A SURVEY ON FIXED POINT THEOREMS
IN GENERALIZED CONVEX SPACES, II

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ABSTRACT. In our previous survey [P12], we reviewed some fixed point theorems which have appeared in our previous works. We follow this line of survey on various types of generalized convex spaces and on recently obtained results in [P14] and others. We add a new unified fixed point theorem.

1. Introduction

Since we introduced the concept of generalized convex spaces (simply, G-convex spaces) in 1993, there have appeared a large number of works contributing mainly to the KKM theory and equilibria theory on those spaces. In our previous survey [P12], which will be called Part I, we reviewed some fixed point theorems which have appeared mainly in our previous works on G-convex spaces.

In a recent work [P14], we introduced new concepts of admissibility (in the sense of Klee) and of Klee approximability for subsets of G-convex uniform spaces and showed that *any compact closed multimap in the class $\mathfrak{B}$ from an admissible G-convex space into itself has a fixed point*. This new theorem contains a large number of known results on topological vector spaces or on various subclasses of the class of admissible G-convex spaces.

In the present survey, we review the contents of [P14] and other new results. In fact, we review various types of G-convex uniform spaces such as LG-spaces, locally G-convex spaces, spaces of the Zima-Hadžić type, G-convex Φ-spaces, and admissible G-convex spaces. Moreover, we obtain a new general fixed point theorem which extends the main result of [P14] and its various consequences. Consequently, many known fixed point theorems for multimaps on topological vector spaces are extended to corresponding ones on G-convex spaces.

We follow the terminology and notations in Part I. Especially, all topological spaces are not necessarily Hausdorff unless explicitly stated otherwise, a t.v.s. means a topological vector space, and co denotes the convex hull. Multimaps are called simply maps.

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2. Generalized convex spaces

Definition. A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A \in \langle D \rangle$ with the cardinality $|A| = n + 1$, there exist a subset $\Gamma(A)$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \in \langle A \rangle$ implies $\phi_A(\Delta_J) \subset \Gamma(J)$.

Here, $\langle D \rangle$ denotes the set of all nonempty finite subsets of D, Δ_n the standard n-simplex with vertices $\{e_i\}_{i=0}^n$, and Δ_J the face of Δ_n corresponding to $J \in \langle A \rangle$; that is, if $A = \{a_0, a_1, \ldots, a_n\}$ and $J = \{a_{i_0}, a_{i_1}, \ldots, a_{i_k}\} \subset A$, then $\Delta_J = \mathrm{co}\{e_{i_0}, e_{i_1}, \ldots, e_{i_k}\}$. We may write $\Gamma_A := \Gamma(A)$. A G-convex space $(X, D; \Gamma)$ with $X \supset D$ is denoted by $(X \supset D; \Gamma)$ and $(X; \Gamma) := (X, X; \Gamma)$. For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$.

For details on G-convex spaces, see the references of [P6,14] and Part I.

Examples 2.1. The following are typical examples of G-convex spaces:
(1) Any nonempty convex subset of a t.v.s.
(2) A convex space due to Lassonde [L].
(3) A C-space (or an H-space) due to Horvath [Ho1,2].
(4) An L-space due to Ben-El-Mechaiekh et al. [B,BC]. The so-called FC-spaces are particular forms of L-spaces.

Examples 2.2. A ϕ_A-space $(X, D; \{\phi_A\}_{A \in \langle D \rangle})$ consisting of a topological space X, a nonempty set D, and a family of continuous functions $\phi_A : \Delta_n \to X$ for $A \in \langle D \rangle$ with $|A| = n + 1$, can be made into a G-convex space [P16].

Definition. A G-*convex uniform space* $(X, D; \Gamma; \mathcal{U})$ is a G-convex space such that $(X, \mathcal{U})$ is a uniform space with a basis $\mathcal{U}$ of the uniformity consisting of symmetric entourages. For each $U \in \mathcal{U}$, let

$$U[x] := \{x' \in X \mid (x, x') \in U\}$$

be the U-*ball* around a given element $x \in X$.

3. The Class $\mathfrak{B}$ of multimaps

Definition. Let $(E, D; \Gamma)$ be a G-convex space, X a nonempty subset of E, and Y a topological space. We define *the better admissible class* $\mathfrak{B}$ of multimaps from X into Y as follows:

$F \in \mathfrak{B}(X, Y) \iff F : X \multimap Y$ is a map such that, for any $\Gamma_N \subset X$, where $N \in \langle D \rangle$ with the cardinality $|N| = n + 1$, and for any continuous function $p : F(\Gamma_N) \to \Delta_n$, the composition

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} F(\Gamma_N) \xrightarrow{p} \Delta_n$$

has a fixed point. Note that Γ_N can be replaced by the compact set $\phi_N(\Delta_n) \subset X$.

We give some subclasses of $\mathfrak{B}$ as follows:

Examples 3.1. For topological spaces X and Y, an *admissible* class $\mathfrak{A}_c^\kappa(X, Y)$ of maps $F : X \multimap Y$ is now well-known; see Part I.

Note that for a G-convex space $(E, D; \Gamma)$, $X \subset E$, and any space Y, an admissible class $\mathfrak{A}_c^\kappa(X, Y)$ is a subclass of $\mathfrak{B}(X, Y)$. An example of maps in $\mathfrak{B}$ not belonging to $\mathfrak{A}_c^\kappa$ is the connectivity map due to Nash and Girolo; see [P3].

Examples 3.2. For a convex space $(X \supset D; \Gamma)$, where $\Gamma = \mathrm{co}$ and ϕ_N is a homeomorphism, the class $\mathfrak{B}(X, Y)$ is originally given in [P2] and investigated in [P2,3].

Examples 3.3. Let X and Y be uniform spaces (with respective bases $\mathcal{U}$ and $\mathcal{V}$ of symmetric entourages). A map $T : X \multimap Y$ is said to be *approachable* [B] whenever T admits a continuous W-approximative selection $s : X \to Y$ for each W in the basis $\mathcal{W}$ of the product uniformity on $X \times Y$; that is, $\mathrm{Gr}(s) \subset W[\mathrm{Gr}(F)]$. A map $T : X \multimap Y$ is said to be *approximable* [B] if its restriction $T|_K$ to any compact subset K of X is approachable.

It is known that if $(X \supset D; \Gamma)$ is a G-convex uniform space and Y is a uniform space, then any compact closed approachable map $F : X \multimap Y$ belongs to $\mathfrak{B}(X, Y)$; see Part I.

Examples 3.4. An important subclass of $\mathfrak{B}$ is the class of Φ-maps (or Fan-Browder maps) as follows:

Definition. Let Y be a topological space and $(X, D; \Gamma)$ a G-convex space. Then a map $T : Y \multimap X$ is called a Φ-*map* (or a *Fan-Browder map*) if there is a map $S : Y \multimap D$ such that

    (i) for each $y \in Y$, $M \in \langle S(y) \rangle$ implies $\Gamma_M \subset T(y)$; and

    (ii) $Y = \bigcup \{ \mathrm{Int}\, S^-(z) \mid z \in D \}$.

Recall that Horvath [Ho1] first defined a Φ-map for a C-space $(X; \Gamma)$.

It is well-known that every Φ-map $T : Y \multimap X$ belongs to $\mathbb{C}_c^\kappa(Y, X) \subset \mathfrak{A}_c^\kappa(Y, X)$. Therefore, if $X = Y$, then a Φ-map $T : X \multimap X$ belongs to $\mathfrak{B}(X, X)$.

4. LG-spaces

In this section, we introduce a particular subclass of G-convex uniform spaces:

Definition. A G-convex uniform space $(X \supset D; \Gamma; \mathcal{U})$ is called an LG-*space* [P7] if D is dense in X and, for each $U \in \mathcal{U}$, the U-neighborhood

$$U[A] = \{ x \in X \mid A \cap U[x] \neq \emptyset \}$$

around a given Γ-convex subset $A \subset X$ is Γ-convex.

Note that a singleton is not necessarily Γ-convex in an LG-space.

Examples 4.1. For a C-space $(X; \Gamma)$, an LG-space reduces to an LC-space [Ho1,2]. Any nonempty convex subset X of a locally convex t.v.s. E is an obvious example of an LC-space $(X; \Gamma)$ with $\Gamma_A = \mathrm{co}\, A$ for $A \in \langle X \rangle$. For other examples, see [Ho1].

Examples 4.2. A G-convex space $(X \supset D; \Gamma)$ is called an *LG-metric space* if X is equipped with a metric d such that (1) D is dense in X, (2) for any $\varepsilon > 0$, the set $\{x \in X \mid d(x, C) < \varepsilon\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, and (3) open balls are Γ-convex. This concept generalizes that of LC-metric spaces due to Horvath [Ho1].

Examples 4.3. Horvath [Ho2] showed that any hyperconvex metric space (H, d) is a complete metric LC-space $(H; \Gamma)$.

We give a general definition of Kakutani maps as follows:

Definition. Let Y be a topological space and $(X \supset D; \Gamma)$ a G-convex space. A map $F : Y \multimap X$ is called a *Kakutani map* if it is u.s.c. and has nonempty compact Γ-convex values.

Theorem 4.1. *Let $(X \supset D; \Gamma; \mathcal{U})$ be an LG-space and $T : X \multimap X$ a compact Kakutani map. If X is Hausdorff, then T has a fixed point.*

This is the main result of [P9].

5. Locally G-convex spaces

This section deals with another subclass of the class of G-convex uniform spaces.

Definition. A G-convex uniform space $(X \supset D; \Gamma; \mathcal{U})$ is said to be *locally G-convex* if D is dense in X and, for each $U \in \mathcal{U}$, there exists a $V \in \mathcal{U}$ such that $V \subset U$ and, for each $x \in X$,

$$N \in \langle V[x] \cap D \rangle \Rightarrow \Gamma_N \subset U[x].$$

In particular, if the U-ball $U[x]$ itself is Γ-convex for each $x \in X$, then $(X \supset D; \Gamma; \mathcal{U})$ is locally G-convex. If X is Hausdorff, every singleton is Γ-convex since $\{x\} = \bigcap_{U \in \mathcal{U}} U[x]$ and the intersection of Γ-convex subsets is Γ-convex.

Examples 5.1. Any convex subset of a locally convex Hausdorff t.v.s. is a locally G-convex space.

Examples 5.2. Every LG-space is locally G-convex if every singleton is Γ-convex (that is, $\Gamma_{\{x\}} = \{x\}$ for each $x \in D$).

Theorem 5.1. *Let X be a convex subset of a locally convex t.v.s. E. Then any compact acyclic map $F : X \multimap X$ has a fixed point.*

This was first obtained in [P1] as a generalization of the Himmelberg theorem and applied to abstract variational inequalities, minimax inequalities, geometric properties of convex sets, and other problems. This has been generalized step by step in a number of works of the author and, finally, to Theorem 9.1 which is the most general form we have; see [P10-14].

6. G-convex spaces of the Zima type

From the KKM principle, we deduced the following almost fixed point theorem [P14]:

Theorem 6.1. *Let* $(X \supset D; \Gamma; \mathcal{U})$ *be a G-convex uniform space, and K a totally bounded subset of X such that* $D \cap K$ *is dense in K. Let* $T : X \multimap X$ *be a u.s.c. [resp., an l.s.c.] map such that* $T(x) \cap K \neq \emptyset$ *for each* $x \in X$. *Suppose that for each* $x \in X$ *and each* $U \in \mathcal{U}$, *there exists* $V \in \mathcal{U}$ *such that*

$$N \in \langle \{y \in D \mid T(x) \cap V[y] \neq \emptyset\} \rangle \Rightarrow \Gamma_N \subset \{y \in X \mid T(x) \cap U[y] \neq \emptyset\}.$$

Then T has the almost fixed point property (that is, for each $U \in \mathcal{U}$, *F has a U-fixed point* $x_U \in X$ *satisfying* $F(x_U) \cap U[x_U] \neq \emptyset$).

Note that, if $\Gamma_N \subset D$ for each $N \in \langle D \rangle$, it is sufficient to assume that, for each $x \in D$ and each $U \in \mathcal{U}$, the set $\{y \in D \mid T(x) \cap U[y] \neq \emptyset\}$ is Γ-convex.

Theorem 6.2. *Under the hypothesis of Theorem 6.1, further if X is Hausdorff and if T is closed and compact, then T has a fixed point.*

Motivated by Theorems 6.1 and 6.2, we introduce the following:

Definition. For a G-convex uniform space $(X \supset D; \Gamma; \mathcal{U})$, a subset Y of X is said to be *of the Zima type* (or *of the Zima-Hadžić type*) if $D \cap Y$ is dense in Y and for each $U \in \mathcal{U}$ there exists a $V \in \mathcal{U}$ such that, for each $N \in \langle D \cap Y \rangle$ and any Γ-convex subset A of Y, we have

$$A \cap V[z] \neq \emptyset \ \forall z \in N \Rightarrow A \cap U[x] \neq \emptyset \ \forall x \in \Gamma_N.$$

Examples 6.1. (1) Hadžić [H1] defined that a nonempty subset K of a t.v.s. E is of the Zima type whenever for any $U \in \mathcal{V}$, there exists a $V \in \mathcal{V}$ satisfying $\mathrm{co}(V \cap (K - K)) \subset U$, where $\mathcal{V}$ is a neighborhood system of the origin of E.

Note that any nonempty subset of a locally convex t.v.s. is of the Zima type, and that there exists a subset of the Zima type in a non-locally convex topological vector space; see Hadžić [H2,3].

(2) For a C-space, our definition reduces to that of Hadžić [H3].

Examples 6.2. For an LG-space $(X \supset D; \Gamma; \mathcal{U})$, any nonempty subset Y of X is of the Zima type.

From Theorems 6.1 and 6.2, we have the following:

Theorem 6.3. *Let* $(X \supset D; \Gamma; \mathcal{U})$ *be a G-convex uniform space. Let* $T : X \multimap X$ *be a u.s.c. [resp., an l.s.c.] map with nonempty* Γ-convex values such that $T(X)$ is totally bounded and of the Zima type. Then T has the almost fixed point property.*

In Theorem 6.3, X is not necessarily Hausdorff. From Theorem 6.3, we have the following fixed point theorem for Kakutani maps:

Theorem 6.4. *Let* $(X \supset D; \Gamma; \mathcal{U})$ *be a Hausdorff G-convex uniform space. Let* $T : X \multimap X$ *be a compact Kakutani map such that* $T(X)$ *is of the Zima type. Then T has a fixed point.*

For more results on the Zima type in t.v.s., see [KP] and references therein.

7. G-convex Φ-spaces

In this section, we deal with a subclass of the class of G-convex uniform spaces containing preceding ones.

Definition. For a G-convex uniform space $(X, D; \Gamma; \mathcal{U})$, a subset Y of X is called a Φ-*set* if for each entourage $U \in \mathcal{U}$, there exists a Φ-map $T : Y \multimap X$ such that $\mathrm{Gr}(T) \subset U$ (that is, $T(y) \subset U[y]$ for all $y \in Y$). If X itself is a Φ-set, then it is called a Φ-*space*.

Note that every subset Y of a Φ-space is a Φ-set.

Examples 7.1. Horvath [Ho1] first defined a Φ-space for a C-space $(X; \Gamma)$ and gave examples as follows:

(1) A particular type of uniform spaces including locally convex t.v.s.

(2) Convex metric spaces in the sense of Takahashi with a metric satisfying certain property.

Examples 7.2. An important subclass of Φ-sets is that of locally convex sets in a t.v.s. For nontrivial examples of convex and locally convex subsets, see Hadžić [H2]. Moreover, there is an example of a nonconvex, admissible, locally convex subset of a non-locally convex t.v.s.; see Hahn [Hh].

Proposition 7.1. *Every locally convex subset Y of a convex subset X of a t.v.s. E is a Φ-subset of X.*

Note that the concept of local G-convexity does not generalize that of local convexity of a subset of a t.v.s. in Examples 7.2.

Examples 7.3. Any subset of the Zima type in a G-convex uniform space $(X \supset D; \Gamma; \mathcal{U})$ such that every singleton is Γ-convex is a Φ-set.

Examples 7.4. For a locally G-convex space $(X \supset D; \Gamma; \mathcal{U})$, any nonempty subset Y of X is a Φ-set. A locally G-convex space $(X \supset D; \Gamma; \mathcal{U})$ is a Φ-space.

Examples 7.5. Let $(X \supset D; \Gamma)$ be a metric G-convex space such that (1) D is dense in X; and (2) every open ball is Γ-convex. Then $(X \supset D; \Gamma)$ is a Φ-space.

8. Admissible G-convex spaces

For more general purposes, we generalize the admissibility of subsets of t.v.s. to subsets of G-convex spaces as follows:

Definition. For a G-convex uniform space $(X, D; \Gamma; \mathcal{U})$, a subset Y of X is said to be *admissible* (in the sense of Klee) if, for each nonempty compact subset K of Y and for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to Y$ satisfying

(1) $(x, h(x)) \in U$ for all $x \in K$;

(2) $h(K) \subset \Gamma_N$ for some $N \in \langle D \rangle$; and

(3) there exists a continuous function $p : K \to \Delta_n$ such that $h = \phi_N \circ p$, where $\phi_N : \Delta_n \to \Gamma_N$ and $|N| = n + 1$.

Examples 8.1. Examples of admissible subsets of a t.v.s. E can be seen in the references in [H2].

Definition. Let $(X, D; \Gamma; \mathcal{U})$ be a G-convex uniform space. A subset K of X is said to be *Klee approximable* if, for each entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to X$ satisfying conditions (1)-(3) in the preceding definition. Especially, for a subset Y of X, K is said to be Klee approximable *into* Y whenever the range $h(K) \subset \Gamma_N \subset Y$ for some $N \in \langle D \rangle$ in condition (2).

Examples 8.2. Every nonempty compact Φ-set of a G-convex uniform space is Klee approximable, and every Φ-space $(X, D; \Gamma; \mathcal{U})$ is admissible.

Examples 8.3. In a t.v.s. E, we gave examples of Klee approximable sets in [P15].

The following summarizes the mutual relations among the various subclasses of G-convex uniform spaces [P14]:

Theorem 8.1. *In the class of G-convex uniform spaces, the following hold:*

(1) *Any LG-space is of the Zima-Hadžić type.*

(2) *Every LG-space is locally G-convex whenever every singleton is Γ-convex.*

(3) *Any nonempty subset of a locally G-convex space is a Φ-set.*

(4) *Any Zima-Hadžić type subset of a G-convex uniform space such that every singleton is Γ-convex is a Φ-set.*

(5) *Every Φ-space is admissible. More generally, every nonempty compact Φ-subset is Klee approximable.*

9. Fixed point theorems

We have the following main result in this paper:

Theorem 9.1. *Let $(X, D; \Gamma; \mathcal{U})$ be a G-convex uniform space, Y a subset of X, and $F \in \mathfrak{B}(Y, Y)$ a map such that $F(Y)$ is Klee approximable into Y. Then F has the almost fixed point property.*

Further if Y is Hausdorff and if F is closed and compact, then F has a fixed point $x_0 \in Y$.

Proof. Since $K := F(Y)$ is Klee approximable into Y, for each symmetric entourage $U \in \mathcal{U}$, there exists a continuous function $h : K \to Y$ satisfying conditions (1) - (3) of the definition of Klee approximable subsets, and we have

$$\Delta_n \xrightarrow{\phi_N} \Gamma_N \xrightarrow{F|_{\Gamma_N}} K \xrightarrow{p} \Delta_n$$

for some $N \in \langle D \rangle$ with $|N| = n + 1$ and $\Gamma_N \subset Y$. Let $p' := p|_{F(\Gamma_N)}$. Since $F \in \mathfrak{B}(Y, Y)$, the composition $p' \circ (F|_{\Gamma_N}) \circ \phi_N : \Delta_n \multimap \Delta_n$ has a fixed point $a_U \in \Delta_n$. Let $x_U := \phi_N(a_U)$. Then

$$a_U \in (p' \circ F \circ \phi_N)(a_U) = (p' \circ F)(x_U)$$

and hence

$$x_U = \phi_N(a_U) \in (\phi_N \circ p' \circ F)(x_U).$$

Since $h = \phi_N \circ p$ by definition, we have

$$x_U = h(y_U) \quad \text{for some} \quad y_U \in (F|_{\Gamma_N})(x_U).$$

Therefore, for each entourage $U \in \mathcal{U}$, there exist points $x_U \in Y$ and $y_U \in F(x_U)$ such that $(x_U, y_U) = (h(y_U), y_U) \in U$. So, for each U, there exist $x_U, y_U \in Y$ such that $y_U \in F(x_U)$ and $y_U \in U[x_U]$.

Now suppose F is closed and compact. Since $F(Y)$ is relatively compact, we may assume that the net y_U converges to some $x_0 \in \overline{F(Y)}$. Then, by the Hausdorffness of Y, the net x_U also converges to x_0. Since the graph of F is closed in $Y \times \overline{F(Y)}$ and $(x_U, y_U) \in \mathrm{Gr}(F)$, we have $(x_0, x_0) \in \mathrm{Gr}(F)$ and hence we have $x_0 \in F(x_0)$. This completes our proof.

For $X = Y$, Theorem 9.1 reduces to the following main result of [P14]:

Theorem 9.2. *Let* $(X, D; \Gamma; \mathcal{U})$ *be a G-convex uniform space such that* X *is Hausdorff and* $F \in \mathfrak{B}(X, X)$ *a multimap such that* $F(X)$ *is Klee approximable. Then* F *has the almost fixed point property. Further if* F *is closed and compact, then* F *has a fixed point* $x_0 \in X$.

Theorem 9.3. *Let* $(X, D; \Gamma; \mathcal{U})$ *be a Hausdorff admissible G-convex space. Then any compact closed map* $F \in \mathfrak{B}(X, X)$ *has a fixed point.*

Corollary 9.4. *Let* $(X, D; \Gamma; \mathcal{U})$ *be a compact admissible G-convex space such that* X *is Hausdorff. Then any map* $F \in \mathfrak{A}_c^\kappa(X, X)$ *has a fixed point.*

In view of Theorem 8.1, Theorems 9.1-9.3 and Corollary 9.4 can be applied to various subclasses of the class of admissible G-convex spaces.

Especially, an admissible convex subset of a t.v.s. is an admissible G-convex space, and hence we have the following from Theorem 9.3:

Corollary 9.5. *Let* X *be an admissible convex subset of a Hausdorff t.v.s.* E. *Then any compact closed map* $F \in \mathfrak{B}(X, X)$ *has a fixed point.*

Corollary 9.5 was given in [P3], where we listed more than sixty papers in chronological order, from which we could deduce particular forms. Especially, from Corollary 9.5, we obtain

Corollary 9.6. *Let* X *be an admissible convex subset of a Hausdorff t.v.s.* E. *Then any compact map* $F \in \mathbb{V}_c(X, X)$ *(that is, a finite composition of acyclic maps) has a fixed point.*

This generalizes Theorem 5.1.

In the following, we are mainly concerned with Φ-maps and Φ-spaces.

Lemma 9.7. [Ho1, P4] *Let* Y *be a paracompact space,* $(X, D; \Gamma)$ *an H-space, and* $T : Y \multimap X$ *a* Φ-map. *Then* T *has a continuous selection.*

From Lemma 9.7 and Theorem 9.3, we have the following:

Theorem 9.8. *Let* $(X, D; \Gamma; \mathcal{U})$ *be an admissible paracompact H-space. Then any compact* Φ-map $T : X \multimap X$ *has a fixed point.*

From Theorems 8.1 and 9.8, we have

Corollary 9.9. *Let* $(X, D; \Gamma; \mathcal{U})$ *be an* H-*space. If it is also a paracompact* Φ-*space, then any compact* Φ-*map* $T : X \multimap X$ *has a fixed point.*

Some applications of Corollary 9.9 were given in [Ho1] and [P8].

From Theorems 8.1 and 9.2, we have the following:

Theorem 9.10. *Let* $(X, D; \Gamma; \mathcal{U})$ *be a* G-*convex uniform space and* $F \in \mathfrak{B}(X, X)$ *a map such that* $\overline{F(X)}$ *is a compact Hausdorff* Φ-*subset of* X. *If* F *is closed, then* F *has a fixed point.*

Since every locally convex set is a Φ-set, we have the following:

Corollary 9.11. *Let* X *be a nonempty convex subset of a Hausdorff t.v.s. Then any compact closed map* $F \in \mathfrak{B}(X, X)$ *such that* $\overline{F(X)}$ *is locally convex has a fixed point.*

From Theorems 8.1 and 9.3, we have the following in [P5]:

Theorem 9.12. *Let* $(X, D; \Gamma; \mathcal{U})$ *be a Hausdorff* Φ-*space. Then any compact closed map* $F \in \mathfrak{B}(X, X)$ *has a fixed point.*

Particular forms of Theorem 9.12 were known by Horvath [Ho1] and Park and Kim [PK]. Moreover, Ben-El-Mechaiekh et al. [BC] obtained a particular form of Theorem 9.12 for approachable maps. In our previous works, it was shown that Theorem 9.12 subsumes a large number of fixed point theorems related to approachable maps on G-convex spaces, acyclic maps on locally G-convex spaces, and Kakutani maps on Φ-spaces or on hyperconvex metric spaces; see Part I.

For a non-closed map, we have the following:

Corollary 9.13. *Let* $(X, D; \Gamma; \mathcal{U})$ *be a compact* Φ-*space such that* X *is Hausdorff and* $F \in \mathfrak{A}_c^\kappa(X, X)$. *Then* F *has a fixed point.*

From Examples 7.5, Lemma 9.7, and Theorem 9.12, we have

Corollary 9.14. *Let* $(X \supset D; \Gamma)$ *be a metric* G-*convex space such that* D *is dense in* X *and every open ball is* Γ-*convex. Then every compact* Φ-*map* $F : X \multimap X$ *has a fixed point.*

REFERENCES

[B]   H. Ben-El-Mechaiekh, *Spaces and maps approximation and fixed points*, J. Comp. Appl. Math. **113** (2000), 283–308.

[BC]   H. Ben-El-Mechaiekh, S. Chebbi, M. Florenzano, and J. Llinares, *Abstract convexity and fixed points*, J. Math. Anal. Appl. **222** (1998), 138–151.

[H1]   O. Hadžić, *Some fixed point and almost fixed point theorems for multivalued mappings in topological vector spaces*, Nonlinear Anal. TMA **5** (1981), 1009–1019.

[H2]   _____, *Fixed Point Theory in Topological Vector Spaces*, Univ. of Novi Sad, Novi Sad, 1984, 337pp.

[H3]   _____, *Almost fixed point and best approximation theorems in H-spaces*, Bull. Austral. Math. Soc. **53** (1996), 447–454.

[Hh]   S. Hahn, *A fixed point theorem for multivalued condensing mappings in general topological vector spaces*, Review of Research, Univ. of Novi Sad **15** (1985), 97–106.

[Ho1] C. D. Horvath, *Contractibility and generalized convexity*, J. Math. Anal. Appl. **156** (1991), 341–357.

[Ho2] ———, *Extension and selection theorems in topological spaces with a generalized convexity structure*, Ann. Fac. Sci. Toulouse **2** (1993), 253–269.

[KP]   J.-H. Kim and S. Park, *Almost fixed point theorems of the Zima type*, J. Korean Math. Soc. **41** (2004), 737–746.

[L]   M. Lassonde, *On the use of KKM multifunctions in fixed point theory and related topics*, J. Math. Anal. Appl. **97** (1983), 151–201.

[P1]   Sehie Park, *Some coincidence theorems on acyclic multifunctions and applications to KKM theory*, Fixed Point Theory and Applications (K.-K. Tan, ed.), 248–277, World Sci., River Edge, NJ, 1992.

[P2]   ———, *Coincidence theorems for the better admissible multimaps and their applications*, Nonlinear Anal. **30** (1997), 4183-4191.

[P3]   ———, *A unified fixed point theory of multimaps on topological vector spaces*, J. Korean Math. Soc. **35** (1998), 803–829. *Corrections*, ibid. **36** (1999), 829–832.

[P4]   ———, *Continuous selection theorems in generalized convex spaces*, Numer. Funct. Anal. and Optimiz. **25** (1999), 567–583.

[P5]   ———, *Fixed points of better admissible multimaps on generalized convex spaces*, J. Korean Math. Soc. **37** (2000), 885–899.

[P6]   ———, *Elements of the KKM theory for generalized convex spaces*, Korean J. Comput. & Appl. Math. **7** (2000), 1–28.

[P7]   ———, *Remarks on fixed point theorems for generalized convex spaces*, Fixed Point Theory and Applications (Y.J. Cho, ed.), 135–144, Nova Sci. Publ., New York, 2000.

[P8]   ———, *New topological versions of the Fan–Browder fixed point theorem*, Nonlinear Anal. **47** (2001), 595–606.

[P9]   ———, *Fixed point theorems in locally G-convex spaces*, Nonlinear Anal. **48** (2002), 869–879.

[P10] ———, *Remarks on acyclic versions of generalized von Neumann and Nash equilibrium theorems*, Appl. Math. Letters **15** (2002), 641–647.

[P11] ———, *Fixed points of multimaps in the better admissible class*, J. Nonlinear Convex Anal. **5** (2004), 369–377.

[P12] ———, *A survey on fixed point theorems in generalized convex spaces*, RIMS Kôkyûroku **1484** (2006), 124–133.

[P13] ———, *Fixed points of approximable or Kakutani maps in generalized convex spaces*, J. Nonlinear Convex Anal. **7** (2006), 1–17.

[P14] ———, *A unified fixed point theory in generalized convex spaces*, Acta Math. Sinica, English Ser. **23** (2007), 1509–1536.

[P15] ———, *Fixed point theorems for better admissible multimaps on almost convex sets*, J. Math. Anal. Appl. **329** (2007), 690–702.

[P16] ———, *Various subclasses of abstract convex spaces for the KKM theory*, Proc. Nat. Inst. Math. Sci. **2**(4) (2007), 35–47.

[PK]   S. Park and H. Kim, *Coincidences of composites of u.s.c. maps on H-spaces and applications*, J. Korean Math. Soc. **32** (1995), 251–264.

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