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On p-valently uniformly starlike functions

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Abstract

Let $\mathcal{A}_p$ be the class of analytic and multivalent functions $f(z)$ in the open unit disk $\mathbb{U}$. Furthermore, let $\mathcal{SD}_p(\alpha,\beta)$ be the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ related to uniformly starlikeness. The object of the present paper is to derive coefficient inequalities for $f(z)$ belonging to the class $\mathcal{SD}_p(\alpha,\beta)$ and consider the generalized convolution for the class $\mathcal{SD}_p(\alpha,\beta)$ by using Hölder-type inequality.

1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z)=z^p+\sum_{n=p+1}^{\infty}a_nz^n \qquad (p=1,2,3,\cdots)$$

which are analytic and multivalent in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$. A function $f(z)\in\mathcal{A}_p$ is said to be in the class $\mathcal{SD}_p(\alpha,\beta)$ if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\left|\frac{zf'(z)}{f(z)}-p\right|+\beta \qquad (z\in\mathbb{U})$$

for some α $(\alpha\geqq 0)$ and β $(0\leqq\beta<p)$. If $p=1$ for $f(z)\in\mathcal{A}_1\equiv\mathcal{A}$, then $f(z)\in\mathcal{SD}_1(\alpha,\beta)$ is equivalent to

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\left|\frac{zf'(z)}{f(z)}-1\right|+\beta \qquad (z\in\mathbb{U})$$

for some α $(\alpha\leqq 0)$ and β $(0\leqq\beta<1)$. This class $\mathcal{SD}_1(\alpha,\beta)\equiv\mathcal{SD}(\alpha,\beta)$ was introduced by Shams, Kullcarni and Jahangiri [6]. Lately, it was studied by Nishiwaki and Owa [3].

Remark 1.1. For $f(z)\in\mathcal{SD}_p(\alpha,\beta)$, we write $w(z)=zf'(z)/f(z)=u+iv$. If $\alpha>1$, then w lies in the domain which is the part of the complex plane which contains $w=p$ and is bounded by the elliptic domain such that

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$$\frac{\left(u-\dfrac{\alpha^2 p-\beta}{\alpha^2-1}\right)^2}{\left(\dfrac{\alpha(p-\beta)}{\alpha^2-1}\right)^2}+\frac{v^2}{\left(\dfrac{p-\beta}{\sqrt{\alpha^2-1}}\right)^2}<1.$$

If $\alpha = 1$, then w lies in the domain which is the part of the complex plane which contains $w = p$ and is bounded by the parabolic domain such that

$$u > \frac{v^2}{2(p-\beta)} + \frac{p+\beta}{2}.$$

If $0 \leqq \alpha < 1$, then w lies in the domain which is the part of the complex plane which contains $w = p$ and is bounded by the right side of the hyperbolic domain such that

$$\frac{\left(u+\dfrac{\alpha^2 p-\beta}{1-\alpha^2}\right)^2}{\left(\dfrac{\alpha(p-\beta)}{1-\alpha^2}\right)^2}-\frac{v^2}{\left(\dfrac{p-\beta}{\sqrt{1-\alpha^2}}\right)^2}>1.$$

lemma 1.1. *If $f(z) \in \mathcal{A}_p$ satisfies*

$$\sum_{n=p+1}^{\infty} \{(p-\beta)+(1+\alpha)(n-p)\}|a_n| \leqq p-\beta \tag{1.1}$$

for some α $(\alpha \geqq 0)$ and β $(0 \leqq \beta < p)$, then $f(z) \in \mathcal{SD}_p(\alpha, \beta)$.

We define the subclass $\mathcal{SD}_p^*(\alpha,\beta)$ of $\mathcal{A}_p$ consisting of functions $f(z)$ which satisfy the coefficient inequality (1.1). In view of Lemma 1.1, we know that $\mathcal{SD}_p^*(\alpha,\beta) \subset \mathcal{SD}_p(\alpha,\beta) \subset \mathcal{A}_p$

In the purpose of this paper, we investigate some interesting properties for functions $f(z)$ in the class $\mathcal{SD}_p^*(\alpha,\beta)$.

2 Convolution properties for functions in the class $\mathcal{SD}_p^*(\alpha,\beta)$

In this section, some generalized convolution properties for functions $f(z)$ to be in the class $\mathcal{SD}_p(\alpha,\beta)$ are discussed. First of all, for functions $f_j(z) \in \mathcal{A}_p$ given by

$$(f_1 * f_2 * f_3 * \cdots * f_m)(z) = z^p + \sum_{n=p+1}^{\infty}\left(\prod_{j=1}^{m} a_{n,j}\right) z^n \qquad (j = 1, 2, 3, \cdots, m)$$

we define the following generalization of the Hadamard product (or convolution):

$$H_{p,m}(z) = z^p + \sum_{n=p+1}^{\infty} \left(\prod_{j=1}^{m} a_{n,j}^{p_j} \right) z^n \qquad (p_j > 0)$$

The generalized convolution $H_{p,m}(z)$ was considered by Choi, Kim and Owa [1]. Lately, it was studied by Srivastava and Owa [5] (also see [2][4]).

For functions $f_j(z) \in \mathcal{A}_p$, Hölder inequality is given by

$$\sum_{n=p+1}^{\infty} \left(\prod_{j=1}^{m} |a_{n,j}| \right) \leqq \prod_{j=1}^{m} \left(\sum_{n=p+1}^{\infty} |a_{n,j}|^{p_j} \right)^{\frac{1}{p_j}} \qquad (j = 1, 2, 3, \cdots, m),$$

where $p_j > 1$ and $\sum_{j=1}^{m} \frac{1}{p_j} \leqq 1$.

Our first result for $H_{p,m}(z)$ is contained in

Theorem 2.1. *If $f_j(z) \in \mathcal{SD}_p^*(\alpha, \beta_j)$ for each $j = 1, 2, 3, \cdots, m$ $(\alpha \geqq 0\ ,\ 0 \leqq \beta_j < p)$, then $H_{p,m}(z) \in \mathcal{SD}_p^*(\alpha, \beta^*)$ with*

$$\beta^* = p - \frac{(1+\alpha) \prod_{j=1}^{m} (p-\beta_j)^{p_j}}{\prod_{j=1}^{m} \{(p-\beta_j) + (1+\alpha)\}^{p_j} - \prod_{j=1}^{m} (p-\beta_j)^{p_j}},$$

where $\sum_{j=1}^{m} p_j \geqq 1 + \frac{p-\beta_j^*}{1+\alpha}$ $(\beta_j^* = \min\{\beta_j\})$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

Letting $\beta_j = \beta$ $(j = 1, 2, 3, \cdots, m)$ in Theorem 2.1, we obtain

Corollary 2.1. *If $f_j(z) \in \mathcal{SD}_p^*(\alpha, \beta)$ for each $j = 1, 2, 3, \cdots, m$ $(\alpha \geqq 0\ ,\ 0 \leqq \beta < p)$, then $H_{p,m}(z) \in \mathcal{SD}_p^*(\alpha, \beta^*)$ with*

$$\beta^* = p - \frac{(1+\alpha)(p-\beta)^s}{\{(p-\beta) + (1+\alpha)\}^s - (p-\beta)^s},$$

where $s = \sum_{j=1}^{m} p_j \geqq 1 + \frac{p-\beta}{1+\alpha}$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

If we take $p = 1$ in Theorem 2.1, we deduce our next result.

Corollary 2.2. *If $f_j(z) \in \mathcal{SD}^*(\alpha, \beta_j)$ for each $j = 1, 2, 3 \cdots, m$ ($\alpha \geqq 0$, $0 \leqq \beta_j < 1$), then $H_{1,m}(z) \in \mathcal{SD}^*(\alpha, \beta^*)$ with*

$$\beta^* = 1 - \frac{(1+\alpha)\prod_{j=1}^{m}(1-\beta_j)^{p_j}}{\prod_{j=1}^{m}\{(1-\beta_j)+(1+\alpha)\}^{p_j} - \prod_{j=1}^{m}(1-\beta_j)^{p_j}},$$

where $\sum_{j=1}^{m} p_j \geqq 1 + \frac{1-\beta_j^*}{1+\alpha}$ $(\min\{\beta_j\} = \beta_j^*)$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

On setting $\beta_j = \beta$ in Corollary 2.2, we have the next result besides.

Corollary 2.3. *If $f_j(z) \in \mathcal{SD}^*(\alpha, \beta)$ for each $j = 1, 2, 3 \cdots, m$ ($\alpha \geqq 0$, $0 \leqq \beta < 1$), then $H_{1,m}(z) \in \mathcal{SD}^*(\alpha, \beta^*)$ with*

$$\beta^* = 1 - \frac{(1+\alpha)(1-\beta)^s}{\{(1-\beta)+(1+\alpha)\}^s - (1-\beta)^s},$$

where $s = \sum_{j=1}^{m} p_j \geqq 1 + \frac{1-\beta}{1+\alpha}$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

By using $\mathcal{SD}_p^*(\alpha_j, \beta)$ instead of $\mathcal{SD}_p^*(\alpha, \beta_j)$ in Theorem 2.1, we also derive Theorem 2.2 below.

Theorem 2.2. *If $f_j(z) \in \mathcal{SD}_p^*(\alpha_j, \beta)$ for each $j = 1, 2, 3, \cdots, m$ ($\alpha_j \geqq 0$, $0 \leqq \beta < p$), then $H_{p,m}(z) \in \mathcal{SD}_p^*(\alpha^*, \beta)$ with*

$$\alpha^* = \frac{\prod_{j=1}^{m}\{(p-\beta)+(1+\alpha_j)\}^{p_j} - \prod_{j=1}^{m}(p-\beta)^{p_j}}{\prod_{j=1}^{m}(p-\beta)^{p_j-1}} - 1$$

where $\sum_{j=1}^{m} p_j \geqq 1 + \frac{p-\beta}{1+\alpha_j^*}$ $(\alpha_j^* = \min\{\alpha_j\})$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

Taking $\alpha_j = \alpha$ in Theorem 2.2, we get

Corollary 2.4. *If $f_j(z) \in \mathcal{SD}_p^*(\alpha, \beta)$ for each $j = 1, 2, 3, \cdots, m$ ($\alpha \geqq 0$, $0 \leqq \beta < p$), then $H_{p,m}(z) \in \mathcal{SD}_p^*(\alpha^*, \beta)$ with*

$$\alpha^* = \frac{\{(p-\beta)+(1+\alpha)\}^s - (p-\beta)^s}{(p-\beta)^{s-1}} - 1$$

where $s = \sum_{j=1}^{m} p_j \geqq 1 + \frac{p-\beta}{1+\alpha}$, $p_j \geqq \frac{1}{q_j}$, *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

By setting $p=1$ in Theorem 2.2, we can derive

Corollary 2.5. *If* $f_j(z) \in \mathcal{SD}^*(\alpha_j, \beta)$ *for each* $j = 1,2,3,\cdots,m$ $(\alpha_j \geqq 0\ ,\ 0 \leqq \beta < 1)$, *then* $H_{1,m}(z) \in \mathcal{SD}^*(\alpha^*, \beta)$ *with*

$$\alpha^* = \frac{\prod_{j=1}^{m}\{(1-\beta)+(1+\alpha_j)\}^{p_j} - \prod_{j=1}^{m}(1-\beta)^{p_j}}{\prod_{j=1}^{m}(1-\beta)^{p_j-1}} - 1$$

$\sum_{j=1}^{m} p_j \geqq 1 + \frac{1-\beta}{1+\alpha_j^*}$ $(\min\{\alpha_j\} = \alpha_j^*)$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

Finally, putting $\alpha_j = \alpha$ in Corollary 2.5, we obtain the following result

Corollary 2.6. *If* $f_j(z) \in \mathcal{SD}^*(\alpha, \beta)$ *for each* $j = 1,2,3,\cdots,m$ $(\alpha \geqq 0\ ,\ 0 \leqq \beta < 1)$, *then* $H_{1,m}(z) \in \mathcal{SD}^*(\alpha^*, \beta)$ *with*

$$\alpha^* = \frac{\{(1-\beta)+(1+\alpha_j)\}^s - (1-\beta)^s}{(1-\beta)^{s-1}} - 1$$

$\sum_{j=1}^{m} p_j \geqq 1 + \frac{1-\beta}{1+\alpha_j^*}$ $(\min\{\alpha_j\} = \alpha_j^*)$, $p_j \geqq \frac{1}{q_j}$ *and* $\sum_{j=1}^{m} \frac{1}{q_j} \geqq 1.$

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