数理解析研究所講究録
第 1618 巻 2008 年 111-126

Explicit and direct representations of the solutions of nonlinear simultaneous equations

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Abstract

In this paper we shall give practical, numerical and explicit representations of inverse mappings of n-dimensional mappings (of the solutions of n-nonlinear simultaneous equations) and show their numerical experiments by using computers. We derive those concrete formulas from very general ideas for the representation of the inverse functions.

Keywords: Inverse function, inverse mapping, integral transform, integral representation, Green-Stokes formula, singular integral, fundamental solution, reproducing kernel, Sobolev space, non-linear mapping, general nonlinear equation.

Mathematics Subject Classification (2000): Primary 93B30, 93C10, 35A35

1 Introduction

The 2nd author of this paper considered that for any mapping ϕ from an arbitrary abstract set into an arbitrary set, he tried to consider the representation of the inversion ϕ^{-1} in terms of the direct mapping ϕ and he obtained some simple concrete formulas from some general ideas in ([1]). In this paper, from its general ideas, we shall give practical representation formulas of some general functions. Here, we shall give furthermore some general methods and ideas for the inversion formulas for some general non-linear mappings. We shall first state the principles for our methods for the representations of inverses of non-linear mappings based on ([1]):

We shall consider some representation of the inversion ϕ^{-1} in terms of some integral form - at this moment, we shall need a natural assumption for the mapping ϕ . Then, we shall transform the integral representation by the mapping ϕ to the original space that is the defined domain of the mapping ϕ. Then, we will be able to obtain the representation of the inverse ϕ^{-1} in terms of the direct mapping ϕ. In [1], we considered the representation of the inverse ϕ^{-1} in some reproducing kernel Hilbert spaces, and in [4], we considered the representations of the inverse ϕ^{-1} for a very concrete situation and we gave a very fundamental representation of the inverse for some general functions on 1 dimensional spaces.

Indeed, note that

$$K(y_1, y_2) = \frac{1}{2}e^{-|y_1 - y_2|} \qquad y_1, \quad y_2 \in [A, B] \tag{1}$$

is the reproducing kernel for the Sobolev Hilbert space H_K whose members are real-valued and absolutely continuous functions on $[A, B]$ and whose inner product is given by

$$(f_1, f_2)_{H_K} = \int_A^B (f_1'(y)f_2'(y) + f_1(y)f_2(y))dy + f_1(A)f_2(A) + f_1(B)f_2(B) \tag{2}$$

([2]).

For a function $y = f(x)$ that is of C^1 class and a strictly increasing function and $f'(x)$ is not vanishing on $[a, b]$ $(f(a) = A, f(b) = B)$, of course, the inverse function $f^{-1}(y)$ is a single-valued function and it belongs to the space H_K and from the reproducing property, we obtain the representation, for any $y_0 \in [f(a), f(b)]$

$$\begin{aligned} f^{-1}(y_0) &= \left(f^{-1}(\cdot), K(\cdot, y_0)\right)_{H_K} \\ &= \int_{f(a)}^{f(b)} ((f^{-1})'(y)K_y(y, y_0) + f^{-1}(y)K(y, y_0))dy \\ &\quad + aK(f(a), y_0) + bK(f(b), y_0). \end{aligned} \tag{3}$$

From this representation, we obtained in ([4]) the very simple representation

$$f^{-1}(y_0) = \frac{a+b}{2} + \frac{1}{2}\int_a^b \operatorname{sign}(y_0 - f(x))dx. \tag{4}$$

Furthermore, by using the several reproducing kernel Hilbert spaces from [2] as in (3), we calculated similarly with the related assumptions, however, surprisingly enough, we obtain the same formula (4). For the formula (4), we note directly that we do not need any smoothness assumptions for the function $f(x)$, indeed, we need only the strictly increasing assumption. The assumption of integrability does not, even, need for the formula (4).

Now, we would like to obtain some multi-dimensional versions. At this moment, it seems that we can not find some simple representations as in (3) by some concrete known reproducing kernels for some general domains, and indeed, we know the reproducing kernels only for special domains and for special reproducing kernel Hilbert spaces.

In order to consider some general integral representations for some general functions, we shall recall the fundamental facts:

We can represent a function f in terms of the delta function δ in the form

$$f(q) = \int_D f(p)\delta(p-q)dp \tag{5}$$

in some domain, symbolically. Meanwhile, a fundamental solution $G(p-q)$ for some linear differential operator L is given by the equation, symbolically

$$LG(p-q) = \delta(p-q). \tag{6}$$

So, from (5) we obtain the representation

$$f(q) = \int_D f(p)LG(p-q)dp. \tag{7}$$

Then, we can obtain the representation symbolically, by using the Green-Stokes formula, for some adjoint operator L^* for L,

$$f(q) = \int_D L^* f(p)G(p-q)dp + \text{some boundary integrals.} \tag{8}$$

We shall firstly use this type representation. In this approach, we will meet the singular integral representation in the first term of (8), however, if $G(p-q)$ is integrable, then by a simple regularization for $G(p-q)$ we will be able to realize the representation in numerical treatments. In the separate paper [3] we discussed the natural regularization in the form, for example, for the singularity

$$\frac{1}{(|x-y|)^{\alpha}},$$

we consider the regularization

$$\frac{1}{(|x-y|+\delta)^{\alpha}}$$

for a small δ and we considered their error estimates.

We are interested in some very concrete results that may be realized by computers. So, we considered very concrete cases in the 2 dimensional spaces. It seems that the results will depend on dimensions, domains and functions spaces dealing with.

In [6], we considered the following typical problem:

Let $D \subset \mathbf{R}^2$ be a bounded domain with a finite number of piecewise C^1 class boundary components. Let f be a one-to-one C^1 class mapping from $\bar{D}$ into $\mathbf{R}^2$ and we assume that its Jacobian $J(x)$ is positive on D. We shall represent f as follows:

$$\begin{aligned} y_1 &= f_1(x) = f_1(x_1, x_2) \\ y_2 &= f_2(x) = f_2(x_1, x_2) \end{aligned} \tag{9}$$

and the inverse mapping f^{-1} of f as follows:

$$\begin{aligned} x_1 &= (f^{-1})_1(y) = (f^{-1})_1(y_1, y_2) \\ x_2 &= (f^{-1})_2(y) = (f^{-1})_2(y_1, y_2). \end{aligned} \tag{10}$$

Then, we represented

$$\begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix} \tag{11}$$

in terms of the direct mapping (9).

Of course, we are interested in some numerical and practical solutions of the non-linear simultaneous equations (9) and we obtained

Proposition 1 *([6]) For the mappings (9) and (10) with (11), we obtain the representation, for any $y^* = (y_1^*, y_2^*) \in f(D)$,*

$$\begin{aligned} \begin{pmatrix} (f^{-1})_1(y^*) \\ (f^{-1})_2(y^*) \end{pmatrix} &= \frac{1}{2\pi} \oint_{\partial D} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\,\mathrm{Arctan} \frac{f_2(x) - y_2^*}{f_1(x) - y_1^*} \\ &- \frac{1}{2\pi} \int\int_D \frac{1}{|f(x) - y^*|^2} \mathrm{adj}\, J(x) \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1 dx_2. \end{aligned} \tag{12}$$

In this paper, we shall first give the natural version for the 3 dimensional case by using the well-known Poisson integral formula and in the last, surprisingly enough we shall give some unified and natural inversion formulas for the general dimensions that are better than the formula derived from the Poisson integral formula.

2 3-dimensional formula derived from the Poisson integral formula

Let D be a bounded domain in $\mathbf{R}^3$ with a finite number of C^1 boundary components ∂D. Let f be a one to one C^2 class mapping of D onto $f(D)$ in $\mathbf{R}^3$ with sense preserving and we assume that its Jacobian is positive on D. We set

$$y = f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix}$$

and its inversion f^{-1} as follows:

$$x = (f^{-1})(y) = \begin{pmatrix} (f^{-1})_1(y) \\ (f^{-1})_2(y) \\ (f_3^{-1})_3(y) \end{pmatrix} = \begin{pmatrix} (f^{-1})_1(y_1, y_2, y_3) \\ (f^{-1})_2(y_1, y_2, y_3) \\ (f^{-1})_3(y_1, y_2, y_3) \end{pmatrix}.$$

Let $\Delta = \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}$ and $\frac{\partial y}{\partial x}(x)$ be the Jacobian of $y = (y_1, \cdots, y_n)$ with respect to $x = (x_1, \cdots, x_n)$. For a matrix A, let $(A)_i$ be the i low vector of A and $(A)_{ij}$ the i, j element of A.

We set the vector fields

$$S_i(x) = \sum_{j=1}^{3} \frac{\mathrm{adj}({}^t(\frac{\partial y}{\partial x}(x))(\frac{\partial y}{\partial x}(x))_{ij}}{\det(\frac{\partial y}{\partial x}(x))} \frac{\partial}{\partial x_j} \tag{13}$$

$$T_i(x) = \frac{x_i}{|y_0 - f(x)|^2} \mathrm{adj}(\frac{\partial y}{\partial x}(x))_i \cdot (y_0 - f(x)) \sum_{j=1}^{3} \frac{\partial}{\partial x_j}. \tag{14}$$

Then, we obtain the theorem:

Theorem 1 For any point $y_0 \in f(D)$, we obtain the representation

$$(f^{-1})_i(y_0) = -\frac{1}{4\pi}\int\int\int_D \frac{1}{|y_0 - f(x)|}\operatorname{div} S_i(x)\, dx_1 dx_2 dx_3 + \frac{1}{4\pi}\int\int_{\partial D} \frac{1}{|y_0 - f(x)|}(S_i - T_i)(x)\cdot d\mathbf{A}_x \qquad i = 1, 2, 3. \tag{15}$$

Let U and V be bounded domains in $\mathbf{R}^n$ and we write a C^2 class and one to one mapping from U onto V as follows: $y_i = y_i(x), \quad i = 1, \cdots, n$. We denote its inversion by $x_i = x_i(y)$. Then, we obtain, directly

Lemma 1 For the pull back y^* of the mapping y, we have

$$y^*(\Delta x_i(y)\, dy_1 \wedge \cdots \wedge dy_n) = \operatorname{div} \frac{\left(\operatorname{adj}\left\{{}^t\left(\frac{\partial y}{\partial x}(x)\right)\frac{\partial y}{\partial x}(x)\right\}\right)_i}{\det \frac{\partial y}{\partial x}(x)}\, dx_1 \wedge \cdots \wedge dx_n \qquad i = 1, \cdots, n. \tag{16}$$

Proof. We consider the differential $n-1$ form on V

$$\omega_i = \sum_{j=1}^{n}(-1)^{j-1}\frac{\partial x_i}{\partial y_j}(y)\, dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n, \quad i = 1, \cdots, n.$$

Then,

$$d\omega_i = \sum_{j=1}^{n}(-1)^{2(j-1)}\frac{\partial^2 x_i}{\partial y_j^2}(y)\, dy_1 \wedge \cdots \wedge dy_n,$$

that is the part in ( ) in the right hand side in (16). Meanwhile,

$$y^*\omega_i = \sum_{j=1}^{n}(-1)^{j-1}y^*\left(\frac{\partial x_i}{\partial y_j}(y)\right) y^*(dy_1 \wedge \cdots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \cdots \wedge dy_n)$$

and by the inverse function theorem

$$y^*\left(\frac{\partial x_i}{\partial y_j}(y)\right) = \left(\frac{\partial y}{\partial x}(x)^{-1}\right)_{ij} = \frac{1}{\det \frac{\partial y}{\partial x}(x)}\left(\operatorname{adj} \frac{\partial y}{\partial x}(x)\right)_{ij}$$

where

$$\left(\operatorname{adj} \frac{\partial y}{\partial x}(x)\right)_{ij} = (-1)^{i+j}\det \frac{\partial y}{\partial x}(x)\left(\det \frac{\partial(y_1,\cdots,y_{j-1},y_{j+1},\cdots,y_n)}{\partial(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_n)}(x)\right).$$

Furthermore,

$$\begin{aligned}
&y^*(dy_1\wedge\cdots\wedge dy_{j-1}\wedge dy_{j+1}\wedge\cdots\wedge dy_n)\\
&=\sum_{k=1}^{n}\det\frac{\partial(y_1,\cdots,y_{j-1},y_{j+1},\cdots,y_n)}{\partial(x_1,\cdots,x_{k-1},x_{k+1},\cdots,x_n)}(x)\,dx_1\wedge\cdots\wedge dx_{k-1}\wedge dx_{k+1}\wedge\cdots\wedge dx_n\\
&=\sum_{k=1}^{n}(-1)^{j+k}\left(\operatorname{adj}\frac{\partial y}{\partial x}(x)\right)_{kj} dx_1\wedge\cdots\wedge dx_{k-1}\wedge dx_{k+1}\wedge\cdots\wedge dx_n.
\end{aligned}$$

Hence,

$$\begin{aligned}
y^*\omega_i &= \frac{1}{\det\frac{\partial y}{\partial x}(x)}\sum_{j,k=1}^{n}(-1)^{(j-1)+(j+k)}\left(\operatorname{adj}\frac{\partial y}{\partial x}(x)\right)_{ij}\left(\operatorname{adj}\frac{\partial y}{\partial x}(x)\right)_{kj}\\
&\quad\times dx_1\wedge\cdots\wedge dx_{k-1}\wedge dx_{k+1}\wedge\cdots\wedge dx_n\\
&=\frac{1}{\det\frac{\partial y}{\partial x}(x)}\sum_{k=1}^{n}(-1)^{k-1}\left(\operatorname{adj}\frac{\partial y}{\partial x}(x)\operatorname{adj}{}^t\left(\frac{\partial y}{\partial x}(x)\right)\right)_{ik}\\
&\quad\times dx_1\wedge\cdots\wedge dx_{k-1}\wedge dx_{k+1}\wedge\cdots\wedge dx_n\\
&=\frac{1}{\det\frac{\partial y}{\partial x}(x)}\sum_{k=1}^{n}(-1)^{k-1}\left(\operatorname{adj}\left\{{}^t\left(\frac{\partial y}{\partial x}(x)\right)\frac{\partial y}{\partial x}(x)\right\}\right)_{ik}\\
&\quad\times dx_1\wedge\cdots\wedge dx_{k-1}\wedge dx_{k+1}\wedge\cdots\wedge dx_n.
\end{aligned}$$

Meanwhile,

$$d(y^*\omega_i) = \sum_{k=1}^{n}(-1)^{2(k-1)}\frac{\partial}{\partial x_k}\frac{\left(\operatorname{adj}\left\{{}^t\left(\frac{\partial y}{\partial x}(x)\right)\frac{\partial y}{\partial x}(x)\right\}\right)_{ik}}{\det\frac{\partial y}{\partial x}(x)}\,dx_1\wedge\cdots\wedge dx_n$$

which is the right hand side of (16). Therefore, from $y^*(d\omega_i) = d(y^*\omega_i)$, we have the desired result.

Example. $n = 1$

$$y^*(\Delta x(y)\,dy) = \frac{d}{dx}\frac{1}{\frac{dy}{dx}(x)}\,dx.$$

Example. $n = 2$

$$y^*(\Delta x_1(y)\,dy_1 \wedge dy_2) = \left(\frac{\partial}{\partial x_1}\frac{\frac{\partial y}{\partial x_2}(x)\cdot\frac{\partial y}{\partial x_2}(x)}{\det\frac{\partial y}{\partial x}(x)} - \frac{\partial}{\partial x_2}\frac{\frac{\partial y}{\partial x_1}(x)\cdot\frac{\partial y}{\partial x_2}(x)}{\det\frac{\partial y}{\partial x}(x)}\right)dx_1 \wedge dx_2$$

$$y^*(\Delta x_2(y)\,dy_1 \wedge dy_2) = \left(-\frac{\partial}{\partial x_1}\frac{\frac{\partial y}{\partial x_1}(x)\cdot\frac{\partial y}{\partial x_2}(x)}{\det\frac{\partial y}{\partial x}(x)} + \frac{\partial}{\partial x_2}\frac{\frac{\partial y}{\partial x_1}(x)\cdot\frac{\partial y}{\partial x_1}(x)}{\det\frac{\partial y}{\partial x}(x)}\right)dx_1 \wedge dx_2.$$

Here, $\cdot$ denotes the inner product and

$$\frac{\partial y}{\partial x_l}(x) = \begin{pmatrix}\frac{\partial y_1}{\partial x_l}(x)\\ \frac{\partial y_2}{\partial x_l}(x)\end{pmatrix}.$$

Proof of Theorem 1.

By the Poisson integral formula, we have, when $\Delta f^{-1} = 0$ on $f(D)$

$$f^{-1}(y_0) = \frac{1}{4\pi}\int\int_{\partial f(D)}\left\{\frac{1}{|y_0-y|}\frac{\partial f^{-1}(y)}{\partial \nu_y} - f^{-1}(y)\frac{\partial}{\partial \nu_y}\frac{1}{|y_0-y|}\right\}dA_y$$

(where ν denotes the inner normal derivative)

$$= \frac{1}{4\pi}\int\int_{\partial f(D)}\left\{\frac{1}{|y_0-y|}\operatorname{grad} f^{-1}(y) - f^{-1}(y)\operatorname{grad}\frac{1}{|y_0-y|}\right\}\cdot \nu_y dA_y$$

($* : A^p(\partial f(D)) \to A^{3-p}(\partial f(D)), p = 1, 2, 3$ denotes the Hodge $*$ operator)

$$= \frac{1}{4\pi}\int\int_{\partial f(D)} *\left\{\frac{1}{|y_0-y|}d\,f^{-1}(y) - f^{-1}(y)\,d\left(\frac{1}{|y_0-y|}\right)\right\}$$

($\psi_i : V_i \to \partial f(D)$ denotes the local coordinates)

$$
\begin{aligned}
&= \frac{1}{4\pi}\sum_i \int\int_{V_i} \psi_i^* * \left\{\frac{1}{|y_0 - y|} d\, f^{-1}(y) - f^{-1}(y)\, d\Big(\frac{1}{|y_0 - y|}\Big)\right\} \\
&= \frac{1}{4\pi}\sum_i \int\int_{V_i} \frac{1}{|y_0 - \psi_i|}\left\{(f^{-1})'(\psi_i) - f^{-1}(\psi_i)\frac{y_0 - \psi_i}{|y_0 - \psi_i|^2}\right\}\begin{pmatrix} d\psi_{i2}\wedge d\psi_{i3} \\ d\psi_{i3}\wedge d\psi_{i1} \\ d\psi_{i1}\wedge d\psi_{i2}\end{pmatrix} \\
&= \frac{1}{4\pi}\sum_j \int\int_{U_j} \frac{1}{|y_0 - f(\phi_j)|}\left\{f'(\phi_j)^{-1} - \phi_j\frac{y_0 - f(\phi_j)}{|y_0 - f(\phi_j)|^2}\right\}\begin{pmatrix} df_2(\phi_j)\wedge df_3(\phi_j) \\ df_3(\phi_j)\wedge df_1(\phi_j) \\ df_1(\phi_j)\wedge df_2(\phi_j)\end{pmatrix} \\
&= \frac{1}{4\pi}\sum_j \int\int_{U_j} \frac{1}{|y_0 - f(\phi_j)|}\left\{\frac{\operatorname{adj} f'(\phi_j)}{\det f'(\phi_j)} - \phi_j\frac{y_0 - f(\phi_j)}{|y_0 - f(\phi_j)|^2}\right\}\operatorname{adj}{}^t f'(\phi_j)\begin{pmatrix} d\phi_{j2}\wedge d\phi_{j3} \\ d\phi_{j3}\wedge d\phi_{j1} \\ d\phi_{j1}\wedge d\phi_{j2}\end{pmatrix} \\
&= \frac{1}{4\pi}\sum_j \int\int_{U_j} \frac{1}{|y_0 - f(\phi_j)|}\left\{\frac{\operatorname{adj}{}^t f'(\phi_j) f'(\phi_j)}{\det f'(\phi_j)}\right. \\
&\qquad \left. - \frac{\phi_j}{|y_0 - f(\phi_j)|^2}{}^t\Big(\big(\operatorname{adj} f'(\phi_j)\big)\big(y_0 - f(\phi_j)\big)\Big)\right\}\begin{pmatrix} d\phi_{j2}\wedge d\phi_{j3} \\ d\phi_{j3}\wedge d\phi_{j1} \\ d\phi_{j1}\wedge d\phi_{j2}\end{pmatrix}. \qquad (17)
\end{aligned}
$$

Therefore by using $S_i, T_i, i = 1, 2, 3$ we have

$$
\begin{aligned}
(f^{-1})_i(y_0) &= \frac{1}{4\pi}\int\int_{\partial D} \frac{1}{|y_0 - f(x)|}(S_i - T_i)(x)\cdot \nu_x\, dA_x \\
&= \frac{1}{4\pi}\int\int_{\partial D} \frac{1}{|y_0 - f(x)|}(S_i - T_i)(x)\cdot d\mathbf{A}_x \qquad i = 1, 2, 3.
\end{aligned}
$$

In general, by using Lemma 1, we obtain the theorem.

3 n-dimensional formulas

We shall give the very beautiful representation

Theorem 2 Let D be a bounded domain in $\mathbf{R}^n$ with a finite number ∂D of C^1 class boundary components. Let f be a C^1 class real-valued function on $\bar{D}$. For any $\hat{x} \in D$ and for any $n \in \mathbf{N}$ we have the representation

$$
f(\hat{x}) = -c_n\big(df(x), dG_n(x - \hat{x})\big) + c_n \int_{\partial D} f(x) * dG_n(x - \hat{x}) \qquad (18)
$$

Here, for $n \leq 2, c_n = 1$ and for $n \geq 3, c_n = n-2$. $*$ is the Hodge star operator, G_n the fundamental solution of the Laplacian $\Delta_n = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$, and $(\ ,\ )$ the inner product of the vector space $A^k(D)$ comprising of the k order differential forms over D with finite L^2 norms that is

$$(\omega, \eta) = \int_D \omega \wedge *\eta = \int_D \eta \wedge *\omega \quad (\omega, \eta \in A^k(D)).$$

Lemma 2 Let $U_\epsilon(0)$ be an ϵ neighbourhood with centre 0, then

$$\int_{\partial U_\epsilon(0)} *dG_n(x) = \frac{1}{c_n}.$$

Proof. Let A_n be $A_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$, the surface measure of the n dimensional unit disk. Then,

$$G_n(x) = \frac{1}{c_n A_n} \begin{cases} |x| & (n=1) \\ \log|x| & (n=2) \quad \text{(logarithmic kernel)} \\ -\frac{1}{|x|^{n-2}} & (n \geq 3) \quad \text{(Newton kernel).} \end{cases}$$

Hence, on $\mathbf{R}^n \setminus U_\epsilon(0)$ we have $dG_n(x) = \frac{\sum_{i=1}^n x_i dx_i}{c_n A_n |x|^n}$ $(\forall n \in \mathbf{N})$. Then, for $x = (x_1, \cdots, x_n)$

$$*dG_n(x) = \frac{\sum_{i=1}^n (-1)^{i-1} x_i dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge dx_n}{c_n A_n |x|^n} \quad (\forall n \in \mathbf{N}).$$

For a local coordinate $\phi : U_\epsilon(0) \longrightarrow \mathbf{R}^n$, we denote the pull back $\phi^* * dG_n(x)$ of $*dG_n(x)$ by the polar coordinate, by using $x = \phi(\theta) = \epsilon\tilde{\phi}(\theta), \theta = (\theta_1, \cdots, \theta_{n-1}) \in [0,\pi] \times \cdots \times [0,\pi] \times [0, 2\pi]$,

$$\tilde{\phi}(\theta) = \left(\cos\theta_1, \quad \sin\theta_1 \cos\theta_2, \quad \cdots, \quad \sin\theta_1 \cdots \sin\theta_{n-2}\cos\theta_{n-1}, \quad \sin\theta_1 \cdots \sin\theta_{n-2}\sin\theta_{n-1}\right),$$

we have $\phi^* * dG_n(x) = \frac{\sin^{n-2}\theta_1 \cdots \sin\theta_{n-2}}{c_n A_n} d\theta_1 \wedge \cdots \wedge d\theta_{n-1}$. Hence,

$$\int_{\partial U_\epsilon(0)} *dG_n(x) = \int_{[0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi]} \phi^* * dG_n(x) = \frac{1}{c_n}.$$

Proof of Theorem 2.
Let $U_\epsilon(\hat{x})$ be a neighbourhood contained in D. Then, for $G_n(x-\hat{x}) \in C^\infty(D \setminus U_\epsilon(\hat{x}))$ and for $f \in C^1(D)$, $f(x) * dG_n(x-\hat{x}) \in A^1(D \setminus U_\epsilon(\hat{x}))$ that is a C^1 class differential. Hence, on $D \setminus U_\epsilon(\hat{x})$, for $f(x) * dG_n(x-\hat{x})$ we apply the Green-Stokes formula and we have

$$\begin{aligned}&\int_{D\setminus U_\epsilon(\hat{x})} d\{f(x) * dG_n(x-\hat{x})\}\\ &= \int_{\partial D} f(x) * dG_n(x-\hat{x}) - \int_{\partial U_\epsilon(\hat{x})} f(x) * dG_n(x-\hat{x}).\end{aligned}$$

Let δ be the Dirac distribution and ω_v be $dx_1 \wedge \cdots \wedge dx_n$. Then, by $d * dG_n(x-\hat{x}) = \Delta_n G_n(x-\hat{x})\omega_v = \delta(x-\hat{x})\omega_v$, we have

$$\begin{aligned}d\{f(x) * dG_n(x-\hat{x})\} &= df(x) \wedge *dG_n(x-\hat{x}) + (-1)^0 f(x) d * dG_n(x-\hat{x})\\ &= df(x) \wedge *dG_n(x-\hat{x}) + \delta(x-\hat{x}) f(x)\omega_v.\end{aligned}$$

Hence,

$$\lim_{\epsilon\to 0} \int_{D\setminus U_\epsilon(\hat{x})} d\{f(x) * dG_n(x-\hat{x})\} = \big(df(x), dG_n(x-\hat{x})\big).$$

As in the proof of Lemma 2, from the polar coordinate representation $\phi^*(f(x) * dG_n(x-\hat{x})) = \phi^* f(x)\phi^* * dG_n(x-\hat{x}) = f(\hat{x}+\epsilon\tilde{\phi}(\theta))\phi^* * dG_n(x-\hat{x})$ and from Lemma 2,

$$\begin{aligned}\int_{\partial U_\epsilon(\hat{x})} f(x) * dG_n(x-\hat{x}) &= \int_{[0,\pi]\times\cdots\times[0,\pi]\times[0,2\pi]} f(\hat{x}+\epsilon\tilde{\phi}(\theta))\phi^* dG_n(x-\hat{x})\\ &= \frac{f(\hat{x})}{c_n} \quad (\epsilon \to 0).\end{aligned}$$

We thus obtain the desired representation.

Theorem 3 In the situation of Theorem 2 and we assume furthermore that f is a sense preserving C^1 class function on $\bar{D}$ in $\mathbf{R}^n$ with a single-valued inverse. Then, for $\hat{y} \in f(D)$, we obtain the representation

$$f_i^{-1}(y_0) = -\int_D dx_i \wedge f^* * dG_n(y - y_0) + \int_{\partial D} x_i f^* * dG_n(y - y_0).$$

Here, f_i^{-1} denotes the i component of f^{-1}.

Proof. For the function f^{-1} on $f(\bar{D})$, we use the representation in Theorem 2 and we use the transform of the representation by f. Then, by using the formulas $f^* df_i^{-1}(y) = dx_i$, and $f^*(\omega, \eta) = (f^*\omega, f^*\eta)$, we obtain the desired representation.

In particular, for $n = 1$, we obtain (4), directly.

For $n = 2$, we obtain (13) and this formula may be represented as follows, from our general formula:

For any $\hat{y} \in f(D)$, we have

$$f_i^{-1}(\hat{y}) = \frac{1}{2\pi}\left(\int_{\partial D} x_i d\theta_i - \int_D dx_i \wedge d\theta_i\right)$$
$$i = 1, 2.$$

Here, $\theta_1 = \text{Arctan}\,\frac{f_1(x) - \hat{y}_1}{f_2(x) - \hat{y}_2}$, $\theta_2 = -\,\text{Arctan}\,\frac{f_2(x) - \hat{y}_2}{f_1(x) - \hat{y}_1}$. In particular, furthermore, when D is a convex domain, we have the representation

$$f_i^{-1}(\hat{y}) = \frac{\hat{x}_i^{min} + \hat{x}_i^{max}}{2} + \frac{1}{2\pi}\left(\int_{\partial D} \theta_i dx_i - \int_D dx_i \wedge d\theta_i\right) \qquad i = 1, 2.$$

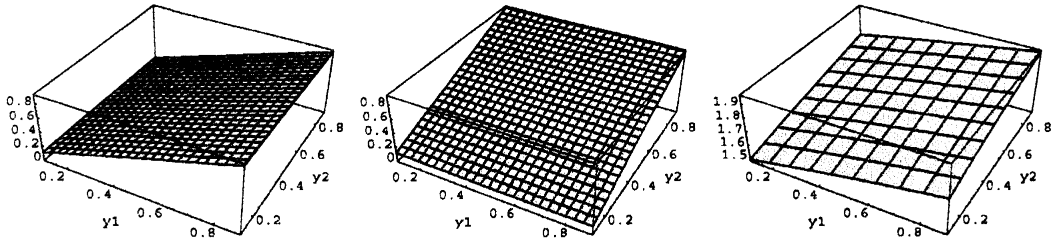
Here, $\hat{x}_i^{min}$ and $\hat{x}_i^{max}$ are determined by $\hat{y}$ as the two points of ∂D ([6]).

4 Numerical experiments

We shall give some simple numerical examples. The integrations are computed by Mathematica$^{\text{TM}}$. Consider the mapping f on $\bar{D} = [0, 1]^2 \times [1, 2]$ as follows:

$$
\begin{aligned}
y_1 &= f_1(x) = x_1, \\
y_2 &= f_2(x) = x_2, \\
y_3 &= f_3(x) = -x_1 - x_2 + x_3^2.
\end{aligned}
$$

Since $\det f'(x) = 2x_3 > 0$ on D and f_3 is subharmonic because of $\Delta f_3(x) = 2 > 0$, Theorem1 can be applied. Fig.1 (a), (b) and (c) shows the graph of $f_1^{-1}|_{[0.1,0.9]^2\times\{2\}}$, $f_2^{-2}|_{[0.1,0.9]^2\times\{2\}}$ and $f_3^{-1}|_{[0.1,0.9]^2\times\{2\}}$ computed by (16), respectively.



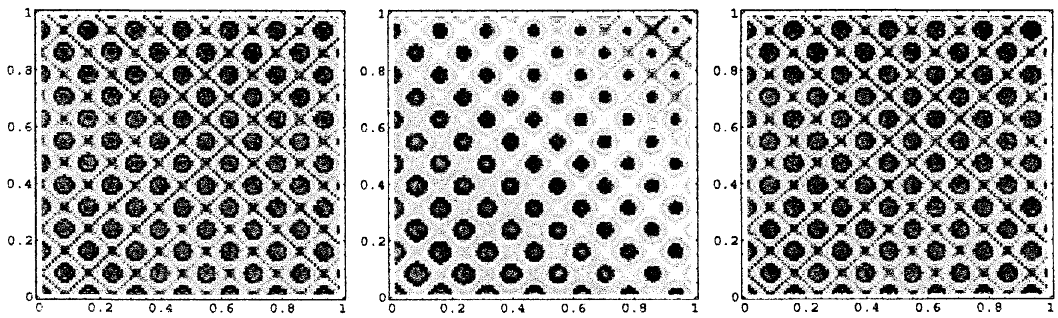
(a) $f_1^{-1}|_{[0.1,0.9]^2\times\{2\}}$ (b) $f_2^{-1}|_{[0.1,0.9]^2\times\{2\}}$ (c) $f_3^{-1}|_{[0.1,0.9]^2\times\{2\}}$

Figure 1: the graph of $f^{-1}|_{[0.1,0.9]^2\times\{2\}}$ computed by (16).

Next, regard the mapping on $[0, 1]^2$

$$
g(u, v) = \frac{1}{2}|\cos(40u) + \cos(40v)| + 1
$$

as an original image data and regard $\tilde{g}(u, v) = f_3(u, v, g(u, v))$ on $[0, 1]^2$ as the transformed image data. Then $\tilde{\tilde{g}}(u, v) = f_3^{-1}(u, v, \tilde{g}(u, v))$ on $[0, 1]^2$ can be considered to be the reconstructed image data because of $\tilde{\tilde{g}}(u, v) = g(u, v)$. Figure 2(a) and Figure 2(b) show $g(u, v)$ and $\tilde{g}(u, v)$, respectively. Figure 2(c) shows $\tilde{\tilde{g}}(u, v)$ computed by (16).



(a) $g(u, v)$ on $[0.01, 0.99]$ (b) $\tilde{g}(u, v)$ on $[0.01, 0.99]$ (c) $\tilde{\tilde{g}}(u, v)$ on $[0.01, 0.99]$

Figure 2: Numerical image reconstruction computed by (16).

For the case of the identity on $D = [0,1]^3$
On $D = [0,1]^3$, we consider the identity mapping

$$\begin{aligned} y_1 &= f_1(x) = x_1, & x_1 &= f_1^{-1}(y) = y_1 \\ y_2 &= f_2(x) = x_2, & x_2 &= f_2^{-1}(y) = y_2 \\ y_3 &= f_3(x) = x_3, & x_3 &= f_3^{-1}(y) = y_3. \end{aligned}$$

Then, for $x \in D, \det f'(x) = 1 > 0$ and from $D = f(\bar{D}) = [0,1]^3, \partial D = \partial f(D) = \cup_{j=1}^{6} \phi_j(U)$. Here, $U = [0,1]^2$, and we assume that

$$\phi_1(u,v) = \begin{pmatrix} 1 \\ u \\ v \end{pmatrix}, \phi_2(u,v) = \begin{pmatrix} 0 \\ v \\ u \end{pmatrix}, \phi_3(u,v) = \begin{pmatrix} v \\ 1 \\ u \end{pmatrix},$$
$$\phi_4(u,v) \doteq \begin{pmatrix} u \\ 0 \\ v \end{pmatrix}, \phi_5(u,v) = \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}, \phi_6(u,v) = \begin{pmatrix} v \\ u \\ 0 \end{pmatrix}.$$

From (2), we have

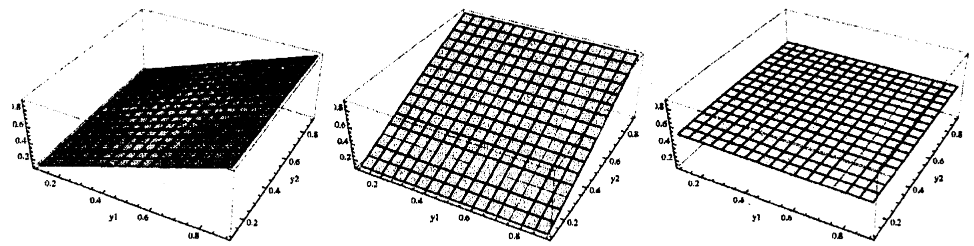
$$f^{*}{*}dG_3(y-\bar{y}) == \frac{1}{4\pi|x-\bar{y}|^3}\left((x_1-\bar{y}_1)dx_2\wedge dx_3+(x_2-\bar{y}_2)dx_3\wedge dx_1+(x_3-\bar{y}_3)dx_1\wedge dx_2\right).$$

Hence, from

$$\int_{\partial D} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} f^* * dG_n(y-\bar{y}) = \int_U \sum_{j=1}^{6} \phi_j^* \left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} f^* * dG_n(y-\bar{y}) \right)$$

we obtain by Theorem 2,

$$f^{-1}(\bar{y}) = \frac{1}{4\pi}\int_0^1\int_0^1\left(\frac{1-\bar{y}_1}{|\phi_1-\bar{y}|^3}\begin{pmatrix}1\\u\\v\end{pmatrix} + \frac{\bar{y}_1}{|\phi_2-\bar{y}|^3}\begin{pmatrix}0\\v\\u\end{pmatrix} + \frac{1-\bar{y}_2}{|\phi_3-\bar{y}|^3}\begin{pmatrix}v\\1\\u\end{pmatrix}\right.$$
$$\left. + \frac{\bar{y}_2}{|\phi_4-\bar{y}|^3}\begin{pmatrix}u\\0\\v\end{pmatrix} + \frac{1-\bar{y}_3}{|\phi_5-\bar{y}|^3}\begin{pmatrix}u\\v\\1\end{pmatrix} + \frac{\bar{y}_3}{|\phi_6-\bar{y}|^3}\begin{pmatrix}v\\u\\0\end{pmatrix}\right)dudv$$
$$- \frac{1}{4\pi}\int_0^1\int_0^1\int_0^1 \frac{1}{|x-\bar{y}|^3}\begin{pmatrix}x_1-\bar{y}_1\\x_2-\bar{y}_2\\x_3-\bar{y}_3\end{pmatrix} dx_1dx_2dx_3.$$



the graphs of f^{-1} on $\{\bar{y}_0 \in [0.1, 0.9]^3 | \bar{y}_3 = 0.5\}$.

Acknowledgements

S. Saitoh is supported in part by the Grant-in-Aid for Scientific Research (C)(2)(No. 19540164) from the Japan Society for the Promotion Science.

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