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A successive approximation method for solving a Lipschitz optimization problem

新潟大学大学院自然科学研究科　山田修司 (Syuuji YAMADA)
Graduate School of Science and Technology, Niigata University
新潟大学大学院自然科学研究科　田中環 (Tamaki TANAKA)
Graduate School of Science and Technology, Niigata University
大阪大学大学院工学研究科　谷野哲三 (Tetsuzo TANINO)
Graduate School of Engineering, Osaka University

Abstract. In this paper, we propose a successive approximation method for solving a Lipschitz optimization problem. The algorithm is based on the outer approximation method proposed by Thach and Tuy (1987). It is shown that the proposed algorithm have the global convergence.

Keywords: Global Optimization, Lipschitz Optimization, Outer Approximation Method.

1 Introduction

In this paper, we consider an optimization problem with a Lipschitz continuous function to be minimized over a compact convex set in the n-dimensional Euclidean space $\mathbb{R}^n$. It is called Lipschitz optimization problem (LOP) and known that many global optimization problems can be converted into (LOP). We propose an algorithm for solving (LOP), which is based on the outer approximation method proposed by Thach and Tuy (1987). However, it is difficult to solve the given original (LOP) by the algorithm directly. Hence, we transform the original problem into an optimization problem with a Lipschitz objective function to be minimized over the projection of the feasible set of (LOP) on a hemisphere in $\mathbb{R}^{n+1}$. The proposed algorithm utilizes an approximation of the hemisphere by a sequence of polytopes from outside. It is shown that every accumulation point for each sequence of provisional solutions becomes an optimal solution of the transformed problem.

The organization of this paper is as follows: In section 2, we introduce (LOP) and describe an equivalent problem to (LOP), where the equivalence is understood in the sense that both optimal values coincide. In section 3, we formulate an outer approximation algorithm for the latter problem and discuss the convergence of the algorithm.

Throughout this paper, we use the following notation: $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product in $\mathbb{R}^n$. For a subset $X \subset \mathbb{R}^n$, int X and bd X denote the interior and boundary sets of X, respectively. Given a convex polyhedral set (or polytope) $X \subset \mathbb{R}^n$, $V(X)$ denotes the set of all vertices of X. Given a function $f : \mathbb{R} \to \mathbb{R}$, $f'(a)$ denotes the derivative of f at $a \in \mathbb{R}$. Given a convex function $f : \mathbb{R}^n \to \mathbb{R}$, $\partial f(\boldsymbol{x})$ denotes the subdifferential of f at $\boldsymbol{x}$, i.e., $\partial f(\boldsymbol{x}) := \{\boldsymbol{u} \in \mathbb{R}^n : \langle \boldsymbol{u}, \boldsymbol{y}-\boldsymbol{x}\rangle + f(\boldsymbol{x}) \leq f(\boldsymbol{y}),\ \boldsymbol{y} \in \mathbb{R}^n\}$. For $\varepsilon > 0$ ($\varepsilon \in \mathbb{R}$) and $\boldsymbol{x} \in \mathbb{R}^n$, $B^n_<(\boldsymbol{x},\varepsilon) := \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y}-\boldsymbol{x}\| < \varepsilon\}$, $B^n_=(\boldsymbol{x},\varepsilon) := \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y}-\boldsymbol{x}\| = \varepsilon\}$,

$B^n_{\leq}(\boldsymbol{x}, \varepsilon) := \{\boldsymbol{y} \in \mathbb{R}^n : \|\boldsymbol{y} - \boldsymbol{x}\| \leq \varepsilon\}$. Given a matrix $Q \in \mathbb{R}^{m \times n}$, $Q^\top$ denotes the transposed matrix of Q.

2 A Lipschitz Optimization Problem

In this paper, we propose a successive approximation method for solving the following Lipschitz optimization problem:

$$\text{(LOP)} \begin{cases} \text{minimize} & f(\boldsymbol{x}), \\ \text{subject to} & \boldsymbol{x} \in D := \{\boldsymbol{x} \in \mathbb{R}^n : g_i(\boldsymbol{x}) \leq 0, i = 1, \ldots, m\}, \\ & \boldsymbol{x} = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n, \end{cases}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz continuous on an open set $A \subset \mathbb{R}^n$ satisfying $A \supset D$, i.e., there exists $L > 0$ such that $|f(\boldsymbol{x}) - f(\boldsymbol{y})| \leq L\|\boldsymbol{x} - \boldsymbol{y}\|$ for each $\boldsymbol{x}, \boldsymbol{y} \in A$, and $g_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, m)$ are continuously differentiable convex functions. We note that the feasible set D is a convex set.

For (LOP), we assume the following conditions:

(A1) $\{\boldsymbol{x} \in \mathbb{R}^n : g_i(\boldsymbol{x}) < 0,\ i = 1, \ldots, m\} \neq \emptyset$ and D is compact.

(A2) $\boldsymbol{0} \in \operatorname{int} D$,

(A3) There exists $r > 0$ such that $D \subset B^n_{\leq}(\boldsymbol{0}, r) \subset A$.

From assumption (A1), D is nonempty and compact. Since the objective function is continuous, (LOP) has a globally optimal solution. Denote by min(LOP) the optimal value of (LOP). Let $g(\boldsymbol{x}) = \max_{i=1,\ldots,m} g_i(\boldsymbol{x})$. Then, $D = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) \leq 0\}$. Moreover, from the convexity of g, we have $\operatorname{int} D = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) < 0\} \neq \emptyset$ and $\operatorname{bd} D = \{\boldsymbol{x} \in \mathbb{R}^n : g(\boldsymbol{x}) = 0\} \neq \emptyset$. In the next section, we will propose an algorithm based on a cutting plane method for solving (LOP).

However, (LOP) has a difficulty as follows: Let $D(\alpha) := \{\boldsymbol{x} \in D : f(\boldsymbol{x}) \leq \alpha\} \neq \emptyset$ and $\boldsymbol{x}' \in \mathbb{R}^n$ satisfying $f(\boldsymbol{x}') > \alpha$ for some $\alpha \geq \min(\text{LOP})$. Since the convexity of $D(\alpha)$ is not guaranteed, a hyper plane strongly separating $\boldsymbol{x}'$ from $D(\alpha)$ does not always exist. To overcome such a difficulty spawned by the nonconvexity of $D(\alpha)$, we consider projections of D and $D(\alpha)$ on a hemisphere in $\mathbb{R}^{n+1}$ as follows:

$$\begin{aligned} &S(\alpha) := S \cap \Lambda(\alpha),\ (\alpha \in \mathbb{R}) \\ &S := \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \bar{g}_i(\boldsymbol{u}) \leq 0,\ i = 1, \ldots, m\} \cap B^{n+1}_{=}(\boldsymbol{0}, r) \cap \{\boldsymbol{u} \in \mathbb{R}^{n+1} : u_{n+1} \geq 0\}, \end{aligned}$$

where $\Lambda(\alpha) := \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \bar{f}(\boldsymbol{u}) \leq \alpha\}$, $\bar{f}(\boldsymbol{u}) := f((u_1, \ldots, u_n)^\top)$ and $\bar{g}_i(\boldsymbol{u}) := g_i((u_1, \ldots, u_n)^\top)$ for each $i = 1, \ldots, m$ $(\boldsymbol{u} = (u_1, \ldots, u_{n+1})^\top \in \mathbb{R}^{n+1})$. Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ satisfy $\pi(\boldsymbol{u}) = (\pi(\boldsymbol{u})_1, \ldots, \pi(\boldsymbol{u})_n)^\top = (u_1, \ldots, u_n)^\top$ for each $\boldsymbol{u} \in \mathbb{R}^{n+1}$. Then, we note that

- for each $\boldsymbol{u} \in S$, $\pi(\boldsymbol{u}) \in D$,
- for each $\boldsymbol{x} \in D$, $(x_1, \ldots, x_n, x_{n+1})^\top \in S$ where $x_{n+1} = \sqrt{r^2 - \|\boldsymbol{x}\|^2}$.

Therefore, (LOP) can be transformed into the following problem:

$$\text{(MP)} \begin{cases} \text{minimize} & \bar{f}(\boldsymbol{u}), \\ \text{subject to} & \boldsymbol{u} \in S. \end{cases}$$

Let $\bar{A} := A \times \mathbb{R} \subset \mathbb{R}^{n+1}$. Then, from the definition of $\bar{f}$, for each $\boldsymbol{u}', \boldsymbol{u}'' \in \bar{A}$,

$$|\bar{f}(\boldsymbol{u}') - \bar{f}(\boldsymbol{u}'')| = |f(\pi(\boldsymbol{u}')) - f(\pi(\boldsymbol{u}''))| \leq L\|\pi(\boldsymbol{u}') - \pi(\boldsymbol{u}'')\| \leq L\|\boldsymbol{u}' - \boldsymbol{u}''\|.$$

Hence, $\bar{f}$ is Lipschitz continuous. Moreover, for each i, $\bar{g}_i$ is convex. Let $\bar{D} := \{\boldsymbol{u} : \bar{g}(\boldsymbol{u}) \leq 0\}$ where $\bar{g}(\boldsymbol{u}) := \max_{i=1,\ldots,m} \bar{g}_i(\boldsymbol{u})$. Then, $\bar{D} = \emptyset$, int $\bar{D} = \{\boldsymbol{u} : \bar{g}(\boldsymbol{u}) < 0\} \neq \emptyset$ and bd $\bar{D} = \{\boldsymbol{u} : \bar{g}(\boldsymbol{u}) = 0\} \neq \emptyset$. By the definition of the feasible set S, S is compact and int $S = \emptyset$. Hence, (MP) has a globally optimal solution. Moreover, $S = (\text{co } S)\backslash B^{n+1}_{<}(\boldsymbol{0}, r)$ (R. Horst and H. Tuy (1990), Proposition XI.7). From the definition of the objective function $\bar{f}$, $\pi(\bar{\boldsymbol{u}})$ is a globally optimal solution of (LOP) if $\bar{\boldsymbol{u}}$ solves (MP). Furthermore, min(MP) = min(LOP).

3 An Outer Approximation Method

In this section, we propose an algorithm based on a cutting plane method for solving (MP). The algorithm renew a provisional solution by generating cutting planes at each iteration. It is shown that every accumulation point of the sequence of such provisional solutions is a globally optimal solution of (MP).

3.1 Cutting Planes

In this subsection, we explain procedures for generating cutting planes.

Let

$$\ell(\boldsymbol{u}, \boldsymbol{u}', \alpha) := \langle \boldsymbol{u}', \boldsymbol{u} \rangle - r^2 + \frac{1}{2}\left(\max\left\{0, \frac{\bar{f}(\boldsymbol{u}') - \alpha}{L}\right\}\right)^2. \tag{1}$$

Then, the following lemmas hold.

Lemma 3.1 *Let $\Lambda(\alpha) \cap B^{n+1}_{=}(\boldsymbol{0}, r) \neq \emptyset$ for some $\alpha \in \mathbb{R}$. Assume that $\boldsymbol{u}' \in B^{n+1}_{=}(\boldsymbol{0}, r)$ satisfies $\bar{f}(\boldsymbol{u}') > \alpha$. Then,*

$$\ell(\boldsymbol{u}', \boldsymbol{u}', \alpha) > 0,$$
$$\ell(\boldsymbol{u}, \boldsymbol{u}', \alpha) \leq 0, \quad \forall \boldsymbol{u} \in \Lambda(\alpha) \cap B^{n+1}_{=}(\boldsymbol{0}, r).$$

Proof. Since $\bar{f}(\boldsymbol{u}') - \alpha > 0$, we have

$$\begin{aligned}\ell(\boldsymbol{u}', \boldsymbol{u}', \alpha) &= \langle \boldsymbol{u}', \boldsymbol{u}' \rangle - r^2 + \frac{1}{2}\left(\max\left\{0, \frac{\bar{f}(\boldsymbol{u}') - \alpha}{L}\right\}\right)^2 \\ &= r^2 - r^2 + \frac{1}{2}\left(\frac{\bar{f}(\boldsymbol{u}') - \alpha}{L}\right)^2 = \frac{1}{2}\left(\frac{\bar{f}(\boldsymbol{u}') - \alpha}{L}\right)^2 > 0.\end{aligned}$$

Moreover, it follows that for each $\boldsymbol{u} \in \Lambda(\alpha) \cap B^{n+1}_{=}(\boldsymbol{0}, r)$,

$$\left(\frac{\bar{f}(\boldsymbol{u}') - \alpha}{L}\right)^2 \leq \left(\frac{\bar{f}(\boldsymbol{u}') - \bar{f}(\boldsymbol{u})}{L}\right)^2 \leq \|\boldsymbol{u}' - \boldsymbol{u}\|^2 = \langle \boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{u}' - \boldsymbol{u} \rangle.$$

Hence,

$$\ell(\boldsymbol{u}, \boldsymbol{u}', \alpha) \leq \langle \boldsymbol{u}', \boldsymbol{u} \rangle - \langle \boldsymbol{u}', \boldsymbol{u}' \rangle + \frac{1}{2}\langle \boldsymbol{u}' - \boldsymbol{u}, \boldsymbol{u}' - \boldsymbol{u} \rangle = \frac{1}{2}\left(\langle \boldsymbol{u}, \boldsymbol{u} \rangle - \langle \boldsymbol{u}', \boldsymbol{u}' \rangle\right) = \frac{1}{2}(r^2 - r^2) = 0.$$

This completes the proof. □

Lemma 3.2 *Let* $\boldsymbol{u}' \in \Lambda(\alpha) \cap B^{n+1}_{=}(\boldsymbol{0}, r)$ *for some* $\alpha \in \mathbb{R}$. *Then,*

$$B^{n+1}_{=}(\boldsymbol{0}, r) \subset \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \ell(\boldsymbol{u}, \boldsymbol{u}', \alpha) \leq 0\}.$$

Proof. Since $\bar{f}(\boldsymbol{u}') - \alpha \leq 0$, we have that for each $\boldsymbol{u} \in B^{n+1}_{=}(\boldsymbol{0}, r)$,

$$\ell(\boldsymbol{u}, \boldsymbol{u}', \alpha) = \langle \boldsymbol{u}', \boldsymbol{u} \rangle - r^2 \leq \|\boldsymbol{u}'\|\|\boldsymbol{u}\| - r^2 = 0.$$

This completes the proof. □

For each $\alpha' \in \mathbb{R}$ and $\boldsymbol{u}' \in \mathbb{R}^{n+1}$ satisfying $S(\alpha') \neq \emptyset$ and $\bar{f}(\boldsymbol{u}') > \alpha'$, from Lemma 3.1, we have

$$\begin{array}{ll} \ell(\boldsymbol{u}', \boldsymbol{u}', \alpha') > 0, & \\ \ell(\boldsymbol{u}, \boldsymbol{u}', \alpha') \leq 0, & \text{for each } \boldsymbol{u} \in S(\alpha'). \end{array} \tag{2}$$

Moreover, let

$$\rho(\boldsymbol{u}, \boldsymbol{t}', \boldsymbol{u}') := \psi(\boldsymbol{u}')\left(\langle \boldsymbol{t}', \boldsymbol{u} - \boldsymbol{u}' \rangle + \bar{g}(\boldsymbol{u}')\right), \tag{3}$$

where $\boldsymbol{u}' \in \mathbb{R}^{n+1}$, $\boldsymbol{t}' \in \partial \bar{g}(\boldsymbol{u}')$ and

$$\psi(\boldsymbol{u}') := \begin{cases} 1, & \text{if } \boldsymbol{u}' \notin \bar{D}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $\boldsymbol{u}' \notin \bar{D}$,

$$\begin{array}{ll} \rho(\boldsymbol{u}', \boldsymbol{t}', \boldsymbol{u}') > 0, & \\ \rho(\boldsymbol{u}, \boldsymbol{t}', \boldsymbol{u}') \leq 0, & \text{for each } \boldsymbol{u} \in \bar{D}. \end{array} \tag{4}$$

3.2 Formulation of the Algorithm

For solving (MP), we propose an outer approximation algorithm as follows:

Algorithm OA

Step 0. Set $\boldsymbol{z}^0 = \boldsymbol{w}^1 := (0, \ldots, 0, r)^\top$, $\alpha_1 := \bar{f}(\boldsymbol{w}^1)$, $P_0 := \{\boldsymbol{u} \in \mathbb{R}^{n+1} : (-r, \ldots, -r, 0)^\top \leq \boldsymbol{u} \leq (r, \ldots, r)^\top\}$, $P_1 := P_0 \cap \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \ell(\boldsymbol{u}, \boldsymbol{z}^0, \alpha_1) \leq 0\}$. Calculate the vertex sets $V(P_1)$ of P_1. Select $\boldsymbol{v}^1 \in \arg\max\{\|\boldsymbol{v}\| : \boldsymbol{v} \in V(P_k)\}$. Set $k = 1$ and go to step 1.

Step 1. If $\|\boldsymbol{v}^k\| \leq r$ (that is, $P_k \subset B^{n+1}_{\leq}(\boldsymbol{0}, r)$), then stop: $\boldsymbol{w}^k$ and $\pi(\boldsymbol{w}^k)$ are globally optimal solutions of (MP) and (LOP), respectively. Otherwise, go to step 2.

Step 2. Set $\boldsymbol{z}^k := \dfrac{r}{\|\boldsymbol{v}^k\|}\boldsymbol{v}^k$, $\boldsymbol{t}^k \in \partial \bar{g}(\boldsymbol{z}^k)$,

$$\boldsymbol{w}^{k+1} := \begin{cases} \boldsymbol{z}^k, & \text{if } \boldsymbol{z}^k \in S \text{ and } \bar{f}(\boldsymbol{z}^k) < \alpha_k, \\ \boldsymbol{w}^k, & \text{otherwise,} \end{cases}$$

$$\alpha_{k+1} := \begin{cases} \bar{f}(\boldsymbol{z}^k), & \text{if } \boldsymbol{z}^k \in S \text{ and } \bar{f}(\boldsymbol{z}^k) < \alpha_k, \\ \alpha_k, & \text{otherwise,} \end{cases}$$

$$P_{k+1} := P_0 \cap \{\boldsymbol{u} \in \mathbb{R}^{n+1} : \ \ell(\boldsymbol{u}, \boldsymbol{z}^i, \alpha_{k+1}) \leq 0, \ \rho(\boldsymbol{u}, \boldsymbol{t}^i, \boldsymbol{z}^i) \leq 0, \ i = 0, \ldots, k\}.$$

Select $\boldsymbol{v}^{k+1} \in \arg\max\{\|\boldsymbol{v}\| : \boldsymbol{v} \in V(P_{k+1})\}$. Set $k \leftarrow k + 1$ and return to step 1.

From the definitions of P_k, α_k and $\boldsymbol{w}^k$, we notice that

$$\begin{array}{l} P_1 \supset P_2 \supset \cdots \supset P_k \supset \cdots, \\ \alpha \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \cdots, \\ \bar{f}(\boldsymbol{w}^1) \geq \bar{f}(\boldsymbol{w}^2) \geq \cdots \geq \bar{f}(\boldsymbol{w}^k) \geq \cdots \geq \min(\mathrm{MP}), \\ f(\pi(\boldsymbol{w}^1)) \geq f(\pi(\boldsymbol{w}^2)) \geq \cdots \geq f(\pi(\boldsymbol{w}^k)) \geq \cdots \geq \min(\mathrm{LOP}). \end{array} \tag{5}$$

Moreover, by Theorem 3.3, every accumulation point of $\{\boldsymbol{w}^k\}$ is a globally optimal solution of (MP). Furthermore by Corollary 3.1, every accumulation point of $\{\pi(\boldsymbol{w}^k)\}$ solves (LOP). However, in many cases, the stopping condition $\|\boldsymbol{v}^k\| \leq r$ in step 1 does not hold, that is, algorithm OA is not finished. Therefore, in order to terminate after finite number of iterations, we set a tolerance $\tau > 0$ and replace step 1 of algorithm OA as follows:

Step 1. If $\|\boldsymbol{v}^k\| \leq r + \tau$, then stop: $\boldsymbol{w}^k$ and $\pi(\boldsymbol{w}^k)$ are approximate solutions of (MP) and (LOP), respectively. Otherwise, go to step 2.

Then, by Theorem 3.2, algorithm OA certainly terminates after finite number of iterations. Moreover, for (MP) and (LOP), approximate solutions contained in the feasible sets can be obtained.

3.3 Global Convergence

In this subsection, in a case that algorithm OA does not terminate after a finite number of iterations, we shall show that if an infinite sequence $\{\boldsymbol{w}^k\}$ is generated by algorithm OA, then every accumulation point of $\{\pi(\boldsymbol{w}^k)\}$ is a globally optimal solution of (LOP).

Theorem 3.1 *Assume that $\{\boldsymbol{z}^k\}$ generated by algorithm* OA *is infinite. Then, every accumulation point of $\{\boldsymbol{z}^k\}$ is contained in S.*

Proof. We note that $B_{=}^{n+1}(\mathbf{0}, r)$ is compact and that $\{\boldsymbol{z}^k\} \subset B_{=}^{n+1}(\mathbf{0}, r)$. Hence, without loss of generality, we can assume that $\boldsymbol{z}^k \to \bar{\boldsymbol{z}}$ as $k \to \infty$. Then, from the definition of $\boldsymbol{z}^k$, $\bar{\boldsymbol{z}} \in B_{=}^{n+1}(\mathbf{0}, r) \cap \{\boldsymbol{z} \in \mathbb{R}^{n+1} : z_{n+1} \geq 0\}$. Since $\cup_{\boldsymbol{z} \in P_0} \partial \bar{g}(\boldsymbol{z})$ are compact(R. T. Rockafellar (1970), Theorem 24.7), there exists $T > 0$ such that $T = \max\{\|\boldsymbol{t}\| : \boldsymbol{t} \in \cup_{\boldsymbol{z} \in P_0} \partial \bar{g}(\boldsymbol{z})\}$.

In order to obtain a contradiction, we suppose that $\bar{\boldsymbol{z}} \notin \bar{D}$. Then, $\bar{g}(\bar{\boldsymbol{z}}) > 0$. By the definitions of $\boldsymbol{z}^k$ and P_k, we get that for each $k_0 > 0$,

$$\{\boldsymbol{z}^k\}_{k \geq k_0} \subset P_{k_0}. \tag{6}$$

From the convergence of $\{\boldsymbol{z}^k\}$, there exists $k' > 0$ such that $\bar{g}(\boldsymbol{z}^{k'}) > \dfrac{\bar{g}(\bar{\boldsymbol{z}})}{2}$ and $\|\boldsymbol{z}^{k'} - \bar{\boldsymbol{z}}\| < \dfrac{\bar{g}(\bar{\boldsymbol{z}})}{4T}$. Then, we have $\psi(\boldsymbol{z}^{k'}) = 1$ and

$$\rho(\bar{\boldsymbol{z}}, \boldsymbol{t}^{k'}, \boldsymbol{z}^{k'}) = \langle \boldsymbol{t}^{k'}, \bar{\boldsymbol{z}} - \boldsymbol{z}^{k'} \rangle + \bar{g}(\boldsymbol{z}^{k'}) > -T\|\bar{\boldsymbol{z}} - \boldsymbol{z}^{k'}\| + \frac{\bar{g}(\bar{\boldsymbol{z}})}{2} = -T\frac{\bar{g}(\bar{\boldsymbol{z}})}{4T} + \frac{\bar{g}(\bar{\boldsymbol{z}})}{2} = \frac{\bar{g}(\bar{\boldsymbol{z}})}{4} > 0.$$

This implies that $\bar{\boldsymbol{z}} \notin P_{k'}$. This contradicts to (6). Therefore, $\bar{\boldsymbol{z}} \in \bar{D}$. Consequently, $\bar{\boldsymbol{z}} \in S$. □

Theorem 3.2 *Assume that $\{\boldsymbol{v}^k\}$ generated by algorithm* OA *is infinite. Then, every accumulation point of $\{\boldsymbol{v}^k\}$ is contained in S.*

Proof. Since $\{v^k\} \subset P_0$ and $\{t^k\} \subset \cup_{z \in P_0} \partial \bar{g}(z)$, without loss of generality, we can assume that $v^k \to \bar{v}$ and $t^k \to \bar{t}$ as $k \to \infty$. In a similar way of Theorem 3.1, we assume that $z^k \to \bar{z}$ as $k \to \infty$. Then, $\bar{z} \in S$ and $\bar{t} \in \partial \bar{g}(\bar{z})$ (R. T. Rockafellar (1970), Theorem 24.4). Moreover, we can assume that the sequence $\{z^k\}$ has a subsequence $\{z^{k_q}\}$ contained in either $\bar{D}$ or $\mathbb{R}^{n+1} \backslash \bar{D}$.

In order to obtain a contradiction, we suppose that $\bar{v} \neq \bar{z}$. Then, there exists $\mu > 1$ such that $\bar{v} = \mu \bar{z}$.

In the case $\{z^{k_q}\} \subset \bar{D}$, we note that $\lim_{q \to \infty} \langle z^{k_q}, \bar{z} \rangle = \langle \bar{z}, \bar{z} \rangle = r^2$. Hence, since $\frac{r^2}{\mu} < r^2$, $\langle \bar{z}, z^{k_{q'}} \rangle > \frac{r^2}{\mu}$. From the definition of α_k, $\bar{f}(z^{k_{q'}}) \geq \alpha_{k_{q'}+1}$. Then,

$$\ell(\bar{v}, z^{k_{q'}}, \alpha_{k_{q'}+1}) = \langle z^{k_{q'}}, \bar{v} \rangle - r^2 + \frac{1}{2} \left(\frac{\bar{f}(z^{k_{q'}}) - \alpha_{k_{q'}+1}}{L} \right)^2$$
$$= \mu \langle z^{k_{q'}}, \bar{z} \rangle - r^2 + \frac{1}{2} \left(\frac{\bar{f}(z^{k_{q'}}) - \alpha_{k_{q'}+1}}{L} \right)^2$$
$$> \mu \frac{r^2}{\mu} - r^2 + \frac{1}{2} \left(\frac{\bar{f}(z^{k_{q'}}) - \alpha_{k_{q'}+1}}{L} \right)^2 \geq 0.$$

This implies that $\bar{v} \notin P_k$ for each $k \geq k_{q'}$. This contradicts to the convergence of $\{v^k\}$.

In the case $\{z^{k_q}\} \subset \mathbb{R}^{n+1} \backslash \bar{D}$, we get that $\bar{g}(z^{k_q}) > 0$ for each q. Hence, $\psi(z^{k_q}) = 1$ for each q. Since $\bar{t} \in \partial \bar{g}(\bar{z})$ and $\bar{g}(0) < 0$, $\langle \bar{t}, \bar{z} \rangle > 0$. Hence, $\langle \bar{t}, \bar{v} - \bar{z} \rangle = (1 - \mu) \langle \bar{t}, \bar{z} \rangle > 0$. From the convergences of $\{t^k\}$ and $\{z^k\}$, there exists $q' > 0$ such that $\langle t^{k_{q'}}, \bar{v} - z^{k_{q'}} \rangle > 0$. Then,

$$\rho(\bar{v}, t^{k_{q'}}, z^{k_{q'}}) = \langle t^{k_{q'}}, \bar{v} - z^{k_{q'}} \rangle + \bar{g}(z^{k_{q'}}) > 0.$$

Therefore, $\bar{v} \notin P_k$ for each $k \geq k_{q'}$. This contradicts to the convergence of $\{v^k\}$.

Consequently, $\bar{v} = \bar{z}$. By Theorem 3.1, $\bar{v} \in S$. □

Let $S(\alpha)' = \{u \in S : \bar{f}(u) < \alpha\}$. Then, the following lemma holds.

Lemma 3.3 *Let* $\alpha > \min(\text{MP})$. *If there is enough iteration of algorithm* OA, *then there exists* $\hat{k} \geq 0$ *satisfying* $z^{\hat{k}} \in S(\alpha)'$.

Proof. In order to obtain a contradiction, we suppose that for some $\alpha > \min(\text{MP})$, there is no $\hat{k} > 0$ satisfying $z^{\hat{k}} \in S(\alpha)'$. Then, $\{z^k\} \cap S \subset \mathbb{R}^{n+1} \backslash S(\alpha)'$ and $S(\alpha)' \neq \emptyset$. By the definition of α_k, $\alpha_k \geq \alpha$ for each k. Therefore, from the definition of P_k, $S(\alpha)' \subset P_k$ for each k. By the continuity of $\bar{f}$ and the definitions of $S(\alpha)'$ and $\bar{D}$, $S(\alpha)' \cap \text{int}\, \bar{D} \neq \emptyset$. Let $z' \in S(\alpha)' \cap \text{int}\, \bar{D}$. Since $z' \in \text{int}\, \bar{D}$, there exists $\delta_1 > 0$ such that $B^{n+1}_{<}(z', \delta_1) \subset \text{int}\, \bar{D}$. Moreover, let $\delta_2 = \frac{\alpha - \bar{f}(z')}{L}$. Then, we have that $\delta_2 > 0$ and that for each $u \in B^{n+1}_{<}(z', \delta_2)$,

$$\bar{f}(u) \leq \bar{f}(z') + L \|z' - u\| < \bar{f}(z') + L\delta_2 = \bar{f}(z') + L \frac{\alpha - \bar{f}(z')}{L} = \alpha.$$

Let $\delta = \min\{\delta_1, \delta_2\}$ and $\mu = 1 + \min\left\{ \frac{\delta}{2r}, \frac{\delta^2}{2(2r^2 - \delta^2)} \right\}$. Then, $\|\mu z' - z'\| = (\mu - 1)\|z'\| = (\mu - 1) r \leq \frac{\delta}{2} < \delta$, that is, $\mu z' \in \text{int}\, \bar{D}$. Since $\bar{g}(\mu z') < 0$, it follows from the convexity of $\bar{g}$ that for each $z \in B_{=}^{n+1}(0, r) \backslash S$,

$$\rho(\mu z', t, z) < 0, \tag{7}$$

where $t \in \partial \bar{g}(z)$. For each $z \in S \backslash S(\alpha)'$, we have

$$\delta^2 \leq \|z - z'\|^2 = \|z\|^2 + \|z'\| - 2\langle z, z'\rangle = 2r^2 - 2\langle z, z'\rangle,$$

and hence

$$\langle z, z'\rangle \leq \frac{2r^2 - \delta^2}{2}.$$

Moreover, for each $z \in S \backslash S(\alpha)'$, there exists $\tilde{z} \in]z, z'[$ satisfying $\bar{f}(\tilde{z}) = \alpha$. Then, since $\tilde{z} \notin B_{<}^{n+1}(z', \delta)$, $\|z - \tilde{z}\| \leq \|z - z'\| - \delta$. Therefore, for each $z \in S \backslash S(\alpha)'$, we have

$$\begin{aligned} \ell(\mu z', z, \alpha) &= \langle z, \mu z'\rangle - r^2 + \frac{1}{2}\left(\frac{\bar{f}(z) - \alpha}{L}\right)^2 = \langle z, \mu z'\rangle - r^2 + \frac{1}{2}\left(\frac{\bar{f}(z) - \bar{f}(\tilde{z})}{L}\right)^2 \\ &\leq \mu\langle z, z'\rangle - r^2 + \frac{1}{2}(\|z - \tilde{z}\|)^2 \leq \mu\langle z, z'\rangle - r^2 + \frac{1}{2}(\|z - z'\| - \delta)^2 \\ &= \mu\langle z, z'\rangle - r^2 + \frac{1}{2}\|z'\|^2 + \frac{1}{2}\|z\|^2 - \langle z, z'\rangle - \delta\|z' - z\| + \frac{1}{2}\delta^2 \\ &= (\mu - 1)\langle z, z'\rangle - \delta\|z' - z\| + \frac{1}{2}\delta^2 \leq (\mu - 1)\langle z, z'\rangle - \frac{1}{2}\delta^2 \\ &\leq \frac{\delta^2}{2(2r^2 - \delta^2)}\frac{2r^2 - \delta^2}{2} - \frac{1}{2}\delta^2 = \frac{1}{4}\delta^2 - \frac{1}{2}\delta^2 = -\frac{1}{4}\delta^2 < 0. \end{aligned} \tag{8}$$

By (7) and (8), $\mu z' \in P_k$ for each k. From the definition of μ, $\|\mu z'\| > r$. This implies that algorithm OA does not terminate after a finite number of iterations and that $\liminf_{k \to \infty} \|v^k\| = \|\bar{v}\| \geq \|\mu z'\| > r$. This contradicts to Theorem 3.2. Consequently, for each $\alpha > \min$ (MP), there exists $\hat{k} > 0$ such that $z^{\hat{k}} \in S(\alpha)'$. □

Theorem 3.3 *Assume that $\{w^k\}$ generated by algorithm* OA *is infinite. Then, every accumulation point of $\{w^k\}$ is a globally optimal solution of* (MP).

Proof. From the definition of w^k, $\{w^k\} \subset S$. Since S is compact, without loss of generality, we can assume that $w \to \bar{w}$ as $k \to \infty$. Then, $\bar{w} \in S$ and hence $\bar{f}(\bar{w}) \geq \min(\text{MP})$. It follows from the definitions of w^k and α_k that for each k, $\alpha_k = \bar{f}(w^k)$. In order to obtain a contradiction, we suppose that $\bar{f}(\bar{w}) > \min$ (MP). Let $\alpha' = \dfrac{\bar{f}(\bar{w}) + \min(\text{MP})}{2}$. Then, from Lemma 3.3, there exists $k' > 0$ such that $\bar{f}(w^{k'}) < \alpha' < \bar{f}(\bar{w})$. This contradicts to (5). Therefore, $\bar{w}$ is a globally optimal solution of (MP). □

Corollary 3.1 *Assume that $\{w^k\}$ generated by algorithm* OA *is infinite. Then, every accumulation point of $\{\pi(w^k)\}$ is a globally optimal solution of* (LOP).

Proof. Since $\{w^k\} \subset S$, it follows from the definitions of S and π that $\{\pi(w^k)\} \subset D$. Hence, by Theorem 3.3, every accumulation point of $\{\pi(w^k)\}$ is a globally optimal solution of (LOP). □

4 Conclusions

In this paper, we have presented an outer approximation algorithm for solving a global optimization problem to minimize a Lipschitz function over a compact convex set. The proposed algorithm generates two kinds of cutting planes at each iteration. By generating two kinds of cutting planes, it is ensured that every accumulation point of the provisional solutions is a globally optimal solution.

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