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The branching structure of trees on directed sets

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1 Introduction

Trees are familiar objects in diverse mathematical fields. In set theory we are mostly concerned with infinite trees. One basic problem on infinite trees is the existence of paths (also called cofinal branches). The tree property for a cardinal κ is the statement that any κ-tree has a path, i.e. that there are no κ-Aronszajn trees. R. Hinnion gave a generalization of trees; he defined the notion of a κ-tree on a directed set D ([3]). He also defined the tree property for such trees. In this paper we exhibit some statements which are derived from an attempt to generalize a characterization of the tree property (see Theorem 4.1). The characterization involves the tree property for directed set $\mathcal{P}_\kappa\lambda$. In the sequel, we collect the definitions and statements which are necessary for the characterization.

2 Directed sets and cofinal types

In 1922, E. H. Moore and H. L. Smith generalized the notion of convergence ([6]). The idea was to replace the domain of a usual sequence, the set of natural numbers, by an arbitrary directed set. This generalized notion of a sequence is called a net. In his book [8], J. W. Tukey studied topological properties such as uniformity and compactness using nets. On the way he introduced an ordering on the class of all directed sets to compare the ability of convergence of nets having different directed sets as domains. This ordering is known as the Tukey ordering. To begin with, we summarize the definition of the Tukey ordering and related cardinal functions on directed sets.

Definition 2.1 ([8]) Let $\langle D, \leq_D\rangle, \langle E, \leq_E\rangle$ be directed sets. A function $f\colon E \to D$ which satisfies

$$\forall d \in D \exists e \in E \forall e' \geq_E e\ [f(e') \geq_D d]$$

is called a *convergent function.* If such a function exists we write $D \leq E$ and say E *is cofinally finer than* D. $\leq$ is transitive and is called the *Tukey ordering* on the class of directed sets. A function $g\colon D \to E$ which satisfies

$$\forall e \in E \exists d \in D \forall d' \in D\ [g(d') \leq_E e \to d' \leq_D d]$$

is called a *Tukey function.*

If there exists a directed set C which has cofinal subsets D' and E', respectively isomorphic to D and E, then we say D *is cofinally similar with* E. In this case we write $D \equiv E$. $\equiv$ is an equivalence relation, and the equivalence classes with respect to $\equiv$ are the *cofinal types.*

Proposition 2.2 ([8], see also [7]) *For directed sets D and E, the following are equivalent.*

(a) $D \equiv E$.
(b) $D \leq E$ *and* $E \leq D$.

So we can regard the Tukey ordering as an ordering on the class of all cofinal types.

Definition 2.3 For a directed set D,

$$\mathsf{add}(D) \stackrel{\text{def}}{=} \min\{|X| \mid X \subseteq D \text{ unbounded}\},$$
$$\mathsf{cof}(D) \stackrel{\text{def}}{=} \min\{|C| \mid C \subseteq D \text{ cofinal}\}.$$

These are the *additivity* and the *cofinality* of a directed set. We restrict ourselves to directed sets D without maximum, so $\mathsf{add}(D)$ is well-defined.

Proposition 2.4 *For a directed set D (without maximum),*

$$\aleph_0 \leq \mathsf{add}(D) \leq \mathsf{cof}(D) \leq |D|.$$

Furthermore, $\mathsf{add}(D)$ *is regular and* $\mathsf{add}(D) \leq \mathsf{cf}(\mathsf{cof}(D))$. *Here* $\mathsf{cf}(\kappa)$ *denotes the cofinality (or additivity, since these are the same) of a cardinal κ.*

Proposition 2.5 *For directed sets D and E, $D \leq E$ implies*

$$\mathsf{add}(D) \geq \mathsf{add}(E) \quad \textit{and} \quad \mathsf{cof}(D) \leq \mathsf{cof}(E).$$

From Proposition 2.2 and 2.5 we see that the cardinal functions add and cof are invariant under cofinal similarity.

In the following, κ is always an infinite regular cardinal. If P is partially ordered set, we use the notation $X_{\leq a} = \{x \in X \mid x \leq a\}$ for X a subset of P and $a \in P$. As usual, $\mathcal{P}_\kappa\lambda = \{x \subseteq \lambda \mid |x| < \kappa\}$ is ordered by inclusion. Note that $\mathsf{add}(\mathcal{P}_\kappa\lambda) = \kappa$.

Definition 2.6 ([4]) We define the *width* of a directed set D by

$$\mathsf{wid}(D) \stackrel{\text{def}}{=} \sup\{|X|^+ \mid X \text{ is a thin subset of } D\},$$

where 'a *thin* subset of D' means that it satisfys

$$\forall d \in D[|X_{\leq d}| < \mathsf{add}(D)].$$

Lemma 2.7 ([4]) *For a directed set D and a cardinal $\lambda \geq \kappa := \mathsf{add}(D)$, the following are equivalent.*

(a) *D has a thin subset of size λ.*
(b) *$D \geq \mathcal{P}_\kappa\lambda$.*
(c) *There exists an order-preserving function $f\colon D \to \mathcal{P}_\kappa\lambda$ with $f[D]$ cofinal in $\mathcal{P}_\kappa\lambda$.*

Corollary 2.8 ([4]) *The width is invariant under cofinal similarity, and*

$$\begin{aligned}\mathsf{wid}(D) &= \sup\{\lambda^+ \mid D \geq \mathcal{P}_\kappa\lambda\}\\ &= \sup\{\lambda^+ \mid \exists f\colon D \to \mathcal{P}_\kappa\lambda \textit{ order-preserving with } f[D] \textit{ cofinal in } \mathcal{P}_\kappa\lambda\}.\end{aligned}$$

Lemma 2.9 $\mathsf{add}(D)^+ \leq \mathsf{wid}(D) \leq \mathsf{cof}(D)^+$.

3 The tree property for directed sets

Definition 3.1 (κ-tree) ([2]) Let D denote a directed set. A triple $\langle T, \leq_T, s\rangle$ is said to be a *κ-tree* on D if the following holds.

1) $\langle T, \leq_T\rangle$ is a partially ordered set.
2) $s\colon T \to D$ is an order preserving surjection.
3) For all $t \in T$, $s \restriction T_{\leq t}\colon T_{\leq t} \xrightarrow{\sim} D_{\leq s(t)}$ (order isomorphism).
4) For all $d \in D$, $|s^{-1}\{d\}| < \kappa$. We call $s^{-1}\{d\}$ the *level d* of T.

Note that under conditions 1)2)4), condition 3) is equivalent to 3'):

3') (downwards uniqueness principle) $\forall t \in T \forall d' \leq_D s(t) \exists!\, t' \leq_T t\ [s(t') = d']$.

If a κ-tree $\langle T, \leq_T, s \rangle$ satisfies in addition

5) (upwards access principle) $\forall t \in T \forall d' \geq_D s(t) \exists t' \geq_T t\ [s(t') = d']$,

then it is called a *κ-arbor* on D.

In the above definition, if we choose D to be an infinite regular cardinal κ, then the function s corresponds to the height function, and thus a 'κ-tree on κ' coincides with the classical 'κ-tree'. Moreover, an 'arbor' is a generalization of a 'well pruned tree'.

Definition 3.2 (tree property) ([2]) Let $\langle D, \leq_D \rangle$ be a directed set and $\langle T, \leq_T, s \rangle$ a κ-tree on D. $f : D \to T$ is said to be a faithful embedding if f is an order embedding and satisfies $s \circ f = \mathrm{id}_D$. If for each κ-tree T on D there is a faithful embedding from D to T, we say that D has the *κ-tree property.* If D has the $\mathsf{add}(D)$-tree property, we say simply D has the *tree property.*

The reason why we are interested in the case $\kappa = \mathsf{add}(D)$ is explained in [2]. Note that if $D = \kappa$ a faithful embedding corresponds to a path on a tree.

Proposition 3.3 ([2]) *Let D be directed set and let $\kappa = \mathsf{add}(D)$. D has the tree property iff for any κ-arbor on D there is a faithful embedding into it.*

Proposition 3.4 ([4], [5]) *If E has the tree property, $D \leq E$ in the Tukey ordering and $\mathsf{add}(D) = \mathsf{add}(E)$, then D also has the tree property. Thus having or not having the tree property depends only on the cofinal type of a directed set.*

Corollary 3.5 ([2]) *If D has the tree property, then $\mathsf{add}(D)$ has the tree property in the classical sense.*

Theorem 3.6 ([2]) *For a strongly inaccessible cardinal κ, the following are equivalent:*

(a) *κ is strongly compact.*
(b) *All directed sets D with $\mathsf{add}(D) = \kappa$ have the tree property.*

Condition (b) *also valid for $\kappa = \aleph_0$.*

4 The branching equivalence

Theorem 4.1 ([4], [5]) *Let D be a directed set and let $\kappa := \mathsf{add}(D)$ be strongly inaccessible. The following are equivalent:*

(a) *D has the tree property.*
(b) *For all $\lambda < \mathsf{wid}(D)$, $\mathcal{P}_\kappa\lambda$ has the tree property.*
(c) *For all $\lambda < \mathsf{wid}(D)$, $\mathcal{P}_\kappa\lambda$ is mildly ineffable.*

If we consider the generalization of the theorem by dropping the assumption that κ be strongly inaccessible, condition (c) does not make sense any more. In the proof of (a) $\Rightarrow$ (b) we use only that κ is regular. So we ask:

Problem 4.2 *Does the implication* (b) $\Rightarrow$ (a) *hold for regular, non-strong limit cardinals κ?*

Answering a question related to the above, Usuba has recently proved the following theorems:

Theorem 4.3 ([9])(PFA) *All directed sets with $\mathsf{add}(D) = \aleph_2$ have the tree property.*

Theorem 4.4 ([9]) *If $\mathcal{P}_\kappa\lambda$ has the tree property for some $\omega < \kappa < \lambda$, then $0^\#$ exists.*

These theorems indicate the consistency strength of the tree property in the non-strong limit case. Thus Problem 4.2 actually makes sense.

To investigate the problem, we look at method used in the proof of (a) $\Rightarrow$ (b) of Theorem 4.1, given in [5]. It tells us that there is a restriction on the way how an κ-arbor branches. We give some definitions needed to analyze the branching.

Definition 4.5 (branching equivalent levels) ([1]) Let $\langle D, \leq_D\rangle$ be a directed set and $\langle T, \leq_T, s\rangle$ a κ-arbor on D. For $d, d' \in D$ define a binary relation $L_{d,d'} \subseteq s^{-1}\{d\} \times s^{-1}\{d'\}$ by:

$$t, t' \in L_{d,d'} \iff \exists e \in D[e \geq_D d, d' \wedge \exists u \in s^{-1}\{e\}[t \leq_T u \wedge t' \leq_T u]].$$

We say that t and t' are *linked*.

Next define an order relation $\preceq$ on D by:

$$d \preceq d' \iff L_{d,d'} \text{ is a function from } s^{-1}\{d'\} \text{ to } s^{-1}\{d\}.$$

If $d \preceq d'$ holds, we say level d' *decides* level d. Clearly $d \leq d'$ implies $d \preceq d'$. The meaning of $d \preceq d'$ is that if the faithful embedding is given at level d', there is exactly one possible choice to extend it to level d.

Finally define an equivalence relation $\sim$ on D by the usual way; $d \sim d' \iff d \preceq d' \wedge d' \preceq d$, and let $\mathcal{B} := D/\sim$ be the set of equivalence classes. We call the elements of $\mathcal{B}$ the *branching equivalence classes*.

Proposition 4.6 *With the notation of Definition 4.5, assume that κ is weakly compact. Then $|\mathcal{B}_{\leq[d]}| < \kappa$ for $d \in D$.*

Proof Suppose on the contrary that $|\mathcal{B}_{\leq[d]}| \geq \kappa$, and let $\langle d_\alpha \mid \alpha < \kappa\rangle \in {}^\kappa D$ be a sequence such that $d_\alpha \preceq d$ for $\alpha < \kappa$ and $d_\alpha \not\sim d_\beta$ for $\alpha < \beta < \kappa$. As $d_\alpha \not\sim d_\beta$, either $d_\alpha \not\preceq d_\beta$ or $d_\beta \not\preceq d_\alpha$ for $\alpha < \beta < \kappa$. Since κ has the partition property $\kappa \to (\kappa)^2_2$, without loss of generality, $d_\alpha \not\preceq d_\beta$ for $\alpha < \beta < \kappa$, or $d_\beta \not\preceq d_\alpha$ for $\alpha < \beta < \kappa$. Assume the first case. (The other case can be treated similarly.)

For each pair $\langle\alpha, \beta\rangle$ with $\alpha < \beta < \kappa$ there are $t_{\langle\alpha,\beta\rangle} \in s^{-1}\{d_\beta\}$, $u^1_{\langle\alpha,\beta\rangle}, u^2_{\langle\alpha,\beta\rangle} \in s^{-1}\{d_\alpha\}$ such that

$$\begin{aligned} &\langle u^1_{\langle\alpha,\beta\rangle}, t_{\langle\alpha,\beta\rangle}\rangle, \langle u^2_{\langle\alpha,\beta\rangle}, t_{\langle\alpha,\beta\rangle}\rangle \in L_{d_\alpha,d_\beta} \\ \text{and} \quad & u^1_{\langle\alpha,\beta\rangle} \neq u^2_{\langle\alpha,\beta\rangle}. \end{aligned}$$

Since $d_\alpha \leq d$ there are $v^1_{\langle\alpha,\beta\rangle}, v^2_{\langle\alpha,\beta\rangle} \in s^{-1}\{d\}$ such that

$$\begin{aligned} & v^1_{\langle\alpha,\beta\rangle} \text{ is linked to } t_{\langle\alpha,\beta\rangle} \text{ and } u^1_{\langle\alpha,\beta\rangle}, \\ & v^2_{\langle\alpha,\beta\rangle} \text{ is linked to } t_{\langle\alpha,\beta\rangle} \text{ and } u^2_{\langle\alpha,\beta\rangle}, \\ \text{and} \quad & v^1_{\langle\alpha,\beta\rangle} \neq v^2_{\langle\alpha,\beta\rangle}. \end{aligned}$$

So, there is a map $\kappa^2 \ni \langle\alpha, \beta\rangle \mapsto \langle v^1_{\langle\alpha,\beta\rangle}, v^2_{\langle\alpha,\beta\rangle}\rangle \in (s^{-1}\{d\})^2$. Since $|(s^{-1}\{d\})^2| < \kappa$, by the partition property again, there are $\alpha, \beta, \gamma \in \kappa$ such that $\alpha < \beta < \gamma$ and $v^i_{\langle\alpha,\beta\rangle} = v^i_{\langle\beta,\gamma\rangle}$ for $i = 1, 2$. Then $v^1_{\langle\alpha,\beta\rangle} = v^1_{\langle\beta,\gamma\rangle}$ is linked to $u^1_{\langle\alpha,\beta\rangle}$ and $t_{\langle\alpha,\beta\rangle}$, so $u^1_{\langle\alpha,\beta\rangle} = t_{\langle\alpha,\beta\rangle}$. Likewise $u^2_{\langle\alpha,\beta\rangle} = t_{\langle\alpha,\beta\rangle}$. But this contradicts the assumption that $u^1_{\langle\alpha,\beta\rangle} \neq u^2_{\langle\alpha,\beta\rangle}$. Thus $|\mathcal{B}_{\leq[d]}| < \kappa$. □

Proposition 4.7 *With the notation of Definition 4.5, let κ be a cardinal and let $\trianglelefteq$ be a pre-ordering (a relation which is reflexive and transitive) on D such that*

$$d \leq_D d' \Rightarrow d \trianglelefteq d'.$$

Assume that

$$|\mathcal{B}'_{\trianglelefteq[d]}| < \kappa \text{ for } d \in D, \tag{$*$}$$

where $\mathcal{B}' := D/\sim$ and $d \sim d' \iff d \trianglelefteq d' \wedge d' \trianglelefteq d$. Then there is a κ-arbor T on D such that $\trianglelefteq$ coincides with the relation $\preceq$ ('decides') on D with respect to T.

Proof Define a κ-arbor $\langle T, \leq_T, s\rangle$ by

$$s^{-1}\{d\} := \{\langle t, d\rangle \mid t \text{ is a function from } \mathcal{B}'_{\trianglelefteq[d]} \text{ to } 2, \text{ and } |t^{-1}\{1\}| \leq 1\},$$
$$\langle t, d\rangle \leq_T \langle t', d'\rangle \iff d \leq_D d' \text{ and } t = t' \upharpoonright \mathcal{B}'_{\trianglelefteq[d]}.$$

By the condition $(*)$, $|s^{-1}\{d\}| < \kappa$ for $d \in D$.

Downwards uniqueness is clear. We have to check upwards access. Given arbitrary $d \leq_D d'$ and $\langle t, d\rangle \in s^{-1}\{d\}$, we have to find some $\langle t', d'\rangle \geq_D \langle t, d\rangle$. Just take $t' \supseteq t$ so that $t' \upharpoonright \mathcal{B}'_{\trianglelefteq[d]} = t$ and $t'([e]) = 0$ for $[e] \in \mathcal{B}'_{\trianglelefteq[d']} \setminus \mathcal{B}'_{\trianglelefteq[d]}$.

Now we shall see that $\trianglelefteq$ coincides with the relation 'decides'.

(1) $d \trianglelefteq d' \Rightarrow d'$ decides d

Consider the map $\varphi\colon s^{-1}\{d'\} \ni \langle t, d'\rangle \mapsto \langle t \upharpoonright \mathcal{B}'_{\trianglelefteq[d]}, d\rangle \in s^{-1}\{d\}$. We check that this map witnesses d' decides d. Given any $\langle t, d'\rangle \in s^{-1}\{d'\}$ and $d, d' \in D$ with $d \trianglelefteq d'$, pick $d_0 \geq d, d'$ in D. By upwards access, there is $\langle t_0, d_0\rangle \in s^{-1}\{d_0\}$ such that $\langle t, d'\rangle \leq \langle t_0, d_0\rangle$, i.e. $t = t_0 \upharpoonright \mathcal{B}'_{\trianglelefteq[d']}$. But $t \upharpoonright \mathcal{B}'_{\trianglelefteq[d]} = t_0 \upharpoonright \mathcal{B}'_{\trianglelefteq[d']}$ since $d \trianglelefteq d'$. Hence $\langle t \upharpoonright \mathcal{B}'_{\trianglelefteq[d]}, d\rangle \leq \langle t_0, d_0\rangle$. Since $\langle t_0, d_0\rangle$ was taken arbitrary, this shows that d' decides d by the map φ.

(2) d' decides $d \Rightarrow d \trianglelefteq d'$

Assume that $d \not\trianglelefteq d'$. Consider the elements $\langle t_i, d\rangle \in s^{-1}\{d\}$ $(i = 0, 1)$, where $t_i([d]) = i$ and $t([e]) = 0$ for $[e] \neq [d]$. Then both of $\langle t_i, d\rangle$ are linked to $\langle 0, d'\rangle$, where 0 is the constant zero function on $\mathcal{B}'_{\trianglelefteq[d']}$. Thus d' does not decide d.

Hence $\trianglelefteq$ is exactly the relation 'decides' on D. $\square$

Theorem 4.8 *Let κ be a weakly compact cardinal. Then for a partition $\mathcal{B}'$ of D into nonempty sets, the following are equivalent:*

(a) *$\mathcal{B}'$ is the set of branching equivalent classes with respect to some κ-arbor T.*

(b) *$\mathcal{B}'$ is obtained from a pre-ordering $\trianglelefteq$ as described in Proposition 4.7 and satisfies $|\mathcal{B}'_{\trianglelefteq[d]}| < \kappa$ for $d \in D$.*

Instead of considering partitions of D, we may take a set of representatives $Y \subseteq D$ with respect to the branching equivalence relation $\sim$. Now, what are the conditions that a set of representatives $Y \subseteq D$ does satisfy? Since subsets of D are easier than pre-orderings on D to handle with, this setting makes the construction of arbors easier. We hope that from considering all possible Y, we get enough information about the branching structure.

Lemma 4.9 *Let D be a directed set with $\kappa = \mathsf{add}(D)$ and let Y be a set of representatives with respect to the branching equivalence relation of some κ-arbor T. Then all increasing κ-chains of Y are unbounded in D.*

Proof Assume that $\langle d_\alpha \mid \alpha < \kappa\rangle$ is a κ-chain in Y with upper bound $e \in D$. By recursion on α we choose elements $\langle u_\alpha \mid \alpha < \kappa\rangle$ in $s^{-1}\{e\}$ such that

$$\forall\alpha < \kappa\forall\beta, \beta' < \alpha\ [\beta \neq \beta' \rightarrow u_\beta \downarrow d_\alpha \neq u_{\beta'} \downarrow d_\alpha]. \qquad (\star)$$

Here $u \downarrow d$ denotes the unique element v at level d which satisfies $v \leq_T u$. Suppose $\alpha_0 < \kappa$ and we have already a sequence $\langle u_\alpha \mid \alpha < \alpha_0\rangle$ satisfying $(\star)$ up to $\alpha = \alpha_0$. Now look at level d_{α_0} and d_{α_0+1}. Since level d_{α_0} does not decide level d_{α_0+1}, there are

$$t \in s^{-1}\{d_{\alpha_0}\},$$
$$v_0, v_1 \in s^{-1}\{d_{\alpha_0+1}\}$$

such that $t \leq_T v_0, v_1$ and $v_0 \neq v_1$. By $(\star)$, not both of v_0, v_1 can be among $u_\alpha \downarrow d_{\alpha_0+1}$ $(\alpha < \alpha_0)$. So, we can choose $u_{\alpha_0} \in s^{-1}\{e\}$ so that $u_{\alpha_0} \downarrow d_{\alpha_0+1} \notin \{u_\alpha \downarrow d_{\alpha_0+1} \mid \alpha < \alpha_0\}$. This ensures the successor step of the construction.

If α is a limit ordinal and for every $\beta < \alpha$ the sequence $\langle u_{\beta'} \mid \beta' < \beta\rangle$ satisfies $(\star)$, then $\langle u_\beta \mid \beta < \alpha\rangle$ also satisfies $(\star)$.

Hence we succeed in constructing the required sequence $\langle u_\alpha \mid \alpha < \kappa\rangle$. Surely these u_α are distinct elements of $s^{-1}\{e\}$. But this contradicts the restriction of the size of $s^{-1}\{e\}$. $\square$

Proposition 4.10 *Let D be a directed set with $\kappa = \mathsf{add}(D)$ and let Y be a set of representatives with respect to the branching equivalence relation of some κ-arbor T. Then*

- *For all $d \in D$, $|Y_{\leq d}| < \sup\{(2^\theta)^+ \mid \theta < \kappa\}$, and*
- *All increasing κ-chains of Y are unbounded in D.*

Proof The first condition is obtained from the proof of [5, Lemma 8.10] □

Problem 4.11 *Are the above two conditions all that can be said about Y? In other words, given a subset $Y \subseteq D$ satisfying the above two conditions, is there always a κ-arbor T on D such that each two distinct elements of Y lie in different branching equivalence classes?*

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