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Koenderink type theorems for fronts

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We shall study Koenderink type theorems for surfaces with singularities. We state them on the terminology of wave fronts and singular curvature measure of cuspidal edges.

1 Introduction

In 1984 and 1990, J. J. Koenderink showed theorems that relates to how one actually sees a surface. Let $f : M^2 \longrightarrow \boldsymbol{R}^3$ be a non-singular smooth surface in Euclidean three-space $\boldsymbol{R}^3$, let $\pi : \boldsymbol{R}^3 \longrightarrow P$ be the orthogonal projection onto a plane P and let $\pi_{\boldsymbol{o}} : \boldsymbol{R}^3 \longrightarrow S$ be the central projection onto a unit sphere centered at $\boldsymbol{o}$. We call $f(S(\pi \circ f))$ (resp. $f(S(\pi_{\boldsymbol{o}} \circ f))$) the *rim* of M as *viewed* by π (resp. $\pi_{\boldsymbol{o}}$) and $\pi \circ f(S(\pi \circ f))$ (resp. $\pi \circ f(S(\pi_{\boldsymbol{o}} \circ f))$) *apparent contour* of $f(M)$ as viewed by π (resp. $\pi_{\boldsymbol{o}}$). Koenderink showed the following:

Theorem 1.1 ([8]). *Suppose $p \in S(\pi \circ f)$. Let κ_1 be the curvature of the plane curve $\pi(S(\pi \circ f))$, let κ_2 be the curvature of the normal section of $f(M)$ at p by the plane that contains the kernel of π and let K be the Gaussian curvature of $f(M)$. Then*

$$K = \kappa_1 \kappa_2$$

holds at p.

Let κ_3 be the geodesic curvature of the curve $\pi_p(S(\pi_p \circ f))$ and let d be the distance of p from $\boldsymbol{o}$. Then

$$K = \kappa_3 \kappa_2 / d$$

holds at p.

Quite independently this was considered by T. Gaffney and M. Ruas [4]. For a unified approach, see J. W. Bruce and P. J. Giblin [2]. See also [7]. If f has a singular point, generically the Gaussian curvature is unbounded. Thus this type theorem does not hold at the singular point of f. Recently, the author, M. Umehara and K. Yamada showed if f be a *front*, then the *Gaussian curvature form* $K d\hat{A}$ is bounded and introduce the *singular curvature function* on cuspidal edge singularities of fronts [9]. Using these notions, we can extend the above theorem as follows:

Theorem 1.2. *Let $f : M \longrightarrow \boldsymbol{R}^3$ be a smooth map, $\gamma : I \longrightarrow M$ be a cuspidal edge, $\hat{\gamma} = f \circ \gamma$ and $p \in \operatorname{Im} \gamma$. Set $\xi_p = \nu_p \times \hat{\gamma}'/|\hat{\gamma}'|$ and $\boldsymbol{v}_\theta = \cos\theta \xi_p + \sin\theta \nu_p$. Let π_θ be the orthonormal projection with respect to $\boldsymbol{v}_\theta$ Suppose that P is a plane normal to $\hat{\gamma}'/|\hat{\gamma}'|$ and $\pi : \boldsymbol{R}^3 \longrightarrow P$ is the orthonormal projection. Let κ_1 be the curvature of the plane curve $\pi \circ \hat{\gamma}$, let κ_2 be the curvature of the normal section of $f(M)$ at p by the plane P. If $\theta \neq 0$ then*

$$K d\hat{A} = \frac{1}{\cos\theta} \left(\sin\theta \kappa_s - \kappa_1 \right) \kappa_2 \, du \wedge dv \tag{1}$$

holds at p, where κ_s is the singular curvature of cuspidal edge defined in Section 2.

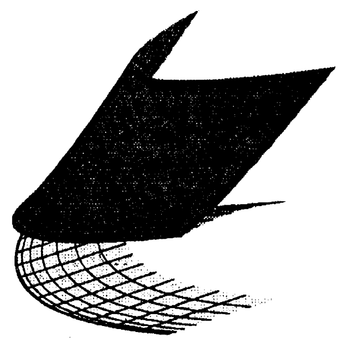

Figure 1: Projection of a front into the plane

We also have the following Koenderink type theorem:

Corollary 1.3. *In the above setting,*

$$K d\hat{A} = \frac{1}{\cos\theta} \left(\sin\theta \kappa_s - \kappa_g / d \right) \kappa_2 \, du \wedge dv \tag{2}$$

holds at p,

2 Cuspidal edges and the singular curvature

In this section, we review the notion of singular curvature given in [9]. A map $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^3$ is called a (*wave*) *front* if it is the projection of a Legendrian immersion

$$L_f : \boldsymbol{R}^2 \longrightarrow T_1^* \boldsymbol{R}^3$$

into the unit cotangent bundle and $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^3$ is called a *frontal* if it is the projection of an isotropic map into the unit cotangent bundle. Cuspidal cross cap is not a front but a frontal.

Let $(U; u, v)$ be a domain in $\boldsymbol{R}^2$ and $f : U \longrightarrow \boldsymbol{R}^3$ a front. Identifying the unit cotangent bundle with the unit tangent bundle $T_1\boldsymbol{R}^3 \sim \boldsymbol{R}^3 \times S^2$, there exists a unit vector field $\nu : U \longrightarrow S^2$ such that the Legendrian lift L_f is expressed as (f, ν). Since $L_f = (f, \nu)$ is Legendrian,

$$\langle df, \nu \rangle = 0 \text{ and } \langle \nu, \nu \rangle = 1$$

hold, where $\langle\ ,\ \rangle$ is the standard Euclidean inner product. Then there exists a function λ such that

$$f_u(u, v) \times f_v(u, v) = \lambda(u, v)\nu(u, v)$$

where $\times$ denotes the exterior product in $\boldsymbol{R}^3$ and $f_u = \partial f/\partial u$, for example. Obviously, $(u, v) \in U$ is a singular point of f if and only if $\lambda(u, v) = 0$.

Definition 2.1. A singular point $p \in \boldsymbol{R}^2$ of a front $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^3$ is *non-degenerate* if $d\lambda \neq 0$ holds at p.

By the implicit function theorem, for a non-degenerate singular point p, the singular set is parameterized by a smooth curve $c : (-\varepsilon, \varepsilon) \longrightarrow \boldsymbol{R}^2$ in a neighborhood of p. Since p is non-degenerate, any $c(t)$ is non-degenerate for sufficiently small t. Then there exists a unique direction $\eta(t) \in T_{c(t)}U$ up to scalar multiplication such that $df(\eta(t)) = 0$ for each t. We call $c'(t)$ the *singular direction* and $\eta(t)$ the *null-direction*. For further details in these notation, see [9].

It has been known the generic singularities of fronts $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^3$ are cuspidal edges and swallowtails [1]. In [6] it has been shown the following useful criteria for cuspidal edges:

Proposition 2.2 ([6],Proposition 1.3). *For a non-degenerate front $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^3$ with singularity at o. f at o is $\mathcal{A}$-equivalent to the cuspidal edge if and only if* $\det(c'(0), \eta(0)) \neq 0$.

The generic singularities of one-parameter fronts are cuspidal lips, cuspidal beaks, butterfly and $D_4^{\pm}$ singularities([1]). Useful criteria for cuspidal lips and cuspidal beaks are given in [5].

Moreover, a useful criteria of cuspidal cross cap is given in [3]. Using this criteria, singularities of maximal surfaces in the Minkowski space and constant mean curvature surfaces in the de Sitter space are investigated([3]).

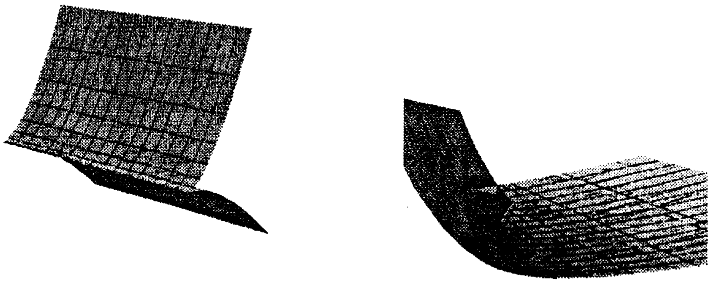

Figure 2: Cuspidal edge and swallowtail

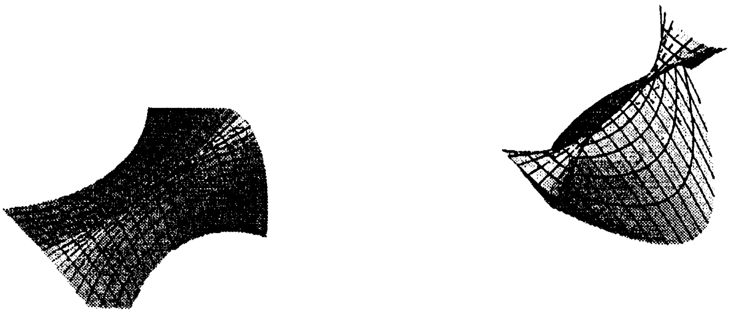

Figure 3: Cuspidal lips and cuspidal beaks

Recently, criteria for A_k-singularities of fronts in general dimensions are obtained ([10]).

In [9], we define the singular curvature on cuspidal edge. We suppose that a singular curve $\gamma(t)$ on $\boldsymbol{R}^2$ consists of cuspidal edges. Then we can choose the null vector fields $\eta(t)$ such that $\bigl(\gamma'(t), \eta(t)\bigr)$ is a positively oriented frame field along γ. We then define the *singular curvature function* along $\gamma(t)$ as follows:

$$\kappa_s(t) := \operatorname{sgn}\bigl(d\lambda(\eta)\bigr)\,\frac{\det\bigl(\hat{\gamma}'(t), \hat{\gamma}''(t), \nu\bigr)}{|\hat{\gamma}'(t)|^3}. \tag{3}$$

Here, we denote $\hat{\gamma}(t) = f(\gamma(t))$. For later computation, it is convenient to take a local coordinate system (u, v) centered at a given non-degenerate singular point $p \in M^2$ as follows:

- the coordinate system (u, v) is compatible with the orientation of M^2,
- the u-axis is the singular curve, and
- there are no singular points other than the u-axis.

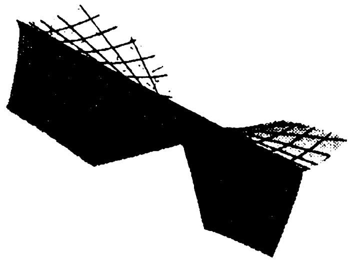

Figure 4: Cuspidal cross cap

We call such a coordinate system (u, v) an *adopted coordinate system* with respect to p. We take an adopted coordinate system (u, v) and write the null vector field $\eta(t)$ as

$$\eta(t) = a(t)\frac{\partial}{\partial u} + e(t)\frac{\partial}{\partial v}, \tag{4}$$

where $a(t)$ and $e(t)$ are C^∞-functions. Since (γ', η) is a positive frame, we have $e(t) > 0$. Here,

$$\lambda_u = 0 \qquad \text{and} \qquad \lambda_v \neq 0 \qquad \text{(on the } u\text{-axis)} \tag{5}$$

hold, and then $d\lambda\big(\eta(t)\big) = e(t)\lambda_v$. In particular, we have

$$\operatorname{sgn}\big(d\lambda(\eta)\big) = \operatorname{sgn}(\lambda_v) = \begin{cases} +1 & \text{if the left-hand side of } \gamma \text{ is } M_+, \\ -1 & \text{if the left-hand side of } \gamma \text{ is } M_-. \end{cases} \tag{6}$$

So we have the following expression: in an adopted coordinate system (u, v),

$$\kappa_s(u) := \operatorname{sgn}(\lambda_v)\frac{\mu_g(f_u, f_{uu}, \nu)}{|f_u|^3}, \tag{7}$$

where $f_{uu} = \partial^2 f/\partial u^2$. Difference between positivity and negativity of the singular curvature relates the following two types of cuspidal edges. The left-hand figure in Figure 5 is positively curved and the right-hand figure is negatively curved.

Now we set

$$d\hat{A} := f^*(\iota_\nu \mu_g) = \lambda(u, v)\, du \wedge dv \tag{8}$$

called the *signed area form*.

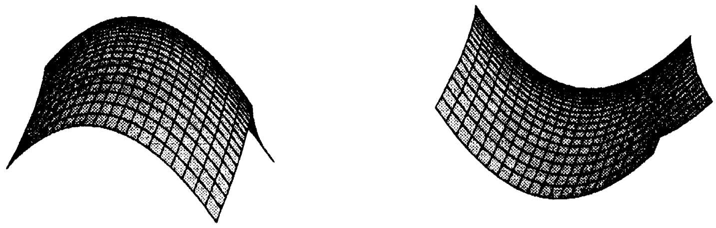

Figure 5: Positively and negatively curved cuspidal edges.

Proposition 2.3 ([9]). *Let $f : \boldsymbol{R}^2 \longrightarrow \boldsymbol{R}^3$ be a front, and K the Gaussian curvature of f which is defined on the set of regular points of f. Then $K\,d\hat{A}$ can be continuously extended as a globally defined 2-form on $\boldsymbol{R}^2$, where $d\hat{A}$ is the signed area form as in* (8).

This also holds for plane curves. Let $c : I \longrightarrow \boldsymbol{R}^2$ be a front, and κ the curvature of c. By the same method one can show that $\kappa\,ds$ can be continuously extend as a globally defined 1-form on I, where ds is the arclength measure. Using (7) and direct calculations, we have Theorem 1.2 and Corollary 1.3. To investigate singularities on projections of fronts and their curvatures and topologies are our future problems.

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