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Simplicial resolutions and their applications

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1 Introduction.

Since Arnold [2] used simplicial resolutions for computing the homology of classical braid groups, it becomes clear that the concept of simplicial resolutions is very powerful and useful in the area of algebraic topology. However, although simplicial resolutions were already used in several papers (e.g. [3], [4], [7], [9], [10]), the properties of simplicial resolutions are not well studied. In this note we shall study the properties of simplicial resolutions and give several examples of the computations which are used. First recall several notations and definitions.

Definition. (i) For a finite set $\mathbf{x}=\{x_1,\cdots,x_m\}\subset\mathbb{R}^N$, let $\sigma(\mathbf{x})$ be the convex hull spanned by the points $x_1,\cdots,x_m$:

$$\sigma(\mathbf{x})=\Big\{\sum_{k=1}^m t_kx_k\in\mathbb{R}^N:\sum_{k=1}^m t_k=1,t_k\geq 0\text{ for any }k\Big\}.$$

If $x_2-x_1,x_3-x_1,\cdots,x_m-x_1$ are linearly independent over $\mathbb{R}$, we say that the set $\mathbf{x}=\{x_1,\cdots,x_m\}$ is *in general position.* Note that $\mathbf{x}$ is in general position if and only if $\sigma(\mathbf{x})$ is an $(m-1)$-dimensional simplex.

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(ii) Let $h : X \longrightarrow \Sigma$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in \Sigma$, and let $i : X \longrightarrow \mathbb{R}^n$ be an embedding. Then we define the the subspace $\mathcal{X}^\Delta \subset \Sigma \times \mathbb{R}^N$ by

$$\mathcal{X}^\Delta = \left\{(y, z) \in \Sigma \times \mathbb{R}^N : z \in \sigma(i(h^{-1}(y)))\right\} \subset \Sigma \times \mathbb{R}^N.$$

We also define the map $h^\Delta : \mathcal{X}^\Delta \longrightarrow \Sigma$ by $h^\Delta(y, z) = y$ for $(y, z) \in \mathcal{X}^\Delta$. The pair $(\mathcal{X}^\Delta, h^\Delta)$ is called *a simplicial resolution of* (h, i).

(iii) A simplicial resolution $(\mathcal{X}^\Delta, h^\Delta)$ is *a non-degenerate* if for each $y \in \Sigma$ any k points of $i(h^{-1}(y))$ span $(k-1)$-dimensional affine subspace of $\mathbb{R}^N$.

Remark. The space X can be regarded as the subspace of $\mathcal{X}^\Delta$ by identifying $x \mapsto (h(x), i(x))$. Moreover, if we identify $X \subset \mathcal{X}^\Delta$ as above, it is easy to see that $h^\Delta|X = h$:

$$\begin{array}{ccc} X & \xrightarrow{h} & \Sigma \\ \cap\downarrow & & \| \\ \mathcal{X}^\Delta & \xrightarrow{h^\Delta} & \Sigma \end{array}$$

2 Properties of simplicial resolutions.

In this section we recall several basic properties of simplicial resolutions.

Theorem 2.1 (([7], [9]). *Let $h : X \longrightarrow \Sigma$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in \Sigma$, let $i : X \longrightarrow \mathbb{R}^n$ be an embedding, and $(\mathcal{X}^\Delta, h^\Delta)$ be a simplicial resolution of (h, i).*

(i) *If X and Σ are closed semi-algebraic spaces, and two maps h and i are polynomial maps, $h^\Delta : \mathcal{X}^\Delta \xrightarrow{\simeq} \Sigma$ is a homotopy equivalence.*

(ii) *Let $i' : X \longrightarrow \mathbb{R}^{N'}$ be an embedding and let $(\mathcal{X}_1^\Delta, h_1^\Delta)$ be a simplicial resolution of (h, i'). If $(\mathcal{X}^\Delta, h^\Delta)$ and $(\mathcal{X}_1^\Delta, h_1^\Delta)$ are non-degenerate, there exists a homeomorphism $\Phi : \mathcal{X}^\Delta \xrightarrow{\cong} \mathcal{X}_1^\Delta$ such that $\Phi|X = id_X$.* □

Theorem 2.2 ([7]). *Let $h : X \longrightarrow \Sigma$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in \Sigma$. If X can be embedded into $\mathbb{R}^{N'}$ for some number N', there exists an embedding $i : X \longrightarrow \mathbb{R}^N$ such that the simplicial resolution $(\mathcal{X}^\Delta, h^\Delta)$ of (h, i) is non-degenerate.* □

Definition. Let $h : X \to \Sigma$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in \Sigma$, let $i : X \to \mathbb{R}^n$ be an embedding, and $(\mathcal{X}^\Delta, h^\Delta)$ be a simplicial resolution of (h, i).

(i) First, assume that $(\mathcal{X}^\Delta, h^\Delta)$ is non-degenerate. In this case, $(h^\Delta)^{-1}(y)$ is a simplex for any $y \in \Sigma$. We denote by $(h^\Delta)^{-1}(y)^{[k-1]}$ the $(k-1)$-dimensional skelton of $(h^\Delta)^{-1}(y)$. Then for each non-negative integer $k \geq 0$, define the subspace $\mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta$ by $\mathcal{X}^\Delta_k = \bigcup_{y \in \Sigma} (h^\Delta)^{-1}(y)^{[k-1]}$.

(ii) Next, consider the general case. In this case, by Theorem 2.2, there exists an embedding $i' : X \to \mathbb{R}^{N'}$ such that the simplicial resolution $(\tilde{\mathcal{X}}^\Delta, \tilde{h}^\Delta)$ of (h, i') is non-degenerate.

Then for each $y \in \Sigma$, the simplicial map $\sigma(i'(h^{-1}(y))) \to \sigma(i(h^{-1}(y)))$ can be easily well-defined. This naturally extends the surjective map $\pi : \tilde{\mathcal{X}}^\Delta \to \mathcal{X}^\Delta$ such that the following diagram is commutative:

$$\begin{array}{ccccc}
X & \xrightarrow[\subset]{} & \tilde{\mathcal{X}}^\Delta & \xrightarrow{\tilde{h}^\Delta} & \Sigma \\
\| & & \pi \downarrow & & \| \\
X & \xrightarrow[\subset]{} & \mathcal{X}^\Delta & \xrightarrow{h^\Delta} & \Sigma
\end{array}$$

Then for each non-negative integer $k \geq 0$, define the subspace $\mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta$ by $\mathcal{X}^\Delta_k = \pi(\tilde{\mathcal{X}}^\Delta_k)$. It is easy to see that there is an increasing filtration

$$\emptyset = \mathcal{X}^\Delta_0 \subset \mathcal{X}^\Delta_1 \subset \cdots \subset \mathcal{X}^\Delta_k \subset \mathcal{X}^\Delta_{k+1} \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}^\Delta_k = \mathcal{X}^\Delta.$$

3 Generalization of simplicial resolutions.

Let $h : X \to Y$ be a surjective map. Even if h is not finite to one, one can define its simplicial resolution in a complete similar way. However, in this case, it is degenerate one. In this case, we need some modification for having a non-degenerate simplicial resolution. Now we recall the following result.

Lemma 3.1. *Let $h : X \to \Sigma$ be a surjective map and let $j : X \to \mathbb{R}^N$ be an embedding. Then for each $k \geq 1$, there is an embedding $j_k : X \to \mathbb{R}^{N_k}$ satisfying the following two conditions:*

(i) *For each* $k \geq 1$, $N_k < N_{k+1}$ *and there is a commutative diagram*

$$\begin{array}{ccc} X & \xrightarrow{j_k} & \mathbb{R}^{N_k} \\ \| & & \cap\downarrow \\ X & \xrightarrow{j_{k+1}} & \mathbb{R}^{N_{k+1}} \end{array}$$

(ii) *The points* $\{j_k(x_1), \cdots, j_k(x_{2k})\}$ *are linearly independent over* $\mathbb{R}$ *for any* $2k$ *distinct points* $\{x_1, \cdots, x_{2k}\} \subset X$. □

Then we can easily see that the following two conditions are satisfied:

(3.1.1) If $\mathbf{x} = \{x_1, \cdots, x_k\} \subset j_k(h^{-1}(y))$, it spans a $(k-1)$ dimensional simplex $\sigma(\mathbf{x})$ in $\mathbb{R}^{N_k}$.

(3.1.2) If $\mathbf{x}_1 = \{x_1, \cdots, x_i\} \subset j_k(h^{-1}(y))$ and $\mathbf{x}_2 = \{y_1, \cdots, y_l\} \subset j_k(h^{-1}(y))$ with $i, l \leq k$, $\sigma(\mathbf{x}_1) \cap \sigma(\mathbf{x}_2) = \emptyset$ if $\mathbf{x}_1 \cap \underline{\mathrm{x}}_2 = \emptyset$.

Then we define the space X_k by

$$\tilde{\mathcal{X}}_k^{\Delta} = \left\{ (y,t) \in \Sigma \times \mathbb{R}^{N_k} \;\middle|\; \begin{array}{c} \{u_1, \cdots, u_l\} \subset j_k(h^{-1}(y)) \\ t \in \sigma(\{u_1, \cdots, u_l\}) \\ l \leq k \end{array} \right\}.$$

By using the commutative diagram (3.1), we can identify $\tilde{\mathcal{X}}_k^{\Delta} \subset \tilde{\mathcal{X}}_{k+1}^{\Delta}$. Then define the space $\tilde{\mathcal{X}}^{\Delta}$ and the map $\tilde{h}^{\Delta} : \tilde{\mathcal{X}}^{\Delta} \to \Sigma$ by $\tilde{\mathcal{X}}^{\Delta} = \bigcup_{k=1}^{\infty} \tilde{\mathcal{X}}_k^{\Delta}$ and $\tilde{h}^{\Delta}(y,t) = y$. One can easily see that $(\tilde{\mathcal{X}}^{\Delta}, \tilde{h}^{\Delta})$ is a non-generate simplicial resolution of h with increasing filtration

$$\emptyset = \tilde{\mathcal{X}}_0^{\Delta} \subset X = \tilde{\mathcal{X}}_1^{\Delta} \subset \tilde{\mathcal{X}}_2^{\Delta} \subset \cdots \subset \tilde{\mathcal{X}}_k^{\Delta} \subset \tilde{\mathcal{X}}_{k+1}^{\Delta} \subset \cdots \subset \bigcup_{k=1}^{\infty} \tilde{\mathcal{X}}_k^{\Delta} = \tilde{\mathcal{X}}^{\Delta}.$$

Theorem 3.2 ([7]). *Let* $h : X \to \Sigma$ *and* $h_1 : W \to \Sigma'$ *be surjective maps,* X *and* W *can be embedded into* $\mathbb{R}^{N'}$ *for some number* N'*, and the following diagram is commutative:*

$$\begin{array}{ccc} X & \xrightarrow{h} & \Sigma \\ f\downarrow & & g\downarrow \\ W & \xrightarrow{k} & \Sigma' \end{array}$$

Then there exists a filtration preserving map $\overline{f} : \tilde{\mathcal{X}}^{\Delta} \to \tilde{\mathcal{W}}^{\Delta}$ such that the diagram

$$\begin{array}{ccccc} X & \xrightarrow[\subset]{} & \tilde{\mathcal{X}}^{\Delta} & \xrightarrow{\tilde{h}^{\Delta}} & \Sigma \\ {\scriptstyle f}\downarrow & & {\scriptstyle \overline{f}}\downarrow & & {\scriptstyle g}\downarrow \\ W & \xrightarrow[\subset]{} & \tilde{\mathcal{W}}^{\Delta} & \xrightarrow{\tilde{h}_1^{\Delta}} & \Sigma' \end{array}$$

is commutative, where $(\tilde{\mathcal{X}}^{\Delta}, \tilde{h}^{\Delta})$ and $(\tilde{\mathcal{W}}^{\Delta}, \tilde{h}_1^{\Delta})$ denote the associated non-degenerate resolutions of the maps h and h_1, respectively. □

4 Spectral sequences of the Vassiliev type.

Let $h : X \to \Sigma$ be a surjective map such that $h^{-1}(y)$ is a finite set for any $y \in \Sigma$ and let $i : X \to \mathbb{R}^n$ be an embedding. Let $(\mathcal{X}^{\Delta}, h^{\Delta})$ denote the simplicial resolution of (h, i) with increasing filtration

$$\emptyset = \mathcal{X}_0^{\Delta} \subset \mathcal{X}_1^{\Delta} \subset \cdots \subset \mathcal{X}_k^{\Delta} \subset \mathcal{X}_{k+1}^{\Delta} \subset \cdots \subset \bigcup_{k=0}^{\infty} \mathcal{X}_k^{\Delta} = \mathcal{X}^{\Delta}.$$

If $h^{\Delta} : \mathcal{X}^{\Delta} \xrightarrow{\simeq} \Sigma$ is a homotopy equivalence, one has the Vassiliev type spectral sequence

$$\{E_t^{r,s}, d_t : E_t^{r,s} \to E_t^{r+t,s-t+1}\} \Rightarrow H_c^{r+s}(\Sigma),$$

where Y_+ denotes the one-point compactification of a locally compact space Y, $H_c^*(Y) := H^*(Y_+)$ (the cohomology with compact supports) and $E_1^{r,s} = \tilde{H}_c^{r+s}(\mathcal{X}_r^{\Delta} \setminus \mathcal{X}_{r-1}^{\Delta})$. We call this type spectral sequence as *the spectral sequence of Vassiliev type.*

Now we give two typical examples of the computations which use the spectral sequences of Vassiliev type.

4.1 Theorem of Arnold-Vassiliev.

Definition. (i) For each integer $d \geq 1$, let P^d denote the space consisting of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{R}[z]$ of degree

d and let $\mathrm{P}_n^d \subset \mathrm{P}_n^d$ be the subspace consisting of all $f(z) \in \mathrm{P}^d$ such that any real root of $f(z)$ has the multiplicity $< n$.

(ii) Let $\Sigma_n^d \subset \mathrm{P}^d$ denote the discriminant of P_n^d defined by $\Sigma_n^d = \mathrm{P}^d \setminus \mathrm{P}_n^d$. Let X_n^d denote the tautological normalization of Σ_n^d defined by

$$X_n^d = \{(f, \alpha) \in \Sigma_n^d \times \mathbb{R} : \alpha \text{ is a root of } f(z) \text{ of multiplicity } \geq n\}.$$

Define the embedding $i : X_n^d \to \mathbb{R}^{d+1+\lfloor d/n \rfloor}$ and the surjective map $p_1 : X_n^d \to \Sigma_n^d$ by $i(f, \alpha) = (j_1(f), \alpha, \alpha^2, \cdots, \alpha^{\lfloor d/n \rfloor})$ and $p_1(f, \alpha) = f$ for $(f, \alpha) \in X_n^d$, where $j_1(f) := (a_1, \cdots, a_d)$ if $f = z^d + a_1 z^{d-1} + \cdots + a_d$.

Let $(\mathcal{X}^\Delta, p_1^\Delta : \mathcal{X}^\Delta \to \Sigma_d^n)$ denote the simplicial resolution of $(p_1.i)$. By Theorem 2.1, p_1^Δ is a homotopy equivalence. Hence, there is the Vassiliev type spectral sequence

$$\{E_t^{r,s}, d_t : E_t^{r,s} \to E_t^{r+t, s-t+1}\} \Rightarrow H_c^{r+s}(\Sigma_n^d, \mathbb{Z}),$$

where $E_1^{r,s} = \tilde{H}_c^{r+s}(\mathcal{X}_r^\Delta \setminus \mathcal{X}_{r-1}^\Delta, \mathbb{Z})$. If we recall that it follows from the Alexander duality that there is a natural isomorphism

$$H_k(\mathrm{P}_n^d, \mathbb{Z}) \cong H_c^{d-k-1}(\Sigma_n^d, \mathbb{Z}) \quad \text{for } 1 \leq k < d-1,$$

by reindexing $E_{r,s}^t = E_t^{d-1-s}$, we have the spectral sequence

$$\{E_{r,s}^t, d^t : E_{r,s}^t \to E_t^{r+t, s+t-1}\} \Rightarrow H_{s-r}(\mathrm{P}_n^d, \mathbb{Z})$$

such that $E_{r,s}^1 = H_c^{d-1+r-s}(\mathcal{X}_r^\Delta \setminus \mathcal{X}_{r-1}^\Delta, \mathbb{Z})$.

It is easy to see that $\mathcal{X}_r^\Delta \setminus \mathcal{X}_{r-1}^\Delta$ is a total space of the real vector bundle over $C_r(\mathbb{R})$ with rank $d-1-r(n-1)$. Hence, by using Thom isomorphism and Poincaré duality, if $1 \leq r \leq d$, there is an isomorphism

$$\begin{aligned} E_{r,s}^1 &= H_c^{d-1-s+r}(\mathcal{X}_r^\Delta \setminus \mathcal{X}_{r-1}^\Delta, \mathbb{Z}) \cong H_c^{rn-s}(C_r(\mathbb{R}), \mathbb{Z}) \\ &\cong H^{rn-s}(S^r, \mathbb{Z}) = \begin{cases} \mathbb{Z} & (s-r = r(n-2),\ 1 \leq r \leq \lfloor d/n \rfloor) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned}$$

By the dimensional reason, it is easy to see that $E_{**}^1 = E_{**}^\infty$ and we have:

Lemma 4.1 (Arnold-Vassiliev; cf. [9], [10]). *If $n \geq 3$, there is an isomorphism*

$$H_k(\mathrm{P}_n^d, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = r(n-2),\ 0 \leq r \leq \lfloor d/n \rfloor \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

If we use the scanning maps (cf. [3]), we have the more precise statement:

Theorem 4.2 (Kozlowski-Yamaguchi; [4]). *If $n \geq 4$, there is a homotopy equivalence $\mathrm{P}^d_n \simeq J_{\lfloor d/n \rfloor}(\Omega S^{n-1})$, where $J_k(\Omega S^m)$ denotes the k-th stage James filtration of ΩS^m defined by*

$$J_k(\Omega S^m) = S^m \cup e^{2m} \cup e^{3m} \cup \cdots \cup e^{km} \subset \Omega S^m = S^m \cup (\bigcup_{j=2}^{\infty} e^{jm}). \quad \square$$

4.2 Theorem of Kozlowski-Yamaguchi.

Definition. (i) For each integer $d \geq 1$, let P^d denote the space consisting of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \cdots + a_d \in \mathbb{R}[z]$ of degree d as before. Let $\mathrm{H}^d = (\mathrm{P}^d)^n$ and let $H^d_n \subset \mathrm{H}^d$ be the subspace consisting of all n-tuples $(f_1(z), \cdots, f_n(z)) \in (\mathrm{P}^d)^n$ of monic polynomials of the same degree d such that $f_1(z), \cdots, f_n(z)$ have no common real root.

(ii) Let $\tilde{\Sigma}^d_n \subset \mathrm{H}^d$ denote the discriminant of H^d_n defined by $\tilde{\Sigma}^d_n = \mathrm{H}^d \backslash H^d_n$. Let $\tilde{X}^d_n$ denote the tautological normalization of $\tilde{\Sigma}^d_n$ defined by

$$\tilde{X}^d_n = \{(f_1, \cdots, f_n, \alpha) \in \tilde{\Sigma}^d_n \times \mathbb{R} : \alpha \text{ is a common root of } f_1, \cdots, f_n\}.$$

Define the embedding $j : \tilde{X}^d_n \to \mathbb{R}^{d+1+dn}$ and the surjective map $q_1 : \tilde{X}^d_n \to \tilde{\Sigma}^d_n$ by $j(f, \alpha) = (j_1(f_1), \cdots, j_1(f_n), 1, \alpha, \alpha^2, \cdots, \alpha^d)$ and $q_1(f, \alpha) = f$ for $(f, \alpha) = (f_1, \cdots, f_n, \alpha) \in \tilde{X}^d_n$. Let $(\tilde{\mathcal{X}}^\Delta, q_1^\Delta : \tilde{\mathcal{X}}^\Delta \to \tilde{\Sigma}^d_n)$ denote the simplicial resolution of $(q_1.j)$. By Theorem 2.1, q_1^Δ is a homotopy equivalence. Hence, there is the Vassiliev type spectral sequence

$$\{E_t^{r,s}, d_t : E_t^{r,s} \to E_t^{r+t, s-t+1}\} \Rightarrow H_c^{r+s}(\tilde{\Sigma}^d_n, \mathbb{Z}),$$

where $E_1^{r,s} = \tilde{H}_c^{r+s}(\tilde{\mathcal{X}}^\Delta_r \setminus \tilde{\mathcal{X}}^\Delta_{r-1}, \mathbb{Z})$. If we recall that it follows from the Alexander duality that there is a natural isomorphism

$$H_k(H^d_n, \mathbb{Z}) \cong H_c^{dn-k-1}(\tilde{\Sigma}^d_n, \mathbb{Z}) \quad \text{for } 1 \leq k < dn - 1,$$

by reindexing $E^t_{r,s} = E_t^{dn-1-s}$, we have the spectral sequence

$$\{E^t_{r,s}, d^t : E^t_{r,s} \to E_t^{r+t, s+t-1}\} \Rightarrow H_{s-r}(H^d_n, \mathbb{Z})$$

such that $E^1_{r,s} = H_c^{dn-1+r-s}(\tilde{\mathcal{X}}_r^{\Delta} \setminus \tilde{\mathcal{X}}_{r-1}^{\Delta}, \mathbb{Z})$.

It is easy to see that $\tilde{\mathcal{X}}_r^{\Delta} \setminus \tilde{\mathcal{X}}_{r-1}^{\Delta}$ is a total space of the real vector bundle over $C_r(\mathbb{R})$ with rank $dn - 1 - r(n-1)$. Hence, by using Thom isomorphism and Poincaré duality, if $1 \leq r \leq d$, there is an isomorphism

$$\begin{aligned} E^1_{r,s} &= H_c^{dn-1-s+r}(\tilde{\mathcal{X}}_r^{\Delta} \setminus \tilde{\mathcal{X}}_{r-1}^{\Delta}, \mathbb{Z}) \cong H_c^{rn-s}(C_r(\mathbb{R}), \mathbb{Z}) \\ &\cong H^{rn-s}(S^r, \mathbb{Z}) = \begin{cases} \mathbb{Z} & (s - r = r(n-2),\ 1 \leq r \leq d) \\ 0 & (\text{otherwise}) \end{cases} \end{aligned}$$

By the dimensional reason, $E^1_{**} = E^{\infty}_{**}$ and we have:

Lemma 4.3 (Kozlowski-Yamaguchi, [4]). *If $n \geq 3$, there is an isomorphism*

$$H_k(H_n^d, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } k = r(n-2),\ 0 \leq r \leq d \\ 0 & \text{otherwise.} \end{cases} \qquad \square$$

If we use the scanning maps (cf. [3]), we have the more precise statement:

Theorem 4.4 (Kozlowski-Yamaguchi; [4], [11]). *If $n \geq 4$ or $n = 3$ with $d \equiv 1 \pmod 2$, there is a homotopy equivalence $H_n^d \simeq J_d(\Omega S^{n-1})$.* $\square$

Remark. If $n \geq 4$, H_n^d is simply connected and it is not so difficult to prove the above result. However, if $n = 3$, $\pi_1(H_3^d) = \mathbb{Z}$ and it seems that the proof for the homotopy stability is not so easy in this case. If $n = 3$ and $d \equiv 1 \pmod 2$, we can show that there is a free S^1-action on H_3^d such that there is a homotopy equivalence $H_3^d \simeq S^1 \times H_3^d/S^1$.

Conjecture. Is there a homotopy equivalence $H_3^d \simeq J_d(\Omega S^2)$ even if $d \equiv 0 \pmod 2$?

5 Generalization of Theorem 4.4.

In this section, we give some generalization of Theorem 4.4.

Definition. From now on, we assume that $2 \leq m < n$ be fixed integers, let $\{z_0, z_1, \cdots, z_m\}$ is a set of fixed variables, and for each $\epsilon \in$

$\{0,1\} = \mathbb{Z}/2 = \pi_0(\mathrm{Map}(\mathbb{RP}^m, \mathbb{RP}^n))$ we denote by $\mathrm{Map}_\epsilon(\mathbb{RP}^m, \mathbb{RP}^n)$ the corresponding path component of $\mathrm{Map}(\mathbb{RP}^m, \mathbb{RP}^n)$.

(i) Let $\mathrm{Map}^*_\epsilon(\mathbb{RP}^m, \mathbb{RP}^n)$ denote the space consisting of all based maps $f \in \mathrm{Map}_\epsilon(\mathbb{RP}^m, \mathbb{RP}^n)$, where $\mathbf{e}_k = [1 : 0 : \cdots : 0] \in \mathbb{RP}^k$ is the base point of $\mathbb{RP}^k$ $(k = m, n)$. Let $\psi_d : \mathbb{RP}^{m-1} \to \mathbb{RP}^n$ denote the map given by $\psi_d([x_0 : \cdots : x_{m-1}]) = [x_0^d : \cdots : x_{m-1}^d : 0 : 0 : \cdots : 0]$. We regard $\mathbb{RP}^{m-1}$ as a subspace of $\mathbb{RP}^m$ by identifying $[x_0 : \cdots : x_{m-1}]$ with $[x_0 : \cdots : x_{m-1} : 0]$, and define the subspace $F_d(m,n) \subset \mathrm{Map}^*(\mathbb{RP}^m, \mathbb{RP}^n)$ by $F_d(m,n) = \{f \in \mathrm{Map}^*(\mathbb{RP}^m, \mathbb{RP}^n) : f|\mathbb{RP}^{m-1} = \psi_d\}$. It is routine to see that there is a homotopy equivalence $F_d(m,n) \simeq \Omega^m S^n$.

(ii) Let $\mathcal{H}_d \subset \mathbb{R}[z_0, \cdots, z_m]$ be the subspace consisting of all homogenous polynomials of degree d, and for $\epsilon \in \{0,1\}$ let $\mathcal{H}_d^\epsilon \subset \mathcal{H}_d$ be the subspace consisting of all homogenous polynomials $f \in \mathcal{H}_d$ such that the coefficient of z_0^d of f is ϵ. For each integer $0 \leq k \leq n$, let $B_k \subset \mathcal{H}_d$ denote the subspace given by

$$B_k = \begin{cases} \{z_k^d + z_m h : h \in \mathcal{H}_{d-1}\} & \text{if } 0 \leq k < m \\ \{z_m h : h \in \mathcal{H}_{d-1}\} & \text{if } m \leq k \leq n \end{cases}$$

and let $A_d(m,n) \subset \mathcal{H}_d^0 \times (\mathcal{H}_d^1)^n$ be the subspace consisting of all $(n+1)$-tuples $(f_0, \cdots, f_n) \in \mathcal{H}_d^0 \times (\mathcal{H}_d^1)^n$ of homogenous polynomials of the same degree d such that $f_0, \cdots, f_n$ have no common *real root* except $\mathbf{0}_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$ (but may have non-trivial common complex roots).

Similarly, let $A_d^*(m,n) \subset A_d(m,n)$ denote the subspace defined by

$$A_d^*(m,n) = A_d(m,n) \cap (B_0 \times B_1 \times \cdots \times B_n).$$

(iii) Let $f = (f_0, \cdots, f_n) \in A_d(m,n)$ be any element and consider the map $i_d(f) : \mathbb{RP}^m \to \mathbb{RP}^n$ given by $i_d(f)([\mathbf{x}]) = [f_0(\mathbf{x}) : \cdots : f_n(\mathbf{x})]$ for $[\mathbf{x}] = [x_0 : \cdots : x_m] \in \mathbb{RP}^m$. This naturally induces the map

$$i_d : A_d(m,n) \to \mathrm{Map}^*_{[d]_2}(\mathbb{RP}^m, \mathbb{RP}^n),$$

where $[d]_2 \in \mathbb{Z}/2$ denotes the integer mod 2.

(iv) If $f \in A_d^*(m,n)$, since $i_d(f)|\mathbb{RP}^{m-1} = \psi_d$, the restriction $j_d = i_d|A_d^*(m,n)$ can be regarded as the map $j_d : A_d^*(m,n) \to F_d(m,n) \simeq \Omega^m S^n$. If we use the spectral sequence induced from the simplicial resolution and Vassiliev spectral sequence given in [9], we can prove the following:

Theorem 5.1 ([1]). *Let $2 \leq m < n$ be integers, and we set*

$$\begin{cases} M(m,n) = 2\lceil \frac{m+1}{n-m} \rceil + 1, & (\lceil x \rceil = \min\{N \in \mathbb{Z} : N \geq x\}) \\ D(d;m,n) = (n-m)\left(\lfloor \frac{d+1}{2} \rfloor + 1\right) - 1. \end{cases}$$

(i) *If $d \geq M(m,n)$, $j_d : A_d(m,n) \to \Omega^m S^n$ is a homotopy equivalence through dimension $D(d;m,n)$ when $m+2 \leq n$ and a homology equivalence through dimension $D(d;m,n)$ when $m+1 = n$.*

(ii) *If $d \geq M(m,n)$ is an even integer, $i_d : A_d(m,n) \to \mathrm{Map}_0^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is a homotopy equivalence through dimension $D(d;m,n)$ when $m+2 \leq n$ and a homology equivalence through dimension $D(d;m,n)$ when $m+1 = n$.* □

Remark. A map $f : X \to Y$ is called a homotopy (resp. homology) equivalence through dimension D if the induced homomorphism

$$f_* : \pi_k(X) \to \pi_k(Y) \quad (\text{resp. } f_* : H_k(X,\mathbb{Z}) \to H_k(Y,\mathbb{Z}))$$

is bijective for any $k \leq D$.

At the moment we cannot prove the homotopy (or homology) unstability theorem for the map $i_d : A_d(m,n) \to \mathrm{Map}_{[d]_2}^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ when $d \equiv 1 \pmod 2$. However, if $d = 1$, we can prove:

Theorem 5.2 ([12], [14]). *If $1 \leq m < n$ and $d = 1$, the map $i_1 : A_1(m,n) \to \mathrm{Map}_1^*(\mathbb{R}\mathrm{P}^m, \mathbb{R}\mathrm{P}^n)$ is a homotopy equivalence through dimension $D(m,n)$, where $D(m,n) := 2(n-m) - 2$.* □

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