j_{n-1}k} = \frac{p_k}{p_k} = 1.
\end{aligned}$$

The orthogonality of $\left\{ \left| e_{[0,n]}(k) \right\rangle \right\}$ is clear because of the orthogonality of $\{ |e_k\rangle \}$.

For any $\mu \in [0, n-1]$ the marginal states $\rho_{\mu]}$ with respect to the Hilbert space $\mathcal{H}_{\mu]}$ has the same decomposition form of $\theta_{[0,n]}$:

$$\begin{aligned}
\rho_{\mu]} &= tr_{\mathcal{H}(\mu} \theta_{[0,n]} \\
&= \sum_k p_k \left| e_{[0,\mu]}(k) \right\rangle \left\langle e_{[0,\mu]}(k) \right|. \qquad (13)
\end{aligned}$$

The another marginal state $\sigma_{(\mu}$ with respect to the Hilbert space $\mathcal{H}_{(\mu}$ is given by

$$\begin{aligned}
\sigma_{(\mu} &= tr_{\mathcal{H}_{\mu]}} \theta_{[0,n]} \\
&= \sum_{\substack{l, j_{\mu+1}, \cdots\cdots, j_{n-1}, \\ i_{\mu+1}, \cdots, i_{n-1}, k}} p_l u^*_{l i_{\mu+1}} u_{l j_{\mu+1}} \prod_{\alpha=\mu+1}^{n-2} u^*_{i_\alpha i_{\alpha+1}} u_{j_\alpha j_{\alpha+1}} \\
&\quad u^*_{i_{n-1}k} u_{j_{n-1}k} \left| e_{j_{\mu+1}}, \cdots, e_{j_{n-1}}, e_k \right\rangle \left\langle e_{i_{\mu+1}}, \cdots, e_{i_{n-1}}, e_k \right| \\
&= \sum_{l,k} p_l \left| e_{(\mu,n-1]}(l,k) \otimes e_k \right\rangle \left\langle e_{(\mu,n-1]}(l,k) \otimes e_k \right| \qquad (14)
\end{aligned}$$

where

$$\left| e_{(\mu,n-1]}(l,k) \right\rangle = \sum_{j_{\mu+1}, \cdots, j_{n-1},} u_{l j_{\mu+1}} \prod_{\alpha=\mu+1}^{n-2} u_{j_\alpha j_{\alpha+1}} u_{j_{n-1}k} \left| e_{j_{\mu+1}}, \cdots, e_{j_{n-1}} \right\rangle.$$

Put $\sigma_{(\mu}(l) = \sum_k \left|e_{(\mu,n-1]}(l,k) \otimes e_k\right\rangle \left\langle e_{(\mu,n-1]}(l,k) \otimes e_k\right|$, then $\sigma_{(\mu}(l)$ can be recognized as densities and

$$\sigma_{(\mu} = \sum_l p_l \sigma_{(\mu}(l) \tag{15}$$

Both decompositions (12) and (13) are Schatten decompositions. Therefore

$$S\left(\theta_{[0,n]}\right) = S\left(\rho_{\mu]}\right) = -\sum_k p_k \log p_k.$$

Before estimating the entropy of $\sigma_{(\mu}$ we recall the following lemma [17].

Lemma 11 *For a density operator ρ given as the convex combination*

$$\rho = \sum_l \lambda_l \rho_l,\ \lambda_l \geq 0, \sum_l \lambda_l = 1$$

of densities ρ_l, the following inequality holds:

$$S(\rho) \leq \sum_l \lambda_l S(\rho_l) - \sum_l \lambda_l \log \lambda_l. \tag{16}$$

The equality holds if $\rho_l \perp \rho_k$ for $l \neq k$.

According to the above lemma one has

$$S\left(\sigma_{(\mu}\right) \leq \sum_l p_l S\left(\sigma_{(\mu}(l)\right) - \sum_l p_l \log p_l. \tag{17}$$

For any number κ included in $(\mu, n-1]$ we separate the Hilbert space $\mathcal{H}_{[0,n]}$ as $\mathcal{H}_{[0,n]} = \mathcal{H}_{[0,\mu]} \otimes \mathcal{H}_{(\mu,\kappa]} \otimes \mathcal{H}_{(\kappa,n]}$. Then the strong subadditivity assert the following:

$$S\left(\theta_{[0,n]}\right) + S\left(\rho_{\mu]}\right) \leq S\left(\rho_{\kappa]}\right) + S\left(\sigma_{(\mu}\right).$$

The Schatten decomposition of $\rho_{\kappa]}$ is given by $\rho_{\kappa]} = \sum_k p_k \left|e_{[0,\kappa]}(k)\right\rangle \left\langle e_{[0,\kappa]}(k)\right|$. Therefor one has

$$-\sum_k p_k \log p_k \leq S\left(\sigma_{(\mu}\right). \tag{18}$$

Notice that the decomposition of $\sigma_{(n-1}$ is given by $\sum_k p_k \left|e_k\right\rangle \left\langle e_k\right|$ so that $S\left(\sigma_{(n-1}\right) = -\sum_k p_k \log p_k$. However the rank of $\sigma_{(\mu}(l)$ is bigger than one when μ is chosen from $[0, n-2]$. This fact means that $S\left(\sigma_{(\mu}(l)\right) > 0$. Summarizing the above argument we have the following theorem which means that $\theta_{[0,n]}$ is 2-entangled state (see also (24) in Appendix A):

Theorem 12 *[4]*

$$\underline{D_{EN}}\left(\theta_{[0,n]}\right) < 0. \tag{19}$$

4.2 Entanglement mapping on EMC

Let us briefly recall the Belavkin–Ohya entanglement condition [7, 8].

Let $\mathcal{H}$ and $\mathcal{K}$ be separable Hilbert spaces. Denote $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ the algebra of all bounded linear operators on $\mathcal{H} \otimes \mathcal{K}$ and let θ be a normal state on $\mathcal{B}(\mathcal{H} \otimes \mathcal{K})$.

The density operator θ satisfies, for all $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$, the identities:

$$\theta(A \otimes B) = tr_{\mathcal{H}\otimes\mathcal{K}}(A \otimes B)\,\theta = tr_{\mathcal{H}} A\,(tr_{\mathcal{K}}(I \otimes B)\,\theta) = tr_{\mathcal{K}}(tr_{\mathcal{H}}(A \otimes I)\,\theta)\,B. \tag{20}$$

Moreover the linear maps $\phi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{H})_*$, $\phi^* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})_*$ defined by

$$\phi^*(A) := tr_{\mathcal{H}}(A \otimes I)\,\theta \;\; ; \;\; \phi(B) := tr_{\mathcal{K}}(I \otimes B)\,\theta$$

(called *entanglements* in [7, 8]) are dual to each other with respect to the Hilbert–Schmidt scalar product:

$$tr_{\mathcal{H}} A\phi(B) = tr_{\mathcal{K}}\phi^*(A)\,B.$$

Both maps are completely co-positive (hence positive), but not always completely positive.

Theorem 13 *[7, 8] If θ is separable, then its entanglements ϕ and ϕ^* are completely positive.*

The equivalence between the above and the PPT condition was proved by Jamiołkowski, Matsuoka and Ohya [12].

Theorem 14 *ϕ or ϕ^* are completely positive if and only if the associated density operator θ satisfies the PPT condition.*

Now we apply these results to EMC [4].

Theorem 15 *For each $n \in \mathbb{N}$ and each $\mu \in [0, n-1]$ the state*

$$\theta_{[0,n]}\left(A_{[0,\mu]} \otimes B_{(\mu,n]}\right) = tr_{\mathcal{H}_{[0,\mu]}\otimes\mathcal{H}_{(\mu,\nu]}}\left(A_{[0,\mu]} \otimes B_{(\mu,n]}\right)\theta_{[0,n]}$$

is a PPT state i.e., its corresponding density operator $\theta_{[0,n]}$ satisfies the PPT condition.

Proof. From theorem 14 it is enough to prove that the operator ϕ^*, defined for any $A_{[0,\mu]}$ by

$$\phi^*(A_{[0,\mu]}) := tr_{\mathcal{H}_{[0,\mu]}}\left((A_{[0,\mu]} \otimes I)\theta_{[0,n]}\right) \tag{21}$$

is completely positive. By Choi's criterium [9] the complete positivity of ϕ^* is equivalent to the positivity of the operator

$$\sum_{i,j} \left|e_{[0,\mu]}(i)\right\rangle\left\langle e_{[0,\mu]}(j)\right| \otimes \phi^*\left(\left|e_{[0,\mu]}(i)\right\rangle\left\langle e_{[0,\mu]}(j)\right|\right) \quad \left(\in \mathcal{B}\left(\mathcal{H}_{[0,n]}\right)\right). \tag{22}$$

Using (12), (13), (14) the density operator can be represented as

$$\begin{aligned}\theta_{[0,n]} &= \sum_{i,j,k} \left|e_{[0,\mu]}(i)\right\rangle \left\langle e_{[0,\mu]}(j)\right| \otimes \\ &\quad \sqrt{p_i}\sqrt{p_j}^* \left|e_{(\mu,n-1]}(i,k) \otimes e_k\right\rangle \left\langle e_{(\mu,n-1]}(j,k) \otimes e_k\right| .\end{aligned}$$

Therefore we see that the operator (22) is equal to

$$\sum_{i,j,k} \left|e_{[0,\mu]}(i)\right\rangle \left\langle e_{[0,\mu]}(j)\right| \otimes \sqrt{p_j}\sqrt{p_i}^* \left|e_{(\mu,n-1]}(j,k) \otimes e_k\right\rangle \left\langle e_{(\mu,n-1]}(i,k) \otimes e_k\right| .$$

Then, for all $x_j = \sum_{j_{\mu+1},\cdots,j_n} a^j_{j_{\mu+1},\cdots,j_n} \left|e_{j_{\mu+1}},\cdots,e_{j_n}\right\rangle$, $a^j_{j_{\mu+1},\cdots,j_n} \in \mathbb{C}$, from the unitarity of $U = (u_{ij})$ one has

$$\begin{aligned}&\sum_{i,j,k} \left\langle x_i, \left(\sqrt{p_j}\sqrt{p_i}^* \left|e_{(\mu,n-1]}(j,k) \otimes e_k\right\rangle \left\langle e_{(\mu,n-1]}(i,k) \otimes e_k\right|\right) x_j \right\rangle \\ &= \sum_{j,k,j_{\mu+1},\cdots,j_{n-1}} p_j \left|a^j_{j_{\mu+1},\cdots,j_{n-1},k}\right|^2 \geq 0.\end{aligned}$$

This means that the operator (22) is positive. Thus ϕ^* is completely positive and so that $\theta_{[0,n]}$ satisfies the PPT condition. ■

Appendix A

If θ on $\mathcal{H} \otimes \mathcal{K}$ is an entangled pure state with marginal states ρ, σ, then von Neumann entropy $S(\theta) = 0$. Moreover, from the Araki-Lieb inequality [6]:

$$|S(\rho) - S(\sigma)| \leq S(\theta) \leq S(\rho) + S(\sigma), \tag{23}$$

the purity of θ implies that $S(\rho) = S(\sigma)$. In general it follows

$$\begin{aligned}I_\theta(\rho,\sigma) &= tr\theta(\log\theta - \log\rho\otimes\sigma) \\ &= tr\theta\log\theta - tr\theta\log\rho\otimes I - tr\theta\log I\otimes\sigma \\ &= S(\rho) + S(\sigma) - S(\theta).\end{aligned} \tag{24}$$

In the case of a pure state θ, $D_{\mathrm{EN}}(\theta)$ can be computed as

$$\begin{aligned}D_{\mathrm{EN}}(\theta) &= \frac{1}{2}\{S(\rho) + S(\sigma)\} - I_\theta(\rho,\sigma) \\ &= S(\rho) - 2S(\rho) \\ &= -S(\rho) \quad (\text{or } = -S(\sigma))\end{aligned} \tag{25}$$

If $D_{EN}(\theta) < 0$, then $S(\rho) = S(\sigma) > 0$ which means that ρ and σ are mixture states. Therefore ρ can be written as $\rho = \sum_i \lambda_i |x_i\rangle\langle x_i|$ where $\{|x_i\rangle\}$ is an ONB in $\mathcal{H}$ and $\sum_i \lambda_i = 1$, $0 \leq \lambda_i \leq 1$ and at least two λ_i are strictly positive. Then due to the Schmidt decomposition there exists an ONB $\{|y_i\rangle\}$ of $\mathcal{K}$ such that θ is given by $\theta = |\Psi\rangle\langle\Psi|$ where

$$|\Psi\rangle = \sum_i \sqrt{\lambda_i}\, |x_i\rangle \otimes |y_i\rangle .$$

Since at least two λ_i are strictly positive, this implies that θ is a pure entangled state. The converse statement obviously holds.

If $D_{\mathrm{EN}}(\theta) = 0$, then $S(\rho) = S(\sigma) = 0$ which means that ρ and σ are pure states respectively. Thus θ is a pure state whose marginals are pure states. This implies that θ is a product of pure states. Conversely, if θ is pure and separable, then it is the product of two pure states, hence $D_{\mathrm{EN}}(\theta) = 0$.

Appendix B

In order to prove theorem 7 we review the monotonicity property of relative entropy of the state ρ with respect to a reference state σ defined in [5, 13, 19] even more general von Neumann algebra $\mathcal{M}$. Its monotonicity property, i.e. nonincrease of the $R(\rho|\sigma)$ after the application of the pre-dual of a normal completely positive unital map $\Lambda : \mathcal{M} \to \mathcal{M}^0$ to the states ρ_0 and σ_0 on a von Numann algebra $\mathcal{M}^0$ is stated as follows [13, 19]:

$$\rho = \rho_0 \Lambda,\ \sigma = \sigma_0 \Lambda \ \Rightarrow\ R(\rho|\sigma) \le R(\rho_0|\sigma_0) . \tag{26}$$

Let θ_s be a separable state on $\mathcal{B}(\mathcal{H}\otimes\mathcal{K})$ with its density θ_s given by

$$\theta_s = \sum_n p_n \rho_n \otimes \sigma_n .$$

Moreover we define the diagonal separable state θ_d as a special case of θ_s, i.e. its density $\theta_{d(\mathcal{H})}$ with respect to $\mathcal{H}$ is given by

$$\theta_{d(\mathcal{H})} = \sum_n p_n |x_n\rangle\langle x_n| \otimes \sigma_n ,$$

where $\{|x_n\rangle\}$ is an ONB in $\mathcal{H}$. Their quasi mutual entropies are defined by

$$I_{\theta_s}(\rho,\sigma) = tr\theta_s(\log\theta_s - \log\rho\otimes\sigma),$$

$$\begin{aligned} I_{\theta_{d(\mathcal{H})}}(\rho_d,\sigma) &= tr\theta_{d(\mathcal{H})}(\log\theta_{d(\mathcal{H})} - \log\rho_d\otimes\sigma) \\ &= \sum_n p_n tr\sigma_n(\log\sigma_n - \log\sigma), \end{aligned}$$

where $\rho = \sum_n p_n\rho_n$, $\sigma = \sum_n p_n\sigma_n$ and $\rho_d = \sum_n p_n |x_n\rangle\langle x_n|$. Then we can introduce the CP map Λ given by

$$\Lambda(A\otimes B) = \sum_n p_n |x_n\rangle\, trA\rho_n \langle x_n| \otimes B,\quad A\otimes B \in \mathcal{B}(\mathcal{H}\otimes\mathcal{K}) \tag{27}$$

into $\mathcal{M}^0 \otimes \mathcal{B}(\mathcal{K})$ where $\mathcal{M}^0$ denotes the diagonal sub-algebra in $\mathcal{B}(\mathcal{H})$. So that due to the monotonicity of relative entropy we have

$$I_{\theta_s}(\rho, \sigma) \leq I_{\theta_{d(\mathcal{H})}}(\rho_d, \sigma). \tag{28}$$

From the inequality (26) it is immediately shown that conditional entropies of θ_s and θ_d satisfy the following :

$$S(\theta_s; \sigma) \geq S\left(\theta_{d(\mathcal{H})}; \sigma\right) = -\sum_n p_n tr \sigma_n \log \sigma_n \geq 0, \tag{29}$$

where $S(\theta; \sigma) = S(\sigma) - I_\theta(\rho, \sigma)$.

For the diagonal separable density $\theta_{d(\mathcal{K})} = \sum_n p_n \rho_n \otimes |y_n\rangle\langle y_n|$ with respect to $\mathcal{K}$, where $\{|y_n\rangle\}$ is an ONB in $\mathcal{K}$, using same argument above we have

$$S(\theta_s; \rho) \geq S\left(\theta_{d(\mathcal{K})}; \rho\right) = -\sum_n p_n tr \rho_n \log \rho_n \geq 0. \tag{30}$$

From (29) and (30) theorem 7 is shown as

$$D_{EN}(\theta_s) = \frac{1}{2}\left(S(\theta_s; \sigma) + S(\theta_s; \rho)\right) \geq 0. \tag{31}$$

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